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Asymptotic solution of a class
of inhomogeneous integral equations

Riccardo Croci

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Abstract

The purpose of this paper is to derive the asymptotic solutions to a class of inhomogeneous integral equations which reduce to algebraic equations when a parameter η goes to zero (the kernel becoming proportional to a Dirac δ -function). This class includes the integral equations obtained from the system of Vlasov and Poisson equations for the Fourier transform in space and the Laplace transform in time of the electric potential, when the equilibrium magnetic field is uniform and the equilibrium plasma density depends on ηx , with the coordinate z being the direction of the magnetic field. In this case the inhomogeneous term is given by the initial conditions and possibly by sources, and the Laplace transform variable ω is the eigenvalue parameter.

Introduction

The system of Vlasov and Poisson equations for the electric potential of a plasma referred to cartesian coordinates, with the coordinate z parallel to the uniform equilibrium magnetic field and the equilibrium plasma density depending on ηx , η being a parameter, yields an integral equation for the Fourier transform in space and the Laplace transform in time of the electric potential. The inhomogeneous term of this equation is given by the initial conditions and possible sources, and the Laplace transform variable ω is the eigenvalue parameter. This equation belongs to a class of integral equations which reduce to algebraic equations when a parameter goes to zero, η in our case, because the kernel becomes proportional to a Dirac δ -function. The purpose of this paper is to derive the asymptotic solutions to this class of integral equations when the parameter η goes to zero. The integration method used requires that, for every value of ω , two linearly independent solutions of the corresponding homogeneous integral equation be known (only one of the solutions being integrable at the eigenvalues). Part of this problem has already been solved; see, for example, Ref. [1], and Ref. [2] for a more general case; indeed, in the last reference the normal modes of a vector system of integral equations were derived by using a version of the WKB method which embodies a variational method for obtaining the equation for the single-wave amplitudes. Here, however, a somewhat different variety of WKB is chosen in order to apply the same method to both the homogeneous and the inhomogeneous integral equations. It consists in deriving asymptotic approximations for the solution $\Phi(k)$ in various superposed k -intervals which are such that one form of the solution matches onto another form in the common subinterval (a 'matched asymptotic procedure'; see, for example, Ref. [3]). In section 1 we derive the asymptotic solutions of the homogeneous integral equation for a particular but relevant equilibrium plasma density profile. In the intervals which do not contain values of k which solve the dispersion relation (i.e. the homogeneous

integral equation with $\eta = 0$), the results are the same as those derived in [2]; in the other intervals the integral equation is approximated by an Airy differential equation; the appropriate linear combination of Airy functions is then determined by the matching condition. The eigenvalue condition is obtained in a compact and general form when it is required that the solution be integrable. In section 2, two linearly independent solutions of the homogeneous integral equation, that are integrable for general ω values, are used to derive the integrable solution of the inhomogeneous integral equation, in the following way. A physical argument leads to an asymptotic approximation for $\Phi(k)$, valid everywhere on the k axis except for the intervals about the real values of k which are solutions of the dispersion relation. In these intervals (of amplitude proportional to $\eta^{2/3}$) the integral equation is approximately written as an inhomogeneous Airy differential equation. After a particular solution is derived by the usual method, the correct solution is obtained by imposing the matching condition in one of the two neighbouring intervals. The matching in the other interval is made possible by adding there a suitable linear combination of the solutions of the homogeneous integral equation. Generalizations of the form of the kernel are introduced and discussed in section 3.

1. Homogeneous electrostatic equation

As is well known, the system of the Vlasov and Poisson equations for a plasma in a uniform magnetic field leads to an integral equation in the (k, ω) space. For an electron plasma with density profile $\exp[-\eta^2 x^2]$ and with electron Larmor radius much smaller than the inhomogeneity length ($k \equiv k_x$ in the inhomogeneity direction and k_z in the magnetic field direction; $k_y = 0$) the integral equation assumes the relatively simple form (see, for example, Ref. [4])

$$\Phi_h(k, \omega) - \frac{\sigma(k, \omega)}{2\eta\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(k'-k)^2/4\eta^2} \Phi_h(k', \omega) dk' = 0, \quad (1)$$

where $\sigma \equiv k^2 \epsilon_{11} + 2kk_z \epsilon_{13} + k_z^2 \epsilon_{33}$, and ϵ_{ik} is the dielectric tensor for the corresponding homogeneous plasma (whose dispersion relation would be $1 - \sigma = 0$); ω is the variable of the time Laplace transform.

In the following we consider integral equations of the form given by eq. (1) for functions σ which are zero for $|k|$ going to infinity and are unity at the values k_i ($i = 1, 2, \dots, 2N$), which are the zeros of the dispersion relation $1 - \sigma = 0$. A particular case is a function σ proportional to $1/(k^2 + K^2)$, where K is a constant, which allows integral equation (1) to be reduced to a differential equation in the x space. Generalizations of the density profile and of the form of σ are considered in section 3.

We write $\Phi_h(k, \omega) \equiv \exp \int_{-k}^k (g/\eta) dk'$ and expand the exponent of the integrand of eq. (1) in powers of $(k' - k)$, because the kernel of eq. (1) is sharply peaked at $k' = k$. In this way we get, with the definition $g' \equiv \partial g / \partial k$,

$$1 - \frac{\sigma}{\sqrt{1 - 2\eta g'}} \exp \left\{ g^2 \frac{1}{1 - 2\eta g'} \right\} = 0 \quad (2)$$

if $2\eta g' < 1$. By neglecting $\eta g'$ we get the reduced equation

$$1 - \sigma \exp \{g^2\} = 0. \quad (3)$$

We shall see later that eq. (3) is a good approximation in the k interval where $|1-\sigma| \gg (\sigma'_i \eta)^{2/3}$, with σ'_i being the value of $\partial\sigma/\partial k$ taken at $k = k_i$. From eq. (3) it follows that integral equation (1) has two solutions, one of which is integrable at the eigenvalues, which we do not yet know. In order to obtain the next-order approximation of g , we take the logarithm of eq. (2):

$$\frac{1}{2} \ln(1 - 2\eta g') = \ln \sigma + g^2 / (1 - 2\eta g') . \quad (4)$$

An expansion of the logarithm now yields

$$-(\ln \sigma + \eta g')(1 - 2\eta g') = g^2 . \quad (5)$$

This equation shows that the approximate solution $g^2 = \ln \sigma$ is valid if $2\eta g' \ll \ln \sigma$, i.e. if $|\sigma'_i| \eta \ll |1 - \sigma|^{3/2}$, as previously stated.

The next approximation, g_1 , follows from eq. (5):

$$2g_0 g_1 = (2 \ln \sigma - 1) \eta g'_0 , \quad (6)$$

which can be written

$$2g_1 = -\eta (g_0^2)' - \eta (\ln |g_0|)' . \quad (7)$$

We finally get the known result

$$\int_{k'}^k (g_1/\eta) dk' = \frac{1}{2} \ln \sigma - \frac{1}{4} \ln |\ln \sigma| . \quad (8)$$

We are now in a position to write explicitly two independent solutions of eq. (1), which we designate 'elementary solutions', in the different intervals we have considered up to now. The L_2 solution of eq. (1) is a linear combination, yet to be determined, of the 'elementary solutions'. First of all we give a name to the different intervals, which, as the figure shows, partly overlap.

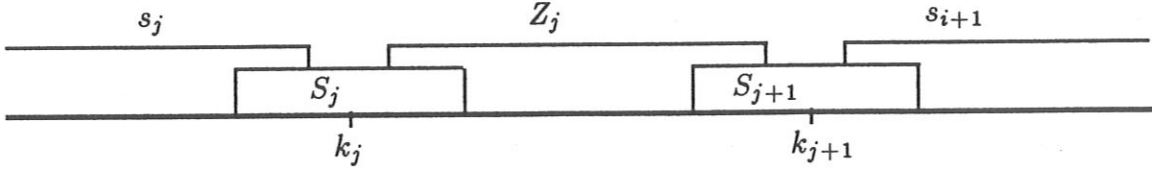


Fig. 1: Overlapping intervals for the matched asymptotic expansion.

Let S_j be the intervals about the points $k = k_j$ where $|1 - \sigma| \ll 1$. The intervals where $|1 - \sigma| \gg |\sigma'_j \eta|^{2/3}$ will be: s_j , where $k < k_j + O(|\sigma'_j \eta|^{2/3})$; Z_j , for k between $k_j + O(|\sigma'_j \eta|^{2/3})$ and $k_{j+1} - O(|\sigma'_{j+1} \eta|^{2/3})$; and finally s_{j+1} , where $k > k_{j+1} + O(|\sigma'_{j+1} \eta|^{2/3})$.

In s_i we choose the following functions as 'elementary solutions':

$$f_1 \equiv \frac{\sqrt{\sigma}}{|\ln \sigma|^{1/4}} \exp \left[\frac{1}{\eta} \int_{k_1}^k \sqrt{|\ln \sigma|} dk' \right], \quad (9)$$

$$f_2 \equiv \frac{\sqrt{\sigma}}{|\ln \sigma|^{1/4}} \exp \left[\frac{-1}{\eta} \int_{k_1}^k \sqrt{|\ln \sigma|} dk' \right]. \quad (10)$$

In the interval Z_i we choose

$$F_1 \equiv \frac{\sqrt{\sigma}}{(\ln \sigma)^{1/4}} \cos (G(k) - (\pi/4)), \quad F_2 \equiv \frac{\sqrt{\sigma}}{(\ln \sigma)^{1/4}} \sin (G(k) - (\pi/4)), \quad (11)$$

where

$$G(k) \equiv (1/\eta) \int_{k_1}^k \sqrt{\ln \sigma} dk'. \quad (12)$$

The solution in S_i is obtained by expanding $\Phi(k')$ in eq. (1) about the value $k' = k$:

$$\Phi_h(k) - \frac{\sigma}{2\eta\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\Phi_h(k) + \frac{(k' - k)^2}{2} \Phi_h''(k) \right) e^{-(k' - k)^2/4\eta^2} dk' = 0. \quad (13)$$

It follows that

$$(1 - \sigma)\Phi_h - \sigma\eta^2\Phi_h'' = 0, \quad (14)$$

which, by introducing the definition $y \equiv \sigma'_i(k - k_i)$, can be written, in the interval $|y| \ll 1$,

$$\frac{\partial^2 \Phi_h}{\partial y^2} + \frac{y}{\eta^2 \sigma'^2_i} \Phi_h = 0. \quad (15)$$

Two independent solutions of eq. (15) for the interval S_i are the functions

$$\sqrt{y} J_{\pm 1/3} \left(\frac{2y^{3/2}}{3\eta|\sigma'_i|} \right), \quad (16)$$

but the following two solutions, proportional to the Airy functions, are better choices as ‘elementary solutions’:

$$\Phi_{1,2}^{(i)} \equiv \sqrt{y} \left(J_{1/3} \left(\frac{2y^{3/2}}{3\eta|\sigma'_i|} \right) \pm J_{-1/3} \left(\frac{2y^{3/2}}{3\eta|\sigma'_i|} \right) \right). \quad (17)$$

Some properties of $\Phi_{1,2}^{(i)}$ which will be useful later are now listed. Their Wronskian is

$$W \equiv (6/\pi) \sin(\pi/3). \quad (18)$$

Their asymptotic form in $(S_i \cap Z_i)$ is

$$\Phi_1^{(i)} \approx 2a^{(i)} \cos(\pi/6) \frac{1}{y^{1/4}} \cos \left(\frac{2y^{3/2}}{3\eta|\sigma'_i|} - (\pi/4) \right), \quad (19)$$

$$\Phi_2^{(i)} \approx 2a^{(i)} \sin(\pi/6) \frac{1}{y^{1/4}} \sin \left(\frac{2y^{3/2}}{3\eta|\sigma'_i|} - (\pi/4) \right), \quad (20)$$

where $a^{(i)} \equiv \sqrt{(3\eta|\sigma'_i|/\pi)}$. These expressions can also be written as a linear combination of the functions F_j , as eqs. (17) and (18) show:

$$\Phi_1^{(i)} = F_1 \cos G^{(i)} + F_2 \sin G^{(i)}, \quad \Phi_2^{(i)} = -F_1 \sin G^{(i)} + F_2 \cos G^{(i)}, \quad (21)$$

where $G^{(i)} \equiv G(k_i)$. In the following, relation (21) will be written as

$$\Phi_j^{(i)} = \beta_{jp}^{(i)} F_p, \quad (22)$$

with the convention that repeated indices have to be summed.

We also need the relation between F_j and $\Phi_j^{(i+1)}$ in the interval $(Z_i \cap S_{i+1})$, where σ'_{i+1} is negative. One easily gets the equations

$$\begin{aligned} F_1 &= \Phi_1^{(i+1)} \sin G^{(i+1)} - \Phi_2^{(i+1)} \cos G^{(i+1)}, \\ F_2 &= -\Phi_1^{(i+1)} \cos G^{(i+1)} - \Phi_2^{(i+1)} \sin G^{(i+1)}, \end{aligned} \quad (23)$$

which in the following are written as

$$F_j = b_{jp}^{(i+1)} \Phi_p^{(i+1)}. \quad (24)$$

Where $y < 0$, i.e. for $k < k_i$ or for $k > k_{i+1}$, eq. (17) is equivalent to

$$\Phi_{1,2}^{(1)} \equiv -\sqrt{|y|} \left(I_{1/3} \left(\frac{2|y|^{3/2}}{3\eta|\sigma'_i|} \right) \mp I_{-1/3}(\cdot) \right), \quad (25)$$

and therefore in $(s_i \cap S_i)$ one has

$$\Phi_1^{(i)} \approx a^{(i)} \sin(\pi/3) \frac{e^{-(2/3\eta|\sigma'_i|)|y|^{3/2}}}{|y|^{1/4}}; \quad \Phi_2^{(i)} \approx -a^{(i)} \frac{e^{(2/3\eta|\sigma'_i|)|y|^{3/2}}}{|y|^{1/4}}. \quad (26)$$

A comparison with the definition of f_j (see eq. (9)) leads to the relation

$$f_j = b_j^{(i)} \Phi_j^{(i)}, \quad (27)$$

where $b_1^{(i)} \equiv \exp[G^{(i)}] / a^{(i)} \sin(\pi/3)$ and $b_2^{(i)} \equiv -\exp[-G^{(i)}] / a^{(i)}$. In the interval $(S_{i+1} \cap s_{i+1})$ one gets the relation

$$\Phi_j^{(i+1)} = c_{jp}^{(i+1)} f_p, \quad (28)$$

where $c_{jj} = 0$, $c_{12} \equiv a^{(i+1)} \sin(\pi/3) \exp[G^{(i+1)}]$ and $c_{21} \equiv -a^{(i+1)} \exp[-G^{(i+1)}]$.

The analytic continuation in the interval s_{i+1} of a linear combination of functions f_j in the interval s_i is again a linear combination of functions f_j , with different

coefficients; we are now in a position to evaluate the new coefficients. The change of k from s_i to successively S_i , Z_i , S_{i+1} and finally to s_{i+1} causes the following changes

$$f_j \rightarrow b_j^{(i)} \Phi_j^{(i)} \rightarrow b_j^{(i)} \beta_{jp}^{(i)} F_p \rightarrow b_j^{(i)} \beta_{jp}^{(i)} b_{pq}^{(i+1)} \Phi_q^{(i+1)} \rightarrow b_j^{(i)} \beta_{jp}^{(i)} b_{pq}^{(i+1)} c_{qs}^{(i+1)} f_s, \quad (29)$$

which are obtained by applying successively eqs. (27), (22), (24) and (28). This procedure can be formalized by introducing the following matrices, defined in a recursive way:

$$T_{pq}^{(i+1)} \equiv T_{ps}^{(i)} B_{sq}^{(i+1)} \quad T_{pq}^{(i+2)} \equiv T_{ps}^{(i+1)} C_{sq}^{(i+2)}, \quad (30)$$

where $T_{pq}^{(1)} \equiv \delta_{pq} \alpha_q$ (α_q being free constants), $C_{sq}^{(i+2)} \equiv c_{sq}^{(i+2)} b_q^{(i+2)}$, and

$$B_{sq}^{(i+1)} \equiv \begin{pmatrix} \sin(G^{(i+1)} - G^{(i)}) & -\cos(G^{(i+1)} - G^{(i)}) \\ -\cos(G^{(i+1)} - G^{(i)}) & -\sin(G^{(i+1)} - G^{(i)}) \end{pmatrix}. \quad (31)$$

The asymptotic approximation of Φ_{hp} in S_i is then given by

$$\Phi_{hp} \approx T_{pq}^{(i)} \Phi_q^{(i)}. \quad (32)$$

The eigenvalues are given by the condition that the f_1 coefficient of Φ_{h1} is zero after the last s interval, since Φ_{h1} is proportional to f_1 in s_1 (and hence goes to zero for k going to minus infinity), and f_1 diverges for k going to plus infinity. This condition is equivalent to

$$T_{12}^{(2N)} = 0, \quad (33)$$

because in every $(S_i \cap s_i)$ interval the function f_1 is matched by $\Phi_2^{(i)}$. We explicitly derive the eigenvalues first in the case of only one Z interval (as already stated, functions σ which allow integral equation (1) to be transformed into a differential equation

belong to this class). Equation (33) is then equivalent to

$$\cos \left(\frac{1}{\eta} \int_{k_1}^{k_2} \sqrt{\ln \sigma} dk' \right) = 0. \quad (34)$$

A first consequence of this equation is that since ω_n is such that $k_{1,2}(\omega_n)$ are real, ω_n is a value of the function $\Omega(k)$, where $\omega = \Omega(k)$ is a solution of $1 - \sigma(k, \omega) = 0$ for real k ; hence the imaginary part of ω_n cannot be larger (more unstable) than the maximum of the imaginary part of $\Omega(k)$, which determines the asymptotic behaviour in time of an homogeneous plasma. A further general consequence can be drawn when σ is symmetric; in this case eq. (34) is equivalent to the two equations

$$\frac{1}{\eta} \int_{k_1}^0 \sqrt{\ln \sigma} dk' = \frac{\pi}{4} + n\pi \quad \text{and} \quad \frac{1}{\eta} \int_{k_1}^0 \sqrt{\ln \sigma} dk' = \frac{\pi}{4} + (n + 1/2)\pi. \quad (35)$$

For the first class of eigenvalues the coefficient of f_2 in eq. (31), T_{11} , is positive, and therefore the corresponding eigenfunctions are symmetric; antisymmetric eigenfunctions correspond to the second class. With two Z intervals the eigenvalue condition (33) is equivalent to

$$\begin{aligned} & \frac{\exp [G^{(3)} - G^{(2)}]}{\sin(\pi/3)} \cos (G^{(2)} - G^{(1)}) \cos (G^{(4)} - G^{(3)}) - \\ & - \sin(\pi/3) \exp [-G^{(3)} + G^{(2)}] \sin (G^{(2)} - G^{(1)}) \sin (G^{(4)} - G^{(3)}) = 0. \end{aligned} \quad (36)$$

When σ is symmetric eq. (36) simplifies to

$$\sin^2 (G^{(2)} - G^{(1)}) = 1 / \left(1 + \sin^2(\pi/3) \exp [-2(G^{(3)} - G^{(2)})] \right). \quad (37)$$

The two equations equivalent to eq. (37) yield the eigenvalues corresponding to symmetric and antisymmetric eigenfunctions, respectively.

2. Inhomogeneous electrostatic equation

We now consider the integral equation

$$\Phi(k, \omega) - \frac{\sigma(k, \omega)}{2\eta\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(k'-k)^2/4\eta^2} \Phi(k', \omega) dk' = H, \quad (38)$$

where H is a given function (in the case of interest for plasma physics H is given by the conditions at $t = 0$ through a Laplace transform or by possible sources). We look for a solution of the form $\Phi \equiv \left[H \exp \int (g/\eta) dk \right]$; if, as we assume, the logarithmic derivative of H is much smaller than $1/\eta$ (also for $|k|$ going to infinity), the function $H(k')$ in the integral of eq. (1) can be approximated by $H(k)$ and taken out of the integral, because the kernel is sharply peaked at $k' = k$. Hence we expand the exponent of the integrand in powers of $(k' - k)$, as was done for the homogeneous equation, and we get

$$1 - \frac{\sigma}{\sqrt{1 - 2\eta g'}} \exp \left\{ g^2 \frac{1}{1 - 2\eta g'} \right\} = \exp \left\{ - \int (g/\eta) dk \right\}. \quad (39)$$

When $|k|$ goes to infinity the solution Φ is expected to assume the same form as in a homogeneous plasma, i.e. $\Phi \approx H/(1 - \sigma)$. Since this solution corresponds to $g \approx -\eta\sigma'/(1 - \sigma)$, the approximate solution is valid as far as $|1 - \sigma| \gg |\sigma'_i \eta|^{2/3}$, i.e. not only for $|k|$ going to infinity, but in every s_i and Z_i interval. In each of these intervals an arbitrary linear combination of Φ_{h1} and Φ_{h2} , the linearly independent solutions of the homogeneous integral equation, can be added to the approximate solution, the coefficients of the combination being in principle different for every interval.

In order to obtain an approximation valid in S_i , we expand the function $\Phi(k')$ in the integral of eq. (1) as we did for the homogeneous equation. Instead of eq. (15) we now get in S_i

$$\frac{\partial^2 \Phi}{\partial y^2} + \frac{y}{\eta^2 \sigma_i'^2} \Phi = - \frac{H}{\eta^2 \sigma_i'^2}. \quad (40)$$

Let us consider the following solution of eq. (40), obtained by using the 'elementary solutions' of the homogeneous differential equation we introduced in the preceding section

$$\phi^{(i)} \equiv (1/\eta^2 \sigma_i'^2 W) \left[\Phi_1^{(i)} \int_0^y H(y') \Phi_2^{(i)}(y') dy' - \Phi_2^{(i)} \int_{-\infty}^y H(y') \Phi_1^{(i)}(y') dy' \right], \quad (41)$$

where W , the Wronskian of $\Phi_{1,2}^{(i)}$, is given by eq. (18). In $(s_i \cap S_i)$ $\phi^{(i)}$ is approximately $-H/y$, i.e. $H/(1 - \sigma)$. Indeed, the largest contribution to the integrals comes from the interval $y' \approx y$; hence both integrals are approximately equal to $-(a_i^2 \sin(\pi/3) \eta \sigma_i' / |y|)$ (see (26)). The foregoing assertion follows from taking into account the value of W and the definition of a_i .

At the other extreme of S_i , in $(S_i \cap Z_i)$, the asymptotic form of $\Phi_j^{(i)}$, given by eqs. (19) and (20), is valid, and therefore, as differential equation (40) shows, $\phi^{(i)}$ has to be the sum of $-H/y$ and some linear combination of $\Phi_1^{(i)}$ and $\Phi_2^{(i)}$. Since $\Phi_j^{(i)}$ now rapidly oscillate and decrease only as $y^{-1/4}$, the coefficients of the linear combination are easily obtained by taking $y = \infty$ in the integrals of eq. (41); hence in $(S_i \cap Z_i)$ we have

$$\phi^{(i)} = -\frac{H}{y} + \frac{\Phi_1^{(i)}}{(\eta \sigma_i')^2} \int_0^\infty H \Phi_2^{(i)} dk' + \frac{\Phi_2^{(i)}}{(\eta \sigma_i')^2} \int_{-\infty}^\infty H \Phi_1^{(i)} dk', \quad (42)$$

an equation which will be written as

$$\phi^{(i)} = -\frac{H}{y} + c_p^{(i)} \Phi_p^{(i)}, \quad (43)$$

where

$$c_1^{(i)} \equiv \frac{1}{(\eta \sigma_i')^2} \int_0^\infty H \Phi_2^{(i)} dk', \quad c_2^{(i)} \equiv \frac{1}{(\eta \sigma_i')^2} \int_{-\infty}^\infty H \Phi_1^{(i)} dk'. \quad (55)$$

The functions $\Phi_p^{(i)}$ in eq. (43) can be related to the solutions of the homogeneous equation, $\Phi_{hp}^{(i)}$, by means of eq. (31); by denoting the inverse matrix of $T_{pq}^{(i)}$ as $\tau_{pq}^{(i)}$

one obtains

$$\phi^{(i)} = -\frac{H}{y} + c_p^{(i)} \tau_{pq}^{(i)} \Phi_{hq}^{(i)}. \quad (45)$$

Let us now suppose that in s_i the solution has the form

$$\Phi \approx \frac{H}{1-\sigma} + \alpha_p \Phi_{hp}, \quad (46)$$

where α_p are constants. In $(s_i \cap S_i)$ eq. (7) is equivalent to

$$\Phi \approx \phi^{(i)} + \alpha_p \Phi_{hp}. \quad (47)$$

Owing to the asymptotic form of $\phi^{(i)}$, given by eq. (43), in $(S_i \cap Z_i)$ Φ is also given by

$$\Phi \approx \frac{H}{1-\sigma} + \left[\alpha_q + c_p^{(i)} \tau_{pq}^{(i)} \right] \Phi_{hq}. \quad (48)$$

As this equation shows, when k goes through an interval S_i where $\sigma'_i > 0$ the asymptotic representation of Φ assumes the additional term $(c_q^{(i)} \tau_{pq}^{(i)}) \Phi_{hq}$; this change is reminiscent of the variation of the form of an analytic function defined through an integral when a pole of the integrand crosses the integration axis. In $(S_{i+1} \cap s_{i+1})$, where $\sigma'_{i+1} < 0$, the equation corresponding to eq. (48) is

$$\Phi \approx \frac{H}{1-\sigma} + \left[\alpha_q + c_p^{(i)} \tau_{pq}^{(i)} - c_p^{(i+1)} \tau_{pq}^{(i+1)} \right] \Phi_{hq}. \quad (49)$$

The foregoing results afford the possibility of constructing the asymptotic approximation of the solution Φ .

In the first interval, s_1 , the function $\alpha \Phi_{h1}$ (where α is a free constant) can be added to $H/(1-\sigma)$, owing to the fact that Φ_{h1} goes to zero more rapidly than H when k goes to minus infinity. In every S_i interval following, solutions of the homogeneous integral equation have to be additionally superposed, according to eqs. (48) and (49). In the last s interval the functions Φ_{hp} are linear superpositions of the functions f_j , as was shown in the preceding section; the coefficient of f_1 must

be zero in order that Φ be integrable. Whereas this condition gives the eigenvalues of the homogeneous equation, it now determines α . We explicitly deduce this coefficient in two cases corresponding to those considered in the previous section.

When there is only one Z interval, one obtains from eq. (49)

$$\alpha T_{12}^{(2)} + c_p^{(1)} \tau_{p2}^{(1)} = 0. \quad (50)$$

It is recalled that $T_{12}^{(2)} = 0$ is the eigenvalue condition for this particular case.

With two Z intervals one gets the condition

$$\alpha T_{12}^{(4)} + \sum_{i=1}^4 c_p^{(i)} \tau_{p2}^{(i)} = 0, \quad (51)$$

where again $T_{12}^{(4)} = 0$ is the eigenvalue condition for this case.

According to the alternative theorem for Fredholm integral equations, when H is orthogonal to the eigenfunctions corresponding to some eigenvalue the inhomogeneous integral equation has an infinity of solutions at this eigenvalue (one particular solution plus a linear superposition of the eigenfunctions). Let us now assume that H is symmetric; we now show that for the eigenvalues corresponding to antisymmetric eigenfunctions, which are therefore orthogonal to H , there is an L_2 solution (and hence infinite). Any two constants $c_p^{(i)}$ and $c_p^{(j)}$ which correspond to a k_i and a k_j symmetric with respect to the origin of the k axis, are equal when H is symmetric. On the other hand, owing to the antisymmetry of the eigenfunctions one has $\tau_{pq}^{(i)} = -\tau_{pq}^{(j)}$ and therefore

$$\sum_{i=1}^4 c_p^{(i)} \tau_{p2}^{(i)} = 0. \quad (52)$$

Hence α is finite, although the coefficient of f_1 in the expression $\alpha \Phi_{h1}$ is by definition equal to zero at the eigenvalues.

3. Generalizations

The results derived in the preceding sections can be extended to the class of density profiles $h(k/\eta)$ which are such that in the limit $\eta = 0$ they yield the Dirac $\delta(k)$ -function, because in the interval $k' \approx k$ and for η sufficiently small they can be represented by the exponential distribution used in the preceding sections.

In general, the class of kernels which can be treated by the method developed here can be enlarged by introducing a k' -dependence in σ (i.e. $\sigma \equiv \sigma(k, k', \omega)$), because in the first order one can substitute k for k' . An important exception is a σ of the form

$$\sigma \equiv a(k, \omega) + (k' - k)b(k, \omega), \quad (53)$$

which is the correct expression in an inhomogeneous plasma if k_y is non-zero. This case will now be considered in some detail. It is not difficult to see that the equation corresponding to eq. (1) is now

$$\Phi(k, \omega) - 2\eta^2 b(k, \omega) \frac{\partial}{\partial k} \Phi(k, \omega) - \frac{a(k, \omega)}{2\eta\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(k'-k)^2/4\eta^2} \Phi(k', \omega) dk' = H. \quad (54)$$

Equation (3) is thus replaced by

$$1 - \frac{a + 2\eta bg}{\sqrt{1 - 2\eta g'}} \exp \left\{ g^2 \frac{1}{1 - 2\eta g'} \right\} = 0, \quad (55)$$

and instead of eq. (4) one gets

$$g^2 = -(\ln a + \eta g' + 2\eta(b/a)g). \quad (56)$$

The zero-order approximation of g^2 is again $-\ln a$. We skip the intermediate steps and write the first-order approximation in the form corresponding to eq. (8):

$$\int_0^k (g_1/\eta) dk' = -\frac{1}{2} \ln a + \frac{1}{4} \ln \ln a - \int_0^k (b/a) dk'. \quad (57)$$

Hence the solution in the intervals (s_i, Z_i) is that derived in section 1, multiplied by the factor

$$\exp \left\{ - \int_{-k_1}^k (b/a) dk' \right\}. \quad (58)$$

The equation corresponding to eq. (13), which determines Φ_h in the intervals S_i is

$$\frac{\partial^2 \Phi}{\partial y^2} + 2(b/a) \frac{\partial \Phi}{\partial y} + \frac{y}{(\eta \sigma'_i)^2} \Phi = 0, \quad (59)$$

which has the solution (cf. eq. (15))

$$\Phi_h = \exp \{ -by/\sigma'_i \} \sqrt{y - (b\eta)^2} J_{\pm 1/3} \left(a_i (y - (b\eta)^2)^{3/2} \right). \quad (60)$$

Since the part of the correction due to g_1 which remains at the exponent is also $-by/\sigma'_i$ (see eq. (58)), one can do the matching of the different intervals as in section 1. One thus gets the same eigenvalue condition, where now the function $G(k)$ is defined as

$$G(k) \equiv (1/\eta) \int_{k_1}^k \sqrt{\ln \sigma} dk' - \int_{k_1}^k (\bar{\sigma}/\sigma) dk'. \quad (61)$$

Conclusion

We have derived the asymptotic solutions to a class of inhomogeneous integral equations which reduce to algebraic equations when a parameter η goes to zero (the kernel becoming proportional to a Dirac δ -function). This class includes the integral equations obtained from the system of Vlasov and Poisson equations for the Fourier transform in space and the Laplace transform in time of the electric potential when the equilibrium magnetic field is uniform and the equilibrium plasma density depends on ηx , z being the direction of the magnetic field. One can therefore take into account the initial conditions, which appear in the inhomogeneous term through the Laplace transform in time, and possible sources.

The matched asymptotic procedure applied here allows the solution and the eigenvalue condition to be deduced in a compact and general form. The corresponding homogeneous integral equation has already been discussed in the x space (see, for example, Ref. [2]). The method presented here, which requires solution of the homogeneous equation in order to get the solution of the inhomogeneous equation, has the advantage of avoiding a cumbersome complex plane analysis. Another advantage is that the derivation of the eigenvalue condition does not require all solutions (in general complex and infinite in number) of the so-called 'local dispersion relation' to be known.

REFERENCES

- [1] T. Watanabe, H. Sanuki, M. Watanabe: J.Phys. Soc. Japan 47 (1979) 286
- [2] H.L. Berk and D. Pfirsch: J. Math. Phys. 21 (1980) 2054
- [3] J.D. Murray in Asymptotic analysis, Chapter 6, Clarendon Press, Oxford, 1974
- [4] R. Croci and R. Saison: Nuclear Fusion 13 (1973) 81
- [5] F.G. Tricomi in Integral equations, Chapter II, Interscience Publishers, Inc., New York, 1957