

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK  
GARCHING BEI MÜNCHEN

Generalized Cherry Oscillators and  
Negative Energy Waves

Dieter Pfirsch

IPP 6/290

February 1990

*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem  
Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über  
die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

# Generalized Cherry Oscillators and Negative Energy Waves

D. Pfirsch

Max-Planck-Institut für Plasmaphysik

EURATOM Association

D-8046 Garching, F. R. Germany

## Abstract

In 1925 Cherry [1] discussed two oscillators of positive and negative energy that are nonlinearly coupled in a special way, and presented exact solutions of the nonlinear equations showing explosive instabilities independent of the strength of the nonlinearity and the initial amplitudes. In this paper Cherry's Hamiltonian is transformed into a form which allows a simple physical interpretation. The new Hamiltonian is generalized to three nonlinearly coupled oscillators; it corresponds to three-wave interaction in a continuum theory, like the Vlasov-Maxwell theory, if there exist linear negative energy waves [2], [3]

## 1 Cherry's Oscillators

In 1925 Cherry [1] discussed nonlinearly coupled oscillators which are described by the Hamiltonian

$$H = -\frac{1}{2} \omega_1 (p_1^2 + q_1^2) + \frac{1}{2} \omega_2 (p_2^2 + q_2^2) + \frac{\alpha}{2} (2q_1 p_1 p_2 - q_2 (q_1^2 - p_1^2)) . \quad (1)$$

The constant  $\alpha$  measures the effect of nonlinearity. For  $\alpha = 0$  one has two uncoupled oscillators of frequencies  $\omega_1 > 0$  and  $\omega_2 > 0$  which possess negative and positive energy, respectively. If  $\omega_2 = 2 \omega_1$ , one has a third-order

resonance. Cherry found for this case the following exact two-parameter solution set:

$$\begin{aligned}
 q_1 &= \frac{\sqrt{2}}{\epsilon - \alpha t} \sin(\omega_1 t + \gamma), & p_1 &= \frac{-\sqrt{2}}{\epsilon - \alpha t} \cos(\omega_1 t + \gamma), \\
 q_2 &= \frac{-1}{\epsilon - \alpha t} \sin(2\omega_1 t + 2\gamma), & p_2 &= \frac{-1}{\epsilon - \alpha t} \cos(2\omega_1 t + 2\gamma),
 \end{aligned}
 \tag{2}$$

where  $\epsilon$  and  $\gamma$  are determined by the initial conditions. These relations show explosive instability for any  $\alpha \neq 0$ , whereas the linearized theory gives only stable oscillations. There is also no threshold amplitude. Small initial amplitudes only mean that it takes a long time for the explosion to occur.

In a continuum theory, like the Vlasov-Maxwell theory, the assumed resonance corresponds to the conservation law

$$\omega_1 + \omega_2 + \omega_3 = 0
 \tag{3}$$

for a three-wave interaction. It is therefore of interest to have a formulation and an example which are closer to the structure of a three-wave interaction. To this end we introduce complex quantities given by

$$\xi = p + i q, \quad \xi^* = p - i q.
 \tag{4}$$

We can do a canonical transformation to  $\xi^*$  as the new momentum and to  $\xi/2i$  as the new coordinate. Cherry's Hamiltonian then becomes

$$H = -\frac{1}{2} \omega_1 \xi_1^* \xi_1 + \frac{1}{2} \omega_2 \xi_2^* \xi_2 + \frac{\alpha}{4i} (\xi_1^2 \xi_2 - \xi_1^{*2} \xi_2^*).
 \tag{5}$$

This exhibits a simple structure of the nonlinear term which also allows a simple physical interpretation: in quantum theoretical language it means the simultaneous annihilation or creation of two quanta of frequency  $\omega_1$  with energy  $-\hbar\omega_1$  each and of one quantum of frequency  $\omega_2$  with energy  $+\hbar\omega_2$ . If  $\omega_2 = 2\omega_1$  these processes leave the energy unchanged and therefore allow

the amplitudes to grow. The same holds for the first two terms in  $H$ . The growth of the amplitudes is, of course, only possible for perturbations with vanishing  $H$ .

## 2 Generalization to Three Coupled Oscillators

In this paper we give a generalization of the two coupled oscillators described by the Hamiltonian (5) to three coupled oscillators corresponding to the mentioned three-wave interaction in a continuum. The Hamiltonian which will be investigated is

$$H = \frac{1}{2} \sum_{k=1}^3 \omega_k \xi_k^* \xi_k + \alpha \xi_1 \xi_2 \xi_3 + \alpha^* \xi_1^* \xi_2^* \xi_3^* . \quad (6)$$

The frequencies  $\omega_k$  are assumed to satisfy the three wave conservation law

$$\sum_{k=1}^3 \omega_k = 0 . \quad (7)$$

The equations of motion corresponding to the Hamiltonian (6) are

$$\dot{\xi}_k = i \omega_k \xi_k + 2 i \alpha^* \xi_1^* \xi_2^* \xi_3^* / \xi_k^* . \quad (8)$$

The ansatz

$$\xi_k(t) = a(t) e^{i\omega_k t + i\varphi_k} , \quad \sum_{k=1}^3 \varphi_k = 0 \quad (9)$$

with  $a(t)$  independent of  $k$  leads to the following equation for this quantity:

$$\dot{a} = 2 i \alpha^* a^{*2} . \quad (10)$$

This can be solved by

$$a = \gamma b(t) , \quad \gamma = \left( \frac{i\alpha^*}{8|\alpha|^4} \right)^{1/3} , \quad b^* = b \quad (11)$$

and

$$\dot{b} = b^2 . \quad (12)$$

This equation has the general solution

$$b = \frac{1}{\hat{\epsilon} - t} \quad (13)$$

with  $\hat{\epsilon}$  being a constant of integration. We therefore obtained a three-parameter solution set

$$\xi_k = \frac{1}{2} \left( \frac{i\alpha^*}{|\alpha|} \right)^{1/3} \frac{1}{\epsilon - |\alpha|t} e^{i\omega_k t + i\varphi_k} \quad , \quad \sum_{i=1}^3 \varphi_k = 0 \quad , \quad (14)$$

where

$$\epsilon = \hat{\epsilon} / |\alpha| \quad . \quad (15)$$

These solutions correspond to Cherry's two-parameter solution set .

## References

- [1] T.M. Cherry, Trans. Cambridge Philos. Soc. **23**, 199 (1925);  
see E.T. Whittaker, Analytical Dynamics (Cambridge, London, 1937),  
Sec. 182, p. 142;  
and A. Wintner, The Analytical Foundations of Celestial Mechanics  
(Princeton University, Princeton, 1947), Sec. 136, p. 101
- [2] P.J. Morrison and M. Kotschenreuther, The Free Energy Principle,  
Negative Energy Modes, and Stability , DOE/ET-53088-280, IFSR  
#280, Austin, Texas 78712, October 1989;
- [3] P.J. Morrison and D. Pfirsch, The Free Energy of Maxwell-Vlasov  
Equilibria , DOE/ET-53088-401, IFSR #401, Austin, Texas 78712,  
October 1989; accepted by Phys. Fluids