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Is the Temperature Gradient or the Derivative of the Density Gradient Responsible for Drift Solitons ?

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Abstract

In conventional drift wave theory the density gradient $\kappa_n = \mathrm{d}\ \ln n/\mathrm{d}x$ determines the linear phase velocity, and the (electron) temperature gradient $\kappa_T = \mathrm{d}\ \ln T/\mathrm{d}x$ gives rise to a nonlinear term which leads to the existence of soliton-type solutions and solitary waves. LAKHIN, MIKHAILOVSKI and ONISHCHENKO, *Phys. Lett. A* 119, 348 (1987) and *Plasma Phys. and Contr. Fus.* 30, 457 (1988), recently claimed that it is not κ_T but essentially the derivative of the density gradient, $\mathrm{d}\kappa_n/\mathrm{d}x$, that is relevant. This claim is refuted by means of an expansion scheme in $\epsilon = e\Phi/T \ll 1$, where Φ is the drift wave potential.

1. Introduction and conclusions

Plasmas with gradients of the density and the temperature immersed in magnetic fields can support nonlinear coherent structures like soliton-type perturbations or solitary waves (ORAEVSKII et al., 1969, PETVIASHVILI, 1977) or two-dimensional vortices (LAEDKE and SPATCHEK, 1986). These nonlinear phenomena, besides being an interesting study in themselves, are also important with regard to, for example non-collisional energy losses. The same is true in the opposite case of drift wave turbulence; see, for example, HASEGAWA and MIMA (1978).

In the simplest case of quasi-one-dimensional drift solitons (ORAEVSKII et al., 1969, OREFICE and POZZOLI, 1970, PETVIASHVILI, 1977, RAHMAN and SHUKLA, 1980, MEISS and HORTON, 1982) the rôles of the density gradient $\kappa_n = \mathrm{d}\,\ln n/\mathrm{d}x$ and temperature gradient $\kappa_T = \mathrm{d}\,\ln T/\mathrm{d}x$ are quite straightforward: κ_n determines the phase velocity of linear drift waves, and κ_T gives rise to the nonlinear term. Another nonlinearity, the so-called Hasegawa-Mima (HM) term contains neither κ_n nor κ_T and vanishes in the strictly one-dimensional case. It came as a surprise when LAKHIN, MIKHAILOVSKII and ONISHCHENKO (1987), (1988) recently claimed that this standard picture is erroneous, and that, owing to the HM term, the temperature gradient should be essentially replaced by the derivative of the density gradient. Since this point is fundamental to nonlinear drift wave theory the basic theory is reconsidered here. It is found that the conclusions of LAKHIN et al. (1987), (1988) are incorrect. Their fault does not lie in the formal derivation but in insufficient consideration of the order of magnitude of terms to begin with.

In the next section, for reference, a short derivation of nonlinear drift wave equations is presented together with the arguments of LAKHIN et al.

In Section 3 a smallness parameter is introduced by assuming $e\Phi/T = \epsilon \ll 1$, and expansions in ϵ are discussed. It is found that the starting equation of LAKHIN et al. does not correspond to a consistent ordering. Two orderings are considered in detail. One

is suitable for long-wavelength solitary structures and waves, $(k_y\rho_s)^2 \ll 1$, and leads to KdV-type equations. The other, with $(k_y\rho_s)^2 \sim \mathcal{O}(1)$, is often used in drift wave turbulence theory but can yield solitary structures of a more complicated type as well. In the first case the pre-Lakhin et al. picture with the κ_T nonlinearity is found to be valid after all. In the second case, for quasi-one-dimensional wave-like solutions, an additional nonlinear term $\sim \partial(\nabla\Phi)^2/\partial t$ is relevant which is lacking in LAKHIN et al. (1987), (1988), so that their theory does not apply as such, and in particular not with a simple replacement of the κ_T nonlinearity. Their insistence on checking the rôle of the HM term, however, is justified and in this second case might lead to modifications of previous results by ORAEVSKII et al. (1969), RAHMAN and SHUKLA (1980).

2. Short review of drift solitons and of Lakhin et al.'s considerations

Let us consider a plasma with density and temperature gradients in the x-direction and a constant magnetic field $\mathbf{B} = B\mathbf{e}_z$ in the z-direction. The continuity equation and the equation of motion for the ions, considered to be cold for simplicity, are

$$\frac{\partial n_i}{\partial t} + \mathrm{div}(n_i\, \mathbf{v}_i) = 0\,, \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla\right)\mathbf{v}_i = \frac{e}{m_i}\left(-\nabla\Phi + \frac{1}{c}\,[\mathbf{v}_i \times \mathbf{B}]\right)\,, \tag{2}$$

where $\mathbf{E} = -\nabla\Phi$ and e, m_i are the charge and mass of the ions. The electrons with temperature $T(x)$ are assumed to obey the Boltzmann distribution

$$n_e = n(x)\, e^{\frac{e\Phi}{T}}\,. \tag{3}$$

The system is closed by the quasineutrality condition $n_e = n_i$ and the assumption, again for simplicity, that $\mathbf{v}\cdot\mathbf{e}_z = \partial\Phi/\partial z = 0$. For drift waves with a frequency much less than the ion gyrofrequency $\Omega_i = eB/(m_i c)$ equ. (2) can be solved iteratively, yielding $\mathbf{v}_i \approx \mathbf{v}_E + \mathbf{v}_P$, where

$$\begin{aligned} \mathbf{v}_E &= \frac{c}{B}\,[\mathbf{e}_z \times \nabla\Phi]\,, \\ \mathbf{v}_P &= -\frac{c}{B\Omega_i}\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla\right)\nabla\Phi\,. \end{aligned} \tag{4}$$

The second term in the polarization drift $\mathbf{v}_P$ gives rise to the so-called Hasegawa-Mima (HM) nonlinearity (HASEGAWA and MIMA, 1978). Other nonlinearities arise from the product $n_i\mathbf{v}_i$ in equ. (1). The simplest one, from $\mathbf{v}_E \cdot \nabla e^{e\Phi/T}$, brings in the temperature gradient and is proportional to $\kappa_T\Phi\partial\Phi/\partial y$. Authors differ somewhat on which of the nonlinear terms to keep or not; compare, for example, TASSO (1967), ORAEVSKII et al. (1969), PETVIASHVILI (1977), HASEGAWA and MIMA (1978), RAHMAN and SHUKLA (1980), MEISS and HORTON (1982), LAEDKE and SPATSCHEK (1986), SHUKLA (1987), LAKHIN et al. (1987), (1988). This reflects the fact that often no systematic expansion is made. ORAEVSKII et al. (1969) and RAHMAN and SHUKLA

(1980) consider more nonlinear terms than the HM and/or the κ_T nonlinearities. If one keeps the two nonlinear terms just mentioned the result is (LAEDKE and SPATSCHEK, 1986, LAKHIN et al. 1987, 1988)

$$\frac{\partial \Phi}{\partial t} - \rho_s^2 \frac{\partial}{\partial t} \nabla^2 \Phi - \kappa_n \rho_s^2 \Omega_i \frac{\partial \Phi}{\partial y} - \rho_s^2 \frac{c}{B} [\nabla \Phi \times \nabla]_z \nabla^2 \Phi + \frac{c \kappa_T}{B} \Phi \frac{\partial \Phi}{\partial y} = 0 , \quad (5)$$

where $\rho_s^2 = T/(m_i \Omega_i^2)$ and $[\nabla \Phi \times \nabla]_z = \partial_x \Phi \cdot \partial_y - \partial_y \Phi \cdot \partial_x$. The stationary wave ansatz $\Phi(x, y, t) = \Phi(x, \eta = y - ut)$, where u is a constant phase velocity, yields

$$D \nabla^2 \Phi - \Lambda \frac{\partial \Phi}{\partial \eta} + 2 S_0 \Phi \frac{\partial \Phi}{\partial \eta} = 0 \quad (6)$$

with

$$D = \frac{\partial}{\partial \eta} - a [\nabla \Phi \times \nabla]_z , \quad (7)$$

$$\Lambda = \frac{1}{\rho_s^2} \left(1 + \frac{\kappa_n \rho_s^2 \Omega_i}{u} \right) , \quad (8)$$

$$S_0 = \frac{c \kappa_T}{2 \rho_s^2 B u}$$

and $a = c/(Bu)$. If the x- dependence of Φ is neglected, D reduces to $\partial/\partial\eta$ and equ. (6) can be integrated once with respect to η. With the boundary condition that Φ and its derivatives vanish at $\eta \to \pm\infty$ one finally obtains (PETVIASHVILI, 1977, MEISS and HORTON, 1982, LAEDKE and SPATSCHEK, 1986, SHUKLA, 1987, HE and SALAT, 1989)

$$\frac{\partial^2 \Phi}{\partial \eta^2} - \Lambda \Phi + S_0 \Phi^2 = 0 , \quad (9)$$

which has well known sech^2 soliton-like solutions; see Appendix A.

LAKHIN et al. (1987), (1988) point out that it is premature to neglect the x- dependence of Φ totally since after all Φ depends on x parametrically via the coefficients. Even for $|\partial^2 \Phi/\partial x^2| \ll |\partial^2 \Phi/\partial y^2|$ the HM term $[\nabla \Phi \times \nabla]_z \nabla^2 \Phi$ need not be small. They manage to integrate equ. (6) once, up to a small correction, and only afterwards do they consider

small x- dependence. The result, equs. (12)-(14) below, indeed differs from equ. (9). LAKHIN et al. realized that equ. (6) can be written as

$$D\,F \;=\; a_1\,\Phi^2\,\frac{\partial\Phi}{\partial\eta} \;=\; \mathcal{O}(\Phi^3) \;\ll\; 1\,, \tag{10}$$

where

$$F \;\equiv\; \nabla^2\Phi \;-\; \Lambda\,\Phi \;+\; \left(S_0 \;+\; \frac{a}{2}\,\frac{\mathrm{d}\Lambda}{\mathrm{d}x}\right)\,\Phi^2 \tag{11}$$

and $a_1 = a\;\mathrm{d}(S_0 + a/2\cdot\mathrm{d}\Lambda/\mathrm{d}x)/\mathrm{d}x$. The r.h.s. of equ. (10) is small of third order in Φ and therefore is neglected. (For the special profile $\mathrm{d}^2\kappa_n/\mathrm{d}x^2 = 0$ $\;a_1$ is even exactly zero.) $DF = 0$ is equivalent to $[\nabla F \times \nabla(a\Phi - x)]_z = 0$ and hence F is an arbitrary function of $a\Phi - x$ only. With the aforementioned boundary condition one finally obtains $F = 0$, i.e.

$$\nabla^2\Phi \;-\; \Lambda\,\Phi \;+\; S\,\Phi^2 \;=\; 0\,, \tag{12}$$

where

$$S \;=\; S_0 \;+\; \frac{a}{2}\,\frac{\mathrm{d}\Lambda}{\mathrm{d}x}\,. \tag{13}$$

If it is now assumed that $|\partial^2\Phi/\partial x^2| \ll |\partial^2\Phi/\partial y^2|$, equ. (12) differs from equ. (9) by the substitution $S_0 \rightarrow S$. When Λ and S_0 are taken from equs. (8), the temperature gradient κ_T exactly cancels, leaving (LAKHIN et al., 1987, 1988)

$$S \;=\; \frac{e}{2m_i u^2}\,\frac{\mathrm{d}\kappa_n}{\mathrm{d}x}\,. \tag{14}$$

This unexpected result, which was derived by LAKHIN et al. (1988) using three different methods, undeniably follows from equ. (5). However, as stated above, the selection of nonlinear terms to work with is not straightforward. It is necessary to set up a consistent expansion scheme in terms of small parameters as done by, for example, OREFICE and POZZOLI (1970), NOZAKI and TANIUTI (1974), TODOROKI and SANUKI (1974). This is done in the next section and will be shown to restore the $\kappa_T\,\Phi\partial\Phi/\partial y$ nonlinearity.

3. Consistent expansion schemes

In order to discuss the relative sizes of terms, it is useful to introduce dimensionless variables. Let T_0, n_0 be the electron temperature and density at $x = 0$, say, and $r_0 = L$ the scale length of the density gradient. The quantity $v_0 = cT_0/(eBL)$ is a velocity of the order of the electron drift velocity $v_{*e} = -cT\kappa_n/(eB)$. We consider the dimensionless quantities $x' = x/r_0$, $y' = y/r_0$, $t' = t/t_0$, where $t_0 = r_0/v_0$, and $\mathbf{v}' = \mathbf{v}_i/v_0$, $n' = n/n_0$, $T' = T/T_0$, $\Phi' = e\Phi/T_0$, $\kappa_n' = r_0\kappa_n$, $\kappa_T' = r_0\kappa_T$. With the definition $\rho_0^2 = T_0/(m_i\Omega_i^2)$ it follows that $t_0 = (L/\rho_0)^2/\Omega_i$. In the following, the primes are again omitted for brevity.

In the new variables the equation of motion of the ions is

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \frac{1}{s}\left(-\nabla\Phi + [\mathbf{v}\times\mathbf{e}_z]\right) \tag{15}$$

with $s = (\rho_0/L)^2$. It is assumed that $\Phi \sim \epsilon \ll 1$ is a small quantity, and similarly that

$$s = \left(\frac{\rho_0}{L}\right)^2 \sim \epsilon^\alpha\,, \qquad \alpha > 0\,, \tag{16}$$

so that $s \ll 1$ also. If the drift solitons or solitary waves are to retain a relation to the physics of drift waves, the ratio of the time derivative to the y- derivative should be of the order of v_0, which in dimensionless units implies $\partial/\partial t \sim \partial/\partial y$. The derivative of Φ in the x-direction is assumed to be weaker than or of the same order as the y-derivative. This implies

$$\begin{aligned} &\frac{\partial}{\partial t} \sim \frac{\partial}{\partial y} \sim \epsilon^\beta\,, \qquad \frac{\partial}{\partial x} \sim \epsilon^{\beta+\xi}\,, \\ &\beta + \xi \le 0\,, \qquad \beta \le 0\,, \qquad \xi \ge 0\,. \end{aligned} \tag{17}$$

The first inequality expresses the fact that $\Phi(x,y,t)$ cannot vary in x more slowly than the equilibrium quantities whose scale lengths are $\sim \mathcal{O}(1)$. The third inequality corresponds to $\partial/\partial x \lesssim \partial/\partial y$, and the second one follows from the other two.

It is now straightforward to see the weak point in the treatment of LAKHIN et al. In dimensionless units their final equation, equ. (12), is

$$\nabla^2\Phi - \frac{1}{sT}\left(1 + \frac{\kappa_n T}{u}\right)\Phi + \frac{1}{2su^2}\frac{\mathrm{d}\kappa_n}{\mathrm{d}x}\Phi^2 = 0\,. \tag{12a}$$

The order of magnitude of the terms is $\epsilon^{1+2\beta}$, $\epsilon^{1-\alpha}$, $\epsilon^{2-\alpha}$, respectively, assuming that in unnormalized units $u \sim v_{*e}$. Hence the nonlinear term is one order of magnitude smaller than the second term. Obviously, the nonlinear term to dominant order should not be kept. If it is kept, for convenience as it were, or by arguing that $1 + \kappa_n T/u$ could be small one should be aware that other small terms could enter as well and modify the nonlinearity. We therefore proceed with a systematic expansion in powers of $\epsilon \ll 1$.

If the equation of motion is solved recursively, it is evident from

$$\mathbf{v} \;=\; [\mathbf{e}_z \times \nabla\Phi] \;-\; s\,\frac{\mathrm{d}}{\mathrm{d}t}\nabla\Phi \;-\; s^2\,[\mathbf{e}_z \times \frac{\mathrm{d}^2}{\mathrm{d}t^2}\nabla\Phi] \;+\; s^3\,\frac{\mathrm{d}^3}{\mathrm{d}t^3}\nabla\Phi \;+\; \cdots \tag{18}$$

that the expansion proceeds pairwise in powers of $(s\mathrm{d}/\mathrm{d}t)^2$. Convergence therefore requires that $s\partial/\partial t \sim \epsilon^{\alpha+\beta} \ll 1$ and $s\nabla\nabla\Phi \sim \epsilon^{1+\alpha+2\beta} \ll 1$, i.e.

$$\alpha \;+\; \beta \;>\; 0 \qquad \text{and} \qquad 1 \;+\; \alpha \;+\; 2\beta \;>\; 0\,. \tag{19}$$

In keeping the usual terms

$$\mathbf{v} \;=\; [\mathbf{e}_z \times \nabla\Phi] \;-\; s\,\frac{\partial\nabla\Phi}{\partial t} \;-\; s\,[\nabla\Phi \times \nabla]_z\nabla\Phi \tag{20}$$

we are aware that the neglected ones are of order $\epsilon^{2(\alpha+\beta)}$ and $\epsilon^{2(1+\alpha+2\beta)}$ smaller than the last two terms. Repeating the procedure which led to equ. (5) one obtains

$$\begin{aligned}
&\frac{1}{T}\frac{\partial\Phi}{\partial t} \;-\; s\frac{\partial}{\partial t}\nabla^2\Phi \;-\; \kappa_n\frac{\partial\Phi}{\partial y} \;-\; s\kappa_n\frac{\partial^2\Phi}{\partial t\partial x} \;-\; s[\nabla\Phi \times \nabla]_z\nabla^2\Phi \;+\; \frac{\kappa_T}{T}\Phi\frac{\partial\Phi}{\partial y} \;-\\
&\frac{s}{T}\nabla\Phi\cdot\frac{\partial}{\partial t}\nabla\Phi \;-\; s\kappa_n[\nabla\Phi \times \nabla]_z\frac{\partial\Phi}{\partial x} \;+\; \frac{s\kappa_T}{T}\Phi\frac{\partial^2\Phi}{\partial t\partial x} \;-\\
&\frac{s}{T}\nabla\Phi\cdot([\nabla\Phi \times \nabla]_z\nabla\Phi) \;+\; \frac{s\kappa_T}{T}\Phi[\nabla\Phi \times \nabla]_z\frac{\partial\Phi}{\partial x} \;=\; 0\,.
\end{aligned} \tag{21}$$

The order of magnitude term by term wise is

$$\begin{aligned}
&\epsilon^{1+\beta} \;\mid\; \epsilon^{1+\alpha+3\beta} \;\mid\; \epsilon^{1+\beta} \;\mid\; \epsilon^{1+\alpha+2\beta+\xi} \;\mid\; \epsilon^{2+\alpha+4\beta+\xi} \;\mid\; \epsilon^{2+\beta} \;\mid\\
&\epsilon^{2+\alpha+3\beta} \;\mid\; \epsilon^{2+\alpha+3\beta+2\xi} \;\mid\; \epsilon^{2+\alpha+2\beta+\xi} \;\mid\\
&\epsilon^{3+\alpha+4\beta+\xi} \;\mid\; \epsilon^{3+\alpha+3\beta+2\xi}\,.
\end{aligned} \tag{22}$$

In view of the inequalities (19) the only terms which can possibly be dominant are those which scale as $\epsilon^{1+\beta}$, $\epsilon^{1+\alpha+3\beta}$, and $\epsilon^{2+\alpha+4\beta+\xi}$. There are four such terms, namely the first three and the fifth. Altogether there are five possible combinations for which at least two terms balance each other. We shall only discuss the two most pronounced cases, namely case a): the first and the third terms (the most basic ones) balance each other, and case b): all four terms contribute to the balancing. The other combinations lead to the same conclusions or describe rather untypical situations.

$$\text{Case a)} \qquad \alpha + 2\beta > 0\,, \qquad 1 + \alpha + 3\beta + \xi > 0\,. \tag{23}$$

This ensures that the dispersion term $s\partial\nabla^2\Phi/\partial t$ and the HM term $s[\nabla\Phi \times \nabla]_z\nabla^2\Phi$ are small. The remaining equation

$$\frac{1}{T}\frac{\partial\Phi}{\partial t} - \kappa_n \frac{\partial\Phi}{\partial y} = 0 \tag{24}$$

implies that to lowest order $\Phi(x,y,t)$ is a function of x and $w = y + \kappa_n T t$ only. In the next order in ϵ a weak dependence on t separately is allowed. With $\Phi = \Phi(x,t,w)$ we make the ansatz

$$\left.\frac{\partial\Phi}{\partial t}\right|_w \sim \epsilon^\tau \frac{\partial\Phi}{\partial w}\,, \qquad \tau > 0\,. \tag{25}$$

In order to obtain contributions from this slow time dependence and from the dispersion term, one has to set $\tau = 1$ and

$$\alpha + 2\beta = 1\,. \tag{26}$$

As a consequence, in unnormalized units one has $(k_y\rho_s)^2 \sim \epsilon^{2\beta+\alpha} \sim \epsilon \ll 1$. The temperature gradient nonlinearity contributes in this order $\sim \epsilon^{2+\beta}$ too. The HM nonlinearity contributes for $1 + \beta + \xi = 0$, which implies $\beta \leq -1$. In the opposite case, $0 \geq \beta > -1$, the HM term is small. It does no harm, however, to keep the term formally in this case too if only correspondingly small terms are finally identified and eliminated. This saves us one more separate discussion of subcases. Thus we consider

$$1 + \beta + \xi \geq 0\,. \tag{27}$$

The term $s\kappa_n \partial^2\Phi/\partial t\partial x \rightarrow s\kappa_n^2 T \partial^2\Phi/\partial w\partial x$ contributes to the present order for $\xi = \beta = 0$ only, i.e. for x- and y-derivatives of the same order of magnitude. In the following, however, for simplicity and for the sake of comparison with LAKHIN et al. (1987), (1988) we focus on quasi-one-dimensional solutions, i.e. $\partial/\partial x \ll \partial/\partial y$, which implies $\xi > 0$. In this case the slow time dependence of $\Phi(x,t,w)$ is governed by the equation

$$\frac{1}{T}\,\frac{\partial\Phi}{\partial t} \;-\; s\kappa_n T\,\frac{\partial^3\Phi}{\partial w^3} \;-\; s\,[\nabla\Phi\times\nabla]_z\,\frac{\partial^2\Phi}{\partial w^2} \;+\; \frac{\kappa_T}{T}\,\Phi\,\frac{\partial\Phi}{\partial w} \;=\; 0\,. \tag{28}$$

If β and ξ are such that $1+\beta+\xi > 0$ the third term can be neglected and the Korteweg-de Vries equation is obtained. Apart from the restrictions (17) and (27) β and ξ are free parameters, while $\alpha = 1 - 2\beta$.

Equation (28) without the third term was also obtained by OREFICE and POZZOLI (1970), with an extension to $\partial/\partial z \neq 0$. The special case $\alpha = 1/2$, $\beta = 0$, $\xi = 1$ was assumed. It was not realized, however, that in this case with $\beta + \xi > 0$ the solution would depend on x more slowly than the equilibrium itself.

Looking again for wave-like solutions, we make the ansatz $\Phi(x,t,w) = \Phi(x,\eta)$ with

$$\eta \;=\; w \;-\; \hat{u}\,t \;=\; y \;+\; (\kappa_n\,T \;-\; \hat{u})\,t\,, \tag{29}$$

where $\hat{u}$ is a constant of order ϵ in order to satisfy the scaling (25) with $\tau = 1$. This yields

$$\frac{\partial^3\Phi}{\partial\eta^3} \;+\; \frac{1}{\kappa_n T}\,[\nabla\Phi\times\nabla]_z\,\frac{\partial^2\Phi}{\partial\eta^2} \;+\; \frac{\hat{u}}{s\kappa_n T^2}\,\frac{\partial\Phi}{\partial\eta} \;-\; \frac{\kappa_T}{s\kappa_n T^2}\,\Phi\,\frac{\partial\Phi}{\partial\eta} \;=\; 0\,. \tag{30}$$

This equation is of the same type as equ. (6), except that $\nabla^2\Phi$ is replaced by $\partial^2\Phi/\partial\eta^2$. Now we have $a = -1/(\kappa_n T)$ and

$$\begin{aligned} \Lambda \;&=\; \frac{-\hat{u}}{s\kappa_n T^2}\,, \\ S_0 \;&=\; \frac{-\kappa_T}{2s\kappa_n T^2}\,. \end{aligned} \tag{31}$$

Repeating the discussion leading to equs. (10)-(13) then yields

$$S \;=\; \frac{-\kappa_T}{2s\kappa_n T^2}\left(1 \;+\; \frac{2\hat{u}}{\kappa_n T}\right) \;-\; \frac{\hat{u}}{2s\kappa_n^3 T^3}\,\frac{\mathrm{d}\kappa_n}{\mathrm{d}x} \tag{32}$$

with $S \sim \epsilon^{-\alpha}$ so that $a_1 = a\mathrm{d}S/\mathrm{d}x \sim \epsilon^{-\alpha}$ and $a_1\Phi^2\partial\Phi/\partial\eta \sim \epsilon^{3-\alpha+\beta} \sim \epsilon^{2+3\beta}$. On the other hand, it holds that $\partial^3\Phi/\partial\eta^3 \sim \epsilon^{1+3\beta}$ so that the r.h.s. of equ. (10) is small compared to the l.h.s. and an equation analogous to equ. (12) is obtained. The nonlinear term $S\Phi^2$ contains both the temperature gradient κ_T and the gradient of κ_n. Since $\hat{u} \sim \epsilon$, however, the $\mathrm{d}\kappa_n/\mathrm{d}x$ term is small and only part of the κ_T nonlinearity matters and $S = S_0 + \textit{higher-order terms}$. The final nonlinear equation resulting is

$$\frac{\partial^2\Phi}{\partial\eta^2} + \frac{\hat{u}}{s\kappa_n T^2}\,\Phi - \frac{\kappa_T}{2s\kappa_n T^2}\,\Phi^2 = 0\,. \tag{33}$$

Although this equation looks different from that of the naive 1-d approach, equs. (8), (9), its soliton-like solutions agree to dominant order (see Appendix B).

With these results we can draw our main conclusion: the nonlinear drift wave equation (5) without the HM term but with the κ_T nonlinearity is well suited to describing long-wavelength ($k_y\rho_0 \sim \sqrt{\epsilon}$) quasi one-dimensional drift solitons (PETVIASHVILI, 1977, MEISS and HORTON, 1982, HE and SALAT, 1989), while the version of LAKHIN et al. (1987), (1988), equs. (12), (14), is not meaningful.

The fact that the HM term $\sim [\nabla\Phi \times \nabla]_z\nabla^2\Phi$ does not leave any trace in equ. (33) is easily understood. $\nabla^2\Phi$ is approximately of the form $\nabla^2\Phi = f(x)\Phi + g(x)\Phi^2$. The operator $[\nabla\Phi\times\nabla]_z$ acting on Φ^n, however, gives zero identically, and only its operation on the slowly varying coefficients $f(x)$, $g(x)$ remains. In the most pronounced case, $1+\beta+\xi = 0$, this gives a reduction by a factor $\epsilon^{-(\beta+\xi)} \sim \epsilon \ll 1$ in relation to the original estimate.

Case b) $\qquad \alpha + 2\,\beta = 0\,, \qquad 1 + \beta + \xi = 0\,.$ (34)

The wave-number in the y direction (in natural units) satisfies $k_y\rho_0 \sim 1$. To dominant order $\sim \epsilon^{1+\beta}$ equ. (21) becomes

$$\frac{1}{T}\,\frac{\partial\Phi}{\partial t} - s\,\frac{\partial}{\partial t}\nabla^2\Phi - \kappa_n\,\frac{\partial\Phi}{\partial y} - s\,[\nabla\Phi\times\nabla]_z\,\nabla^2\Phi = 0\,. \tag{35}$$

This is the Hasegawa-Mima equation, which was originally used to investigate two-dimensional drift wave turbulence (HASEGAWA and MIMA, 1978) and drift vortices

(LAEDKE and SPATSCHEK, 1986). Although it is not in the spirit of these applications we discuss the plane-wave ansatz $\Phi(x,y,t) = \Phi(x,\eta = y - ut)$ in order to see how the arguments of LAKHIN et al. (1987), (1988) work in this scaling. The constant velocity u is here $\mathcal{O}(1)$. The resulting equation

$$\left(\frac{\partial}{\partial\eta} - \frac{1}{u}\left[\nabla\Phi\times\nabla\right]_z\right)\nabla^2\Phi - \frac{1}{sT}\left(1 + \frac{\kappa_n T}{u}\right)\frac{\partial\Phi}{\partial\eta} = 0 \tag{36}$$

is of the form (6) with $a = 1/u$, $S_0 = 0$, and

$$\Lambda = \frac{1}{sT}\left(1 + \frac{\kappa_n T}{u}\right) \tag{37}$$

so that from equ. (13) one obtains

$$S = \frac{1}{2us}\left(\frac{-\kappa_T}{T} + \frac{1}{u}\frac{\mathrm{d}\kappa_n}{\mathrm{d}x}\right). \tag{38}$$

Therefore it holds that $S \sim a_1 \sim \epsilon^{2\beta}$, and the r.h.s. of equ. (10) is a factor ϵ^2 smaller than the l.h.s. Also, the nonlinear term $S\Phi^2$ in equ. (12) is negligible by one order of magnitude, and there only remains

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial x^2}\right)\Phi - \Lambda\,\Phi = 0\,. \tag{39}$$

If it is assumed again that $|\partial^2\Phi/\partial x^2| \ll |\partial^2\Phi/\partial y^2|$, the solution of equ. (39) is

$$\Phi(x,\eta) = A(x)\,\sin(\sqrt{\Lambda}\,\eta + \phi(x)) \equiv \Phi_1 \tag{40}$$

with arbitrary amplitude A and phase ϕ. The situation is different from case a) in that here the functional form of Φ is already fixed by the lowest-order equation. In the next order there is a small dependence of A and ϕ on t and y, and a small deviation from the functional form above: $\Phi = \Phi_1 + \epsilon\Phi_2 + \cdots$. The condition that Φ have no secular behaviour gives rise in the usual way to evolution equations for A and ϕ. In this next order, in equ. (21), there are contributions from the four terms already familiar from the lowest order, and from the terms $\kappa_T\Phi\partial\Phi/\partial y$ and $s\nabla\Phi\cdot\partial\nabla\Phi/\partial t$ as well. Both nonlinear terms are

present in (ORAEVSKII et al., 1969, RAHMAN and SHUKLA, 1980) and have been used to demonstrate the modulational instability of the sinusoidal solution (40) (RAHMAN and SHUKLA, 1980) and the existence of 1-d solitary waves and soliton-like structures of a non-KdV type (ORAEVSKII et al., 1969, RAHMAN and SHUKLA, 1980). The HM nonlinearity, however, was not included although according to the present discussion it contributes at least formally. It remains to be seen whether it affects the next-order results as little as it did the lowest-order ones.

LAKHIN et al. (1987), (1988) do not have the term $\sim \nabla\Phi \cdot \partial\nabla\Phi/\partial t$. They seem to have in mind the case $(k_y\rho_s)^2 \ll 1$, which corresponds to case a). Still, the discussion above underlines the conclusion that only an *ab initio* discussion can decide which terms are relevant in which situations.

Appendix A

Instead of equ. (9) one may consider the slightly generalized equation

$$\frac{\partial^2 \Phi}{\partial \eta^2} \; - \; \Lambda \, \Phi \; + \; S \, \Phi^2 \; = \; C \; , \tag{A1}$$

which reflects the fact that Φ is a potential and may have $\Phi \rightarrow C = const \neq 0$ away from the soliton. Equation (A1) has elliptic function solitary wave solutions and soliton-type solutions. The latter are

$$\Phi \; = \; \Phi_0 \; + \; A \, \mathrm{sech}^2(k\eta) \; , \tag{A2}$$

where

$$A \; = \; 3 \, (\Lambda - 2 \, S\Phi_0) \; / \; (2S) \; , \qquad k^2 \; = \; SA/6 \tag{A3}$$

and $C = \Phi_0(S\Phi_0 - \Lambda)$. The sech^2 solution, however, is not a true soliton. If a collection of such excitations is used as initial condition in equ. (5) (without the HM term) they are destroyed upon collision with each other (ABDULLOEV et al., 1976).

Appendix B

In equ. (33) Φ depends on $\eta = y + (\kappa_n T - \hat{u})t$. For the soliton-type solution $\Phi = \Phi_0 + A \, \mathrm{sech}^2(k\eta)$, equ. (A1), one has $\Lambda = -\hat{u}/(s\kappa_n T^2)$ and $S = -\kappa_T/(2s\kappa_n T^2)$. Inserting this into equs. (A3) and solving for $\hat{u}$, one obtains

$$\hat{u} \; = \; \frac{\kappa_T}{3} \, (A \; + \; 3 \, \Phi_0) \; , \qquad k^2 \; = \; \frac{-\kappa_T A}{12 s \kappa_n T^2} \; . \tag{A4}$$

A and Φ_0 have to be of order ϵ in order to be consistent with $\hat{u} \sim \epsilon$.

In contrast, in the naively derived equ. (9) Φ depends on $\eta = y - ut$, with $u \sim \mathcal{O}(1)$. In dimensionless units Λ and S_0 are given by $\Lambda = (1 + \kappa_n T/u)/(sT)$ and $S_0 = \kappa_T/(2usT)$. Again solving equs. (A3) for u, one obtains

$$u \; = \; - \, \kappa_n T \; + \; \frac{\kappa_T}{3} \, (A \; + \; 3 \, \Phi_0) \; , \tag{A5}$$

$$k^2 = \frac{-\kappa_T A}{12 s \kappa_n T^2} \frac{1}{1 - (A + 3\Phi_0)\kappa_T/(3\kappa_n T)} . \qquad \text{(A6)}$$

Comparison of equs. (A4) - (A6) shows that the total effective phase velocities $-\kappa_n T + \hat{u}$ and u agree. The wave numbers k agree to dominant order in $\epsilon \ll 1$.

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