INTEGRAL CONSTITUTIVE RELATION FOR THE INVESTIGATION OF ION BERNSTEIN WAVES IN NON-HOMOGENEOUS PLASMAS

Marco Brambilla

IPP 5/27 January 1989

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

8046 GARCHING BEI MÜNCHEN

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

INTEGRAL CONSTITUTIVE RELATION FOR THE INVESTIGATION OF ION BERNSTEIN WAVES IN NON-HOMOGENEOUS PLASMAS

Marco Brambilla

IPP 5/27 January 1989

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

INTEGRAL CONSTITUTIVE RELATION FOR THE INVESTIGATION OF ION BERNSTEIN WAVES IN NON HOMOGENEOUS PLASMAS

MARCO BRAMBILLA

ABSTRACT

By integrating the linearised Vlasov equation, we derive an integral form of the constitutive relation (the relation between the h.f. electric field $\vec{E}$ and the h.f. current $\vec{J}$) valid to all orders in the Larmor radius in axially magnetised inhomogeneous plasmas. It has the form of a convolution integral in Fourier $\vec{k}_\perp$-space, whose Kernel can be expanded in Bessel functions, or put in a form analogous to the Gordeev integral form of the usual uniform plasma conductivity tensor, to which it reduces in the uniform plasma limit. Alternatively, it can be formulated as an integral equation in real space. Both formulations can be useful for the investigation of ion Bernstein waves near ion cyclotron resonances and near the Lower Hybrid resonance, where a WKB analysis alone is insufficient.

1. – Introduction.

Ion Bernstein waves (IBW) [1] are of considerable interest in plasma h.f. heating. In harmonic ICR heating at $\omega = 2\Omega_{ci}$ and in ICR minority heating they are excited near resonance as a result of linear mode conversion of the externally launched fast magnetosonic wave [2]; they are usually assumed to damp mainly on the electrons at some distance from the conversion layer. In the Lower Hybrid frequency domain it is also in principle possible to excite high order IBWs by mode conversion of an externally launched slow cold–plasma wave; in this case they are likely to damp on the ions at the nearest cyclotron harmonic [3]. Alternatively, direct launching of IBWs has been proposed as an independent h.f. heating scheme [4], which has been tested with some success on Alcator C [5] and PLT [6].

The physics of mode conversion near the first IC harmonic is well understood [7]. To model this process it is sufficient to solve the wave equations in the finite Larmor radius (FLR) approximation, which can be obtained from the linearised Vlasov equation by expanding in the ratio of the ion Larmor radius to the typical wavelength to order $k_\perp^2 \rho_i^2$ [8–10]. Thank to the availability of the FLR wave equations, direct launching of the lowest IBW has also been modeled in details [11–12].

By contrast, the theoretical description of the excitation of higher order IBWs, either directly or by mode conversion, is much more difficult. In the familiar Bessel function expansion of the hot plasma dielectric tensor [13], the n–th IBW (whith frequency close to $(n+1)\Omega_{ci}$) enters the dispersion relation to order n in $k_\perp^2 \rho_i^2$. As a consequence, the FLR expansion describes only the lowest IBW, $n = 1$; except for some ad–hoc model [14], a set of wave equations adequate to describe higher order IBWs is not yet available.

The situation and the kind of problems one would like to solve can be illustrated with the help of Fig. 1. In this figure we have plotted the solution of the full hot–plasma local dispersion relation in the scrape–off of a hypothetical Tokamak, at a frequency between the third and fourth IC harmonics. Under these conditions, the cold Lower Hybrid resonance is situated quite close to the plasma boundary (in fig. 1 at $r \simeq 42$ cm). For $n_\parallel^2 \geq 1$ the slow cold plasma wave propagates from the edge to the LH resonance, and is evanescent at higher densities. Finite temperature effects on the other hand connect this wave to an IBW which propagates with a much shorter wavelength further inside the plasma, where it should be damped either linearly at the nearest cyclotron harmonic, or via parametric decay for example near half–harmonics [15]. For $n_\parallel^2 < 1$ the slow cold plasma wave is evanescent on the low density side of the LH layer; the IBW branch on the other hand is practically unaffected by the value of $n_\parallel$. Also shown are the solutions in the FLR approximation. While giving a correct result up to astonishingly large values $k_\perp^2 \rho_i^2 \lesssim 2$, the FLR dispersion relation fails completely to describe the IBW, as expected; instead, it predicts a spurious pressure–driven branch propagating back towards the plasma edge, grossly violating the conditions for the validity of the FLR expansion itself, and having no counterpart in the exact disperion relation.

It is manifestly impossible to use the FLR wave equations to evaluate coupling in this situation. For example, an adaptation of the Grill code to take into account FLR corrections [16] predicts essentially total reflection in a two–waveguide launcher under all conditions, because waves with $n_\parallel^2 > 1$ appear to come back along the spurious FLR branch (for $n_\parallel^2 < 1$ it is very difficult to match the exceedingly short wavelength of this wave, and in any case the surface admittance of the plasma tends to zero for $n_\parallel \to 0$). By contrast, the conventional Grill code [17], which takes into account only the cold plasma slow wave, predicts good coupling in a broad range of the density gradient near the plasma edge (Fig. 2). One can argue that the results of the cold plasma approximation are likely to be more realistic than those of the FLR model, since the LH layer, where the cold approximation breaks down, is already in the far–field region. Nevertheless one would like to be able to investigate theoretically the complicated mode transition occurring near the LH layer, to ascertain how far the expectations of the IBW heating scheme, based partly on the inspection of the dispersion relation alone, can be trusted in practice. For example if the wave field becomes very large near the LH resonance, stochastic ion heating [18] could severely spoil further penetration. Note that the situation illustrated in fig. 1 is quite representative of direct launching of IBWs, since in the low frequency domain ($\omega/\Omega_{ci} \lesssim 10$), the LH resonance density is low and insensitive to the harmonic number, as shown in Fig. 3.

The constitutive relation capable of describing higher order IBWs is necessarily integral in nature. Indeed, a differential expansion to order $n+1$ in $k_\perp^2 \rho_i^2$, as would be needed to describe the n–th IBW, would in addition include $n-3$ spurious waves without physical reality. How these spurious solutions arise is easily understood by coinsidering the corresponding expansion of the local dispersion relation. Although convergent for all values of $k_\perp^2 \rho_i^2$, this expansion replaces the trascendent dispersion relation containing Bessel function, which has normally 3 (and at most 4) roots, with a polinome having $n+1$ roots. While it is relatively simple to exclude the wrong roots from the solution of an algebraic equation, a systematic procedure to avoid spurious waves from a differential system of large order is not available, particularly if the range of integration is large and includes nearly singular layers, as near the cold LH resonance.

In this note we present an integral formulation of the plasma wave equations in slab geometry valid to all orders in the Larmor radius, which could provide a starting point for the description of higher order IBWs. It is derived directly from Vlasov equation, generalising a method first developed by Yasseen and Vaclavik for electrostatic waves [19]. These integral wave equations, although complicated, are not hopelessy so: Sauter and Vaclavik [20] have actually solved them numerically in the electrostatic approximation, in a simple geometry showing the excitation of the IBW just below $\omega = 4\Omega_{ci}$. In addition, the WKB theory of integral wave equations of this kind is available [21], allowing the imposition of appropriate radiation conditions far from singular layers (LH and IC harmonics).

Here we show that these equations can be put in the form used by Sauter and Vaclavik [20] in the general, fully electromagnetic case. This is useful, because the electrostatic approximation, although accurate for most purposes, overestimates electron Landau damping of IBWs by about an order of magnitude [12]. Moreover we give other equivalent formulations of the equations which could be useful as starting point for further approximations. Although we do not attempt to solve the equations here, we put into evidence some of their properties, and suggest some approach for their solution in particular cases. It is hoped to give thereby a contribution to the investigation of h.f. waves of very short wavelength in non-uniform plasmas in domains where the WKB approximation is not valid, a situation in which has received little attention until recently.

2. – The constitutive relation in the Fourier space representation.

We consider a plane–layered plasma with gradients in the (x, y) directions, perpendicular to the static magnetic field (along z). The z-direction is ignorable, so that we can consider fields of the form

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}_\perp; k_z)\, e^{i(k_z z - \omega t)} \tag{1}$$

where $\vec{r}_\perp \equiv (x, y)$. Using primes to denote unperturbed orbits

$$\vec{r}'_\perp = \vec{r}_\perp - \frac{(\vec{v}'_\perp - \vec{v}_\perp) \times \vec{b}}{\Omega_{c\alpha}} \tag{2}$$

we can solve the linearised Vlasov equation to obtain the following integral form of the constitutive relation:

$$\vec{J}(\vec{r}_\perp;k_z) = -\sum_\alpha \frac{e^2 Z_\alpha^2}{m_\alpha} \int \vec{v}d\vec{v} \int_{-\infty}^{t} dt' e^{i(\omega - k_z v_z)(t-t')} \vec{E}(\vec{r}_\perp', k_z) \cdot \frac{\partial F_\alpha}{\partial \vec{v}'} \tag{3}$$

In writing this equation I have neglected very small $\vec{v} \times \vec{B}_o$ terms arising because of the inhomogeneity. More important, I have assumed the drift velocity arising due to the magnetic field inhomogeneity to be negligible, so that (2) is an adequate representation of the particle motion. This assumption concerns only the equilibrium configuration, and could be easily relaxed if needed; it is however adequate for frequencies of the order of Ω_{ci} or larger.

We restrict our considerations to a Maxwellian ion distribution function,

$$F_\alpha = n_\alpha F_{M\alpha}(v^2, \vec{r}_g) \tag{4}$$

where

$$\vec{r}_g = \vec{r}_\perp + \vec{\rho}_g = \vec{r}_\perp + \frac{\vec{v}_\perp \times \vec{b}}{\Omega_{c\alpha}} \tag{5}$$

is the guiding center position, and $\vec{b} = \vec{u}_z = \vec{B}_0/B_0$. Then

$$\frac{\partial F_\alpha}{\partial \vec{v}} = -\frac{2n_\alpha}{v_{th\alpha}^2}\left[\vec{v} + \frac{v_{th\alpha}^2}{2\Omega_{c\alpha}}\vec{b} \times \vec{\nabla}_\perp \ln F_{M\alpha}\right] F_{M\alpha}(v^2, \vec{r}_g)$$

$$\vec{\nabla}_\perp \ln F_{M\alpha} = \frac{1}{L_n}\left[1 + \eta\left(\frac{v^2}{v_{th\alpha}^2} - \frac{3}{2}\right)\right]\vec{u}_\nabla$$

where $\eta = (d\ln T/d\ln n)$ and $\vec{u}_\nabla$ defines the direction of the gradient. Hence

$$\begin{aligned}\vec{J}(\vec{r}_\perp;k_z) = &\sum_\alpha \frac{2e^2 Z_\alpha^2}{m_\alpha} \int d\vec{v}\, n_\alpha(\vec{r}_g) F_{M\alpha}(v^2) \frac{\vec{v}}{v_{th\alpha}} \int_{-\infty}^{t} dt' e^{i(\omega - k_z v_z)(t-t')} \\ &\left\{\frac{\vec{v}'}{v_{th\alpha}} + \frac{v_{th\alpha}}{2\Omega_{c\alpha} L_n}\left[1 + \eta\left(\frac{v^2}{v_{th\alpha}^2} - \frac{3}{2}\right)\right]\left(\vec{b} \times \vec{u}_\nabla\right)\right\} \cdot \vec{E}(\vec{r}_\perp', k_z)\end{aligned} \tag{5}$$

In a plasma of finite cross-section, it is always possible to write

$$\vec{E}(\vec{r}_\perp, k_z) = \int d\vec{k}_\perp e^{i\vec{k}_\perp \cdot \vec{r}_\perp} \vec{E}(\vec{k}_\perp, k_z) \tag{6}$$

$$\vec{J}(\vec{r}_\perp, k_z) = \int d\vec{k}_\perp e^{i\vec{k}_\perp \cdot \vec{r}_\perp} \vec{J}(\vec{k}_\perp, k_z) \tag{7}$$

Substituting for $\vec{E}$ into the previous equation, and inverting the Fourier transform for $\vec{J}$ we obtain:

$$\begin{aligned}\vec{J}(\vec{k}_\perp,k_z) &= \frac{1}{(2\pi)^2}\int d\vec{r}\, e^{-i\vec{k}_\perp\cdot\vec{r}_\perp}\vec{J}(\vec{r}_\perp,k_z)\\ &= \frac{1}{(2\pi)^2}\int d\vec{r}\, e^{-i\vec{k}_\perp\cdot\vec{r}_\perp}\int d\vec{v}\,\frac{\omega_{p\alpha}^2(\vec{r}_g)}{2\pi}F_{M\alpha}(v^2)\frac{\vec{v}}{v_{th\alpha}}\int d\vec{k}'_\perp e^{i\vec{k}'_\perp\cdot\vec{r}_\perp}\\ &\int_{-\infty}^{t} dt' \exp i\left\{(\omega-k_zv_z)(t-t')-\frac{\vec{k}'_\perp\cdot(\vec{v}'_\perp-\vec{v}_\perp)\times\vec{b}}{\Omega_{c\alpha}(\vec{r}_g)}\right\}\\ &\left(\frac{\vec{v}'}{v_{th\alpha}}+\frac{v_{th\alpha}}{2\Omega_{c\alpha}L_n}\left[1+\eta\left(\frac{v^2}{v_{th\alpha}^2}-\frac{3}{2}\right)\right]\vec{b}\times\vec{u}_\nabla\right)\cdot\vec{E}(\vec{k}'_\perp,k_z)\end{aligned} \tag{8}$$

Shifting the argument of the FT space integration by $\vec{\rho}_g$ disentagles completely the space and velocity integrations, and makes the $\vec{k}_\perp$ and $\vec{k}'_\perp$ integrations symmetric. Interchanging the order of the $\vec{r}$ and $\vec{k}'_\perp$ integrations gives:

$$\begin{aligned}\vec{J}(\vec{k}_\perp,k_z) =& \frac{1}{(2\pi)^2}\sum_a\int d\vec{k}'_\perp\int d\vec{r}_\perp\, e^{-i(\vec{k}_\perp-\vec{k}'_\perp)\cdot\vec{r}_\perp}\frac{\omega_{p\alpha}^2(\vec{r}_\perp)}{2\pi}\int d\vec{v}\,F_{M\alpha}(v^2)\frac{\vec{v}}{v_{th\alpha}}\\ &\exp i\left\{\frac{\vec{k}_\perp\cdot(\vec{v}_\perp\times\vec{b})}{\Omega_{c\alpha}}\right\}\int_{-\infty}^{t} dt' \exp\left\{i(\omega-k_zv_z)(t-t')-\frac{\vec{k}'_\perp\cdot\vec{v}'_\perp\times\vec{b}}{\Omega_{c\alpha}}\right\}\\ &\left[\frac{\vec{v}'}{v_{th\alpha}}+\frac{v_{th\alpha}}{2\Omega_{c\alpha}L_n}\left[1+\eta\left(\frac{v^2}{v_{th\alpha}^2}-\frac{3}{2}\right)\right]\vec{b}\times\vec{u}_\nabla\right]\cdot\vec{E}(\vec{k}'_\perp,k_z)\\ =& \frac{1}{(2\pi)^2}\sum_a\int d\vec{k}'_\perp\int d\vec{r}_\perp\, e^{-i(\vec{k}_\perp-\vec{k}'_\perp)\cdot\vec{r}_\perp}\frac{\omega_{p\alpha}^2(\vec{r}_\perp)}{2\pi}\\ &\int d\vec{v}\,F_{M\alpha}(v^2)\frac{\vec{v}}{v_{th\alpha}}\exp i\left\{\frac{k_\perp v_\perp}{\Omega_{c\alpha}}\sin(\phi-\psi)\right\}\\ &\int_0^{\infty} d\tau \exp i\left\{(\omega-k_zv_z)\tau-\frac{k'_\perp v_\perp}{\Omega_{c\alpha}}\sin(\phi-\psi'+\Omega_{c\alpha}\tau)\right\}\\ &\left[\frac{\vec{v}'}{v_{th\alpha}}+\frac{v_{th\alpha}}{2\Omega_{c\alpha}L_n}\left[1+\eta\left(\frac{v^2}{v_{th\alpha}^2}-\frac{3}{2}\right)\right]\vec{b}\times\vec{u}_\nabla\right]\cdot\vec{E}(\vec{k}'_\perp,k_z)\end{aligned} \tag{9}$$

where $\psi = \tan^{-1}(k_y/k_x)$ and $\psi' = \tan^{-1}(k'_y/k'_x)$. Note that $\vec{r}_\perp$ should appear also in $v_{th\alpha}$ and in $\Omega_{c\alpha}$; this is no more indicated explicitly to avoid overcrowding of the formulas. In the following moreover I will for simplicity assume $\eta = 0$ (i.e. neglect temperature gradients with respect to density gradients), but this approximation can be easily avoided if required. The terms proportional to $v_{th\alpha}/2\Omega_{c\alpha}L_n$, wich we will call "drift" terms in the following, are anyhow very small in the ion cyclotron frequency range, and are usually neglected. We nevertheless keep them in this formal derivation, because they might be interesting to investigate wether ion cyclotron drift instabilities might play some role in the damping of IBWs.

In the following, we will make repeated use of the fact that Eq. (9) is of the form

$$\vec{J}(\vec{k}_\perp, k_z) = \int d\vec{k}'_\perp \, \underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}'_\perp) \cdot \vec{E}(\vec{k}'_\perp, k_z) \tag{10}$$

The dependence of the Kernel $\underline{\underline{\sigma}}$ on k_z will be omitted when not required for clarity. Our next goal is to give explicit expressions for this Kernel.

2.1 – Bessel function expansion. Using the generating function for Bessel functions, we have:

$$\begin{aligned}
\vec{J}(\vec{k}_\perp, k_z) = & \frac{1}{(2\pi)^2} \sum_a \int d\vec{k}'_\perp \int d\vec{r}_\perp \, e^{-i(\vec{k}_\perp - \vec{k}'_\perp)\cdot \vec{r}_\perp} \frac{\omega_{p\alpha}^2}{2\pi} \\
& \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel F_{M\alpha}(v^2) \int_0^{2\pi} d\phi \left\{ v_\perp \left[\cos\phi \vec{u}_x + \sin\phi \vec{u}_y\right] + v_\parallel \vec{u}_z \right\} \cdot \\
& \sum_{n=-\infty}^{+\infty} J_n\left(\frac{k_\perp v_\perp}{\Omega_{c\alpha}}\right) e^{in(\phi-\psi)} \int_0^\infty d\tau e^{i(\omega - k_z v_z)\tau} \\
& \qquad \sum_{n'=-\infty}^{+\infty} J_{n'}\left(\frac{k'_\perp v_\perp}{\Omega_{c\alpha}}\right) e^{-in'(\phi-\psi'+\Omega_{c\alpha}\tau)} \\
& \left[v_\perp \left(\cos(\phi + \Omega_{c\alpha}\tau)\vec{u}_x + \sin(\phi+\Omega_{c\alpha}\tau)\vec{u}_y \right) + v_\parallel \vec{u}_z \right. \\
& \qquad \left. + \frac{v_{th\alpha}}{2\Omega_{c\alpha} L_n} (\vec{b} \times \vec{u}_\nabla) \right] \cdot \vec{E}_\perp(\vec{k}'_\perp, k_z)
\end{aligned} \tag{11}$$

Assuming that the equilibrium gradients make an angle θ with the x-axis, and performing the ϕ integrations, we get:

$$\begin{aligned}
\frac{4\pi i}{\omega} \sigma_{ij}(\vec{k}_\perp, \vec{k}'_\perp) = & \frac{1}{(2\pi)^2} \sum_a \int d\vec{k}'_\perp \int d\vec{r}_\perp \, e^{-i(\vec{k}_\perp - \vec{k}'_\perp)\cdot \vec{r}_\perp} \\
& \left\{ -4 \frac{\omega_{p\alpha}^2}{\omega^2} \int_0^\infty e^{-w^2} w dw \int_{-\infty}^{+\infty} \frac{e^{-u^2}}{\sqrt{\pi}} du \right. \\
& \left. \sum_{n=-\infty}^{+\infty} \Pi_{ij}^n(\mu, \mu'; w) e^{-in(\psi-\psi')} \left[-i\omega \int_0^\infty d\tau \, e^{i(\omega - n\Omega_{c\alpha} - k_z v_\parallel)\tau} \right] \right\}
\end{aligned} \tag{12}$$

where we have defined the dimensionless quantities

$$\mu = \frac{k_\perp v_{th\alpha}}{\Omega_{c\alpha}} \qquad \mu' = \frac{k'_\perp v_{th\alpha}}{\Omega_{c\alpha}} \qquad \gamma = \frac{v_{th\alpha}}{2\Omega_{c\alpha} L_n}$$

The matrix $\underline{\underline{\Pi}}^n$ has the form

$$\underline{\underline{\Pi}}^n = \underline{\underline{R}}(\psi) \cdot \vec{\pi}^n(\mu w) \; : \; \{\underline{\underline{R}}^{-1}(\psi') \cdot \vec{\pi}^n(\mu' w) + \underline{\underline{\Gamma}}(\theta) \cdot \vec{\tau}^n(\mu' w)\} \tag{13}$$

where $\underline{\underline{R}}$ is a rotation by the angle ψ,

$$\underline{\underline{R}}(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

($\underline{\underline{R}}^{-1}$ is obtained changing the sign of the angle), and

$$\pi_1^n = \frac{n}{\mu} J_n(\mu w) \qquad \pi_2^n = -iwJ_n'(\mu w) \qquad \pi_3^n = uJ_n(\mu w) \tag{15}$$

Further the "drift" term is

$$\underline{\underline{\Gamma}}(\theta) = \begin{pmatrix} 0 & 0 & -\gamma\sin\theta \\ 0 & 0 & \gamma\cos\theta \\ \gamma\sin\theta & -\gamma\cos\theta & 0 \end{pmatrix} \tag{16}$$

with

$$\tau_1^n = \tau_2^n = 0 \qquad \tau_3^n = J_n(\mu' w) \tag{17}$$

The components of the matrix $\underline{\underline{\Pi}}^n$ are listed explicitly in Appendix A.

The integrations over τ and u (parallel velocity) are elementary. The integrations over perpendicular velocities can also be performed with the help of Weber Bessel function integrals,

$$\begin{aligned}
S^n(\mu,\mu') &\equiv 4\int_0^\infty e^{-w^2} J_n(\mu w) J_n(\mu' w) w\,dw = \\
&= 2I_n\left(\frac{\mu\mu'}{2}\right) \exp\left\{-\left(\frac{\mu^2+\mu'^2}{4}\right)\right\} \\
D^n(\mu,\mu') &\equiv 4\int_0^\infty e^{-w^2} J_n(\mu w) J_n'(\mu' w) w^2\,dw = \\
&= \left(\mu I_n'\left(\frac{\mu\mu'}{2}\right) - \mu' I_n\left(\frac{\mu\mu'}{2}\right)\right) \exp\left\{-\left(\frac{\mu^2+\mu'^2}{4}\right)\right\} \\
D^n(\mu',\mu) &\equiv 4\int_0^\infty e^{-w^2} J_n'(\mu w) J_n(\mu' w) w^2\,dw = \\
&= \left(\mu' I_n'\left(\frac{\mu\mu'}{2}\right) - \mu I_n\left(\frac{\mu\mu'}{2}\right)\right) \exp\left\{-\left(\frac{\mu^2+\mu'^2}{4}\right)\right\} \\
T^n(\mu,\mu') &\equiv 4\int_0^\infty e^{-w^2} J_n'(\mu w) J_n'(\mu' w) w^3\,dw = \\
&= \left\{\left(\frac{2n^2}{\mu\mu'} + \mu\mu'\right) I_n\left(\frac{\mu\mu'}{2}\right) - \frac{\mu^2+\mu'^2}{2} I_n'\left(\frac{\mu\mu'}{2}\right)\right\} \exp\left\{-\left(\frac{\mu^2+\mu'^2}{4}\right)\right\}
\end{aligned} \tag{18}$$

The result can be written

$$\frac{4\pi i}{\omega}\sigma_{ij}(\vec{k}_\perp,\vec{k}'_\perp) = -\frac{1}{(2\pi)^2}\sum_\alpha \int d\vec{r}_\perp\, e^{-i(\vec{k}_\perp-\vec{k}'_\perp)\cdot\vec{r}_\perp}\,\frac{\omega_{p\alpha}^2}{\omega^2}\sum_{n=-\infty}^{+\infty}\hat{\sigma}_{ij}^n(\mu,\mu')e^{-in(\psi-\psi')} \tag{19}$$

where the matrix $\hat{\sigma}_{ij}^n$ has the form

$$\underline{\underline{\sigma}}^n = \underline{\underline{R}}(\psi)\cdot\left\{\underline{\underline{\sigma}}^{o,n}(\mu,\mu')\cdot\underline{\underline{R}}^{-1}(\psi') - \underline{\underline{\tau}}^o(\mu,\mu')\cdot\underline{\underline{\Gamma}}(\theta)\right\} \tag{20}$$

The main term is

$$\begin{aligned}
\sigma_{xx}^{o,n} &= \frac{n^2}{\mu\mu'}S^n(\mu,\mu')\left(-x_0^\alpha Z(x_n^\alpha)\right)\\
\sigma_{xy}^{o,n} &= i\frac{n}{\mu}D^n(\mu,\mu')\left(-x_0^\alpha Z(x_n^\alpha)\right)\\
\sigma_{xz}^{o,n} &= \frac{n}{\mu}S^n(\mu,\mu')\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right)\\
\sigma_{yx}^{o,n} &= -i\frac{n}{\mu'}D^n(\mu',\mu)\left(-x_0^\alpha Z(x_n^\alpha)\right)\\
\sigma_{yy}^{o,n} &= T^n(\mu,\mu')\left(-x_0^\alpha Z(x_n^\alpha)\right)\\
\sigma_{yz}^{o,n} &= -iD^n(\mu,\mu')\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right)\\
\sigma_{zx}^{o,n} &= \frac{n}{\mu'}S^n(\mu,\mu')\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right)\\
\sigma_{zy}^{o,n} &= iD^n(\mu',\mu)\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right)\\
\sigma_{zz}^{o,n} &= S^n(\mu,\mu')\left(x_0^\alpha x_n^\alpha Z'(x_n^\alpha)\right)
\end{aligned} \tag{21}$$

and the drift term is

$$\begin{aligned}
\tau_{xx}^{o,n} &= \tau_{xy}^{o,n} = 0\\
\tau_{xz}^{o,n} &= -\frac{n}{\mu}S^n(\mu,\mu')\left(-x_0^\alpha Z(x_n^\alpha)\right)\\
\tau_{yx}^{o,n} &= \tau_{yy}^{o,n} = 0\\
\tau_{yz}^{o,n} &= iD^n(\mu',\mu)\left(-x_0^\alpha Z(x_n^\alpha)\right)\\
\tau_{zx}^{o,n} &= \tau_{zy}^{o,n} = 0\\
\tau_{zz}^{o,n} &= -S^n(\mu,\mu')\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right)
\end{aligned} \tag{22}$$

where Z is the plasma dispersion function, and $x_n = (\omega - n\Omega_{ci})/k_\parallel v_{thi}$. The components of the matrix $\hat{\underline{\underline{\sigma}}}^n$ are listed explicitly in Appendix B.

2.2 – Gordeev integral form [22]. Alternatively, we can sum the Bessel series before performing the $v_{\parallel}$ and τ integrations:

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}(\vec{k}_{\perp},\vec{k}'_{\perp}) = -\frac{1}{(2\pi)^2}\int d\vec{r}_{\perp}\, e^{-i(\vec{k}_{\perp}-\vec{k}'_{\perp})\cdot\vec{r}_{\perp}} \sum_{\alpha}\frac{\omega_{p\alpha}^2}{\omega^2}\int_{-\infty}^{+\infty}\frac{e^{-u^2}}{\sqrt{\pi}}du$$
$$\left[-i\omega\int_0^{\infty}d\tau\, e^{i(\omega-k_z v_{th\alpha}u)\tau}\exp\left\{-\left[\frac{\mu^2+\mu'^2}{4}-\frac{\mu\mu'}{2}\cos Q\right]\right\}\right. \tag{23}$$
$$\left.\underline{\underline{R}}(\psi)\cdot\left[\underline{\underline{T}}(\mu,\mu',\Omega_{c\alpha}\tau)\cdot\underline{\underline{R}}^{-1}(\psi')-\gamma\underline{\underline{t}}(\mu,\mu',\Omega_{c\alpha}\tau)\cdot\underline{\underline{\Gamma}}(\theta)\right]\right]$$

where $Q=\Omega_{c\alpha}\tau+\psi-\psi'$, and

$$\begin{aligned}
T_{11} &= \cos Q - \frac{\mu\mu'}{2}\sin^2 Q\\
T_{12} &= \left[1+\frac{\mu\mu'}{2}\left(1-\frac{\mu'}{\mu}\right)\right]\sin Q\\
T_{13} &= -iu\mu'\sin Q\\
T_{21} &= -\left[1+\frac{\mu\mu'}{2}\left(1-\frac{\mu}{\mu'}\right)\right]\sin Q\\
T_{22} &= \left(1-\frac{\mu^2+\mu'^2}{2}\right)\cos Q+\frac{\mu\mu'}{2}\left(1+\cos^2 Q\right)\\
T_{23} &= iu\left(\mu-\mu'\cos Q\right)\\
T_{31} &= -iu\mu\sin Q\\
T_{32} &= -iu\left(\mu'-\mu\cos Q\right)\\
T_{33} &= 2u^2
\end{aligned} \tag{24}$$

and

$$t_{13}=-i\mu'\sin Q \qquad t_{23}=i\left(\mu-\mu'\cos Q\right) \qquad t_{33}=2 \tag{25}$$

Taking advantage of the periodicity of the factors depending on $\Omega_{c\alpha}\tau$, we can rewrite (8) (assuming to begin with with $\mathrm{Im}(\omega)>0$, pending the usual analytic continuation with the Landau prescription) as:

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}(\vec{k}_{\perp},\vec{k}'_{\perp}) = -\frac{1}{(2\pi)^2}\int d\vec{r}_{\perp}\, e^{-i(\vec{k}_{\perp}-\vec{k}'_{\perp})\cdot\vec{r}_{\perp}}$$
$$\sum_{\alpha}\frac{\omega_{p\alpha}^2}{\omega^2}\int_{-\infty}^{+\infty}\frac{e^{-u^2}}{\sqrt{\pi}}\frac{\varpi}{\sin\pi(\varpi-\xi u)}du \tag{26}$$
$$\int_0^{\pi}d\tau\exp\left\{-\left[\frac{\mu^2+\mu'^2}{4}-\frac{\mu\mu'}{2}\cos(\tau+\psi-\psi')\right]\right\}$$
$$\underline{\underline{R}}(\psi)\cdot\left[\underline{\underline{\tilde{T}}}(\mu,\mu',\tau)\cdot\underline{\underline{R}}^{-1}(\psi')-\gamma\underline{\underline{\tilde{t}}}(\mu,\mu',\tau)\cdot\underline{\underline{\Gamma}}(\theta)\right]$$

where

$$\tilde{T}_{11} = \cos\left((\varpi - \xi u)(\tau - \pi)\right)\left\{ \cos(\tau + \psi - \psi') - \frac{\mu\mu'}{2}\sin^2(\tau + \psi - \psi')\right\}$$

$$\tilde{T}_{12} = -i\sin\left((\varpi - \xi u)(\tau - \pi)\right)\left\{ \left[1 + \frac{\mu\mu'}{2}\left(1 - \frac{\mu'}{\mu}\right)\right]\sin(\tau + \psi - \psi')\right\}$$

$$\tilde{T}_{13} = -u\mu'\sin\left((\varpi - \xi u)(\tau - \pi)\right)\sin(\tau + \psi - \psi')$$

$$\tilde{T}_{21} = i\sin\left((\varpi - \xi u)(\tau - \pi)\right)\left\{ \left[1 + \frac{\mu\mu'}{2}\left(1 - \frac{\mu}{\mu'}\right)\right] sin(\tau + \psi - \psi')\right\}$$

$$\tilde{T}_{22} = \cos\left((\varpi - \xi u)(\tau - \pi)\right)\left\{ \left(1 - \frac{\mu^2 + \mu'^2}{2}\right)\cos(\tau + \psi - \psi') + \frac{\mu\mu'}{2}\left(1 + \cos^2(\tau + \psi - \psi')\right)\right\} \quad (27)$$

$$\tilde{T}_{23} = iu\cos\left((\varpi - \xi u)(\tau - \pi)\right)\left(\mu - \mu'\cos(\tau + \psi - \psi')\right)$$

$$\tilde{T}_{31} = -u\sin\left((\varpi - \xi u)(\tau - \pi)\right)\mu\sin(\tau + \psi - \psi')$$

$$\tilde{T}_{32} = -iu\cos\left((\varpi - \xi u)(\tau - \pi)\right)\left(\mu' - \mu\cos(\tau + \psi - \psi')\right)$$

$$\tilde{T}_{33} = 2u^2\cos\left((\varpi - \xi u)(\tau - \pi)\right)$$

and

$$\begin{aligned} \tilde{t}_{13} &= -\mu'\sin\left((\varpi - \xi u)(\tau - \pi)\right)\sin(\tau + \psi - \psi') \\ \tilde{t}_{23} &= i\cos\left((\varpi - \xi u)(\tau - \pi)\right)\left(\mu - \mu'\cos(\tau + \psi - \psi')\right) \\ \tilde{t}_{33} &= 2\cos\left((\varpi - \xi u)(\tau - \pi)\right) \end{aligned} \quad (28)$$

Here we have put

$$\varpi = \frac{\omega}{\Omega_{c\alpha}} \qquad \xi = \frac{k_{\|}v_{\|}}{\Omega_{c\alpha}}$$

The Bessel series expression for the Kernel $\underline{\underline{\sigma}}$ could be useful near low order isolated cyclotron harmonics. The Gordeev form derived in this section on the other hand could be useful as starting point for asymptotic approximations in the limit $\vec{k}_{\perp}\rho_{\alpha} \gg 1$, particularly when also $\omega \gg \Omega_{c\alpha}$, as near the Lower Hybrid resonance.

2.3 – Discussion of the Fourier space representation. As already stressed, in the Fourier space representation the constitutive relation is of the form

$$\vec{J}(\vec{k}_\perp, k_z) = \int d\vec{k}'_\perp \, \underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}'_\perp \| k_z) \cdot \vec{E}(\vec{k}'_\perp, k_z) \tag{29}$$

with

$$\underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}'_\perp) = \int d\vec{r}_\perp \, e^{-i(\vec{k}_\perp - \vec{k}'_\perp)\cdot\vec{r}_\perp} \hat{\underline{\underline{\sigma}}}(\vec{k}_\perp, \vec{k}'_\perp; k_z; \vec{r}_\perp) \tag{30}$$

Note the symmetry of the conductivity Kernel,

$$\sigma_{ij}(k_\perp, k'_\perp) = \sigma^\dagger_{ji}(k'_\perp, k_\perp) \tag{31}$$

where † denote the hermitean conjugate

In the uniform limit the $\vec{r}_\perp$ integration gives a factor $\delta(\vec{k}_\perp - \vec{k}'_\perp)$, and $\gamma = 0$; then the Kernel $\underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}'_\perp)$ obtained in this section reduces immediately to the usual h.f. conductivity tensor, either in the form of a Bessel function series, or in the Gordeev integral form:

$$\underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}'_\perp)|_{\vec{k}_\perp = \vec{k}'_\perp} = \underline{\underline{\sigma}}(\vec{k}) \tag{32}$$

Linear space variations in the plasma parameters can also be treated easily. Taking into account the form of the conductivity Kernel, we can write:

$$\hat{\underline{\underline{\sigma}}} = \underline{\underline{\sigma}}^0(\vec{k}_\perp, \vec{k}'_\perp) + \sum_j (\vec{\kappa}_j \cdot \vec{r}_\perp) \underline{\underline{\sigma}}^j(\vec{k}_\perp, \vec{k}'_\perp) \tag{33}$$

where j runs over the plasma parameters X_j (density, magnetic field strength, temperature, etc.), whith

$$\vec{\kappa}_j = \vec{\nabla} X_j \qquad\qquad \underline{\underline{\sigma}}^j = X_j \frac{\partial \hat{\underline{\underline{\sigma}}}}{\partial X_j} \tag{34}$$

Then

$$\begin{aligned} \underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}'_\perp) &= \left\{ \underline{\underline{\sigma}}(\vec{k}) + i\vec{\kappa}_j \cdot \left. \frac{\partial \underline{\underline{\sigma}}^j}{\partial \vec{k}_\perp} \right)_{\vec{k}_\perp = \vec{k}'_\perp} \right\} \delta(\vec{k}_\perp - \vec{k}'_\perp) \\ &= \left\{ \underline{\underline{\sigma}}(\vec{k}) - i\vec{\kappa}_j \cdot \left. \frac{\partial \underline{\underline{\sigma}}^j}{\partial \vec{k}'_\perp} \right)_{\vec{k}_\perp = \vec{k}'_\perp} \right\} \delta(\vec{k}_\perp - \vec{k}'_\perp) \end{aligned} \tag{35}$$

Hence this case can be exhamined by solving an algebraic equation which is a generalisation of the local Dispersion Relation. This particularly simple approach to the investigation of wave propagation in non–uniform plasmas is closely related to recent work by Lashmore–Davies and Dendy [23].

3 – Configuration space equations.

In the general nonuniform case, Eq. (29) is of the convolution type in $\vec{k}_\perp$ space. It is however immediate to transform it into a convolution equation in physical space:

$$\vec{J}(\vec{r}_\perp) = \frac{1}{(2\pi)^2 \mathcal{S}} \int d\vec{r}\,' \underline{\underline{\sigma}}(\vec{r}_\perp, \vec{r}\,'_\perp) \cdot \vec{E}(\vec{r}\,'_\perp) \tag{36}$$

where $\mathcal{S}$ is the area of the plasma cross-section, and

$$\underline{\underline{\sigma}}(\vec{r}_\perp, \vec{r}\,'_\perp) = \mathcal{S} \int d\vec{k}_\perp\, e^{i\vec{k}_\perp \cdot \vec{r}_\perp} \int d\vec{k}\,'_\perp\, e^{-i\vec{k}\,'_\perp \cdot \vec{r}\,'_\perp} \underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}\,'_\perp) \tag{37}$$

According to the results of the previous section, this can be written

$$\begin{aligned} \underline{\underline{\sigma}}(\vec{r}_\perp, \vec{r}\,'_\perp) = & \frac{\mathcal{S}}{(2\pi)^2} \sum_\alpha \int d\vec{r}\,''_\perp \\ & \int d\vec{k}_\perp\, e^{i\vec{k}_\perp \cdot (\vec{r}_\perp - \vec{r}\,''_\perp)} \int d\vec{k}\,'_\perp\, e^{-i\vec{k}\,'_\perp \cdot (\vec{r}\,'_\perp - \vec{r}\,''_\perp)} \hat{\underline{\underline{\sigma}}}^\alpha(\vec{k}_\perp, \vec{k}\,'_\perp; \vec{r}\,''_\perp) \end{aligned} \tag{38}$$

where the dependence of $\hat{\underline{\underline{\sigma}}}^\alpha$ on $\vec{r}\,''_\perp$ is through $\omega^2_{p\alpha}(\vec{r}\,''_\perp)/\omega^2$, $\Omega_{c\alpha}(\vec{r}\,''_\perp)$, etc..

In general the multiple integrations (including those over the angles ψ and ψ') are quite complicated. In the one–dimensional case (gradient only in the x–direction, and $k_y = 0$), we have the somewhat simpler result

$$\vec{J}(x) = \frac{1}{2\pi \mathcal{L}} \int dx' \underline{\underline{\sigma}}(x, x') \cdot \vec{E}(x') \tag{39}$$

with

$$\begin{aligned} \underline{\underline{\sigma}}(x, x') = & \frac{\mathcal{L}}{2\pi} \sum_\alpha \int dx'' \\ & \int_{-\infty}^{+\infty} dk_x\, e^{ik_x(x - x'')} \int_{-\infty}^{+\infty} dk'_x\, e^{-ik'_x(x' - x'')} \hat{\underline{\underline{\sigma}}}^\alpha(k_x, k'_x; x'') \end{aligned} \tag{40}$$

where $\mathcal{L}$ is the thickness of the plasma slab. We can write in this case

$$\frac{4\pi i}{\omega} \hat{\sigma}^\alpha_{ij}(k_x, k'_x; x'') = -\frac{\omega^2_{p\alpha}}{\omega^2} \sum_{n=-\infty}^{+\infty} \tilde{\sigma}^n_{ij}(\mu, \mu') \tag{41}$$

with

$$
\begin{aligned}
\tilde{\sigma}^n_{xx} &= \frac{n^2}{\mu\mu'} S^n(\mu,\mu') \cdot [-x_0^\alpha Z(x_n^\alpha)] \\
\tilde{\sigma}^n_{xy} &= \frac{n}{\mu} [iD^n(\mu,\mu') + \gamma S^n(\mu,\mu')] \cdot [-x_0^\alpha Z(x_n^\alpha)] \\
\tilde{\sigma}^n_{xz} &= \frac{n}{\mu} S^n(\mu,\mu') \cdot \left[\frac{x_0^\alpha}{2} Z'(x_n^\alpha)\right] \\
\tilde{\sigma}^n_{yx} &= -i\frac{n}{\mu'} D^n(\mu',\mu) \cdot [-x_0^\alpha Z(x_n^\alpha)] \\
\tilde{\sigma}^n_{yy} &= -i\,[iT^n(\mu,\mu') - \gamma D^n(\mu',\mu)] \cdot [-x_0^\alpha Z(x_n^\alpha)] \\
\tilde{\sigma}^n_{yz} &= -iD^n(\mu',\mu) \cdot \left[\frac{x_0^\alpha}{2} Z'(x_n^\alpha)\right] \\
\tilde{\sigma}^n_{zx} &= \frac{n}{\mu} S^n(\mu,\mu') \cdot \left[\frac{x_0^\alpha}{2} Z'(x_n^\alpha)\right] \\
\tilde{\sigma}^n_{zy} &= [iD^n(\mu,\mu') + \gamma S^n(\mu,\mu')] \cdot \left[\frac{x_0^\alpha}{2} Z'(x_n^\alpha)\right] \\
\tilde{\sigma}^n_{zz} &= S^n(\mu,\mu') \cdot [x_0^\alpha x_n^\alpha Z'(x_n^\alpha)]
\end{aligned}
\tag{42}
$$

In this case of course $\mu = k_x v_{th\alpha}/\Omega_{c\alpha}$, $\mu' = k'_x v_{th\alpha}/\Omega_{c\alpha}$. These equations are easily generalised to the slab case with $k_y \neq 0$.

The integrals over wavevectors can be simplified using the identity

$$
\begin{aligned}
\tilde{P}^n &= \int_{-\infty}^{+\infty} d\mu\, e^{i\mu\xi} \int_{-\infty}^{+\infty} d\mu' e^{-i\mu\xi'}\, I_n\left(\frac{\mu\mu'}{2}\right) e^{-\frac{\mu^2+\mu'^2}{4}} \\
&= 4\int_0^\pi d\theta \frac{\cos(n\theta)}{\sin\theta} \exp\left\{-\frac{(\xi+\xi')^2}{4\sin^2(\theta/2)} - \frac{(\xi-\xi')^2}{4\cos^2(\theta/2)}\right\}
\end{aligned}
\tag{43}
$$

which can be obtained substituting into the r.h. side the integral representation of the modified Bessel functions,

$$
I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta} \cos n\theta) d\theta
$$

With the help of elementary properties of Bessel functions, we also have

$$
\begin{aligned}
\tilde{S}^n &\equiv \tilde{P}^{n-1} - \tilde{P}^{n+1} = \\
&= 4\int_0^\pi d\theta\, \sin(n\theta) \exp\left\{-\frac{(\xi+\xi')^2}{4\sin^2(\theta/2)} - \frac{(\xi-\xi')^2}{4\cos^2(\theta/2)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}^n &\equiv \tilde{P}^{n-1} + \tilde{P}^{n+1} = \\
&= 4\int_0^\pi d\theta \frac{\cos\theta}{\sin\theta} \cos(n\theta) \exp\left\{-\frac{(\xi+\xi')^2}{4\sin^2(\theta/2)} - \frac{(\xi-\xi')^2}{4\cos^2(\theta/2)}\right\}
\end{aligned}
$$

Since

$$\xi - \xi' = x - x' \qquad\qquad \xi + \xi' = x + x' - 2x''$$

we can then write:

$$\underline{\underline{\sigma}}(x,x') = \frac{2}{\pi\mathcal{L}}\int_{-\infty}^{+\infty} dx''\,\underline{\underline{\tilde{\sigma}}}\left(x - x', \frac{x+x'}{2}\middle\| x''\right) \tag{44}$$

with

$$\frac{4\pi i}{\omega}\tilde{\underline{\underline{\sigma}}}_{ij}\left(x - x', \frac{x+x'}{2}\middle\| x''\right) = -\frac{\mathcal{L}^2\Omega_{c\alpha}^2(x'')}{v_{th\alpha}^2}\frac{\omega_{p\alpha}^2(x'')}{\omega^2}\sum_{n=-\infty}^{+\infty} T_{ij}^n(\theta \,\|\, x'') \tag{45}$$

where

$$\begin{aligned}
T_{xx}^n &= n\tilde{S}^n \cdot (-x_0 Z(x_n)) \\
T_{xy}^n &= \frac{i}{2}\left[n\tilde{D}^n - \frac{\partial^2 \tilde{S}^n}{\partial \xi'^2} + 2\gamma\frac{\partial \tilde{S}^n}{\partial \xi'}\right]\cdot(-x_0 Z(x_n)) \\
T_{xz}^n &= i\frac{\partial \tilde{S}^n}{\partial \xi'}\cdot\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right) \\
T_{yx}^n &= -\frac{i}{2}\left[\tilde{D}^n - \frac{\partial^2 \tilde{S}^n}{\partial \xi^2}\right]\cdot(-x_0 Z(x_n)) \\
T_{yy}^n &= \left[n\tilde{S}^n - 2\frac{\partial^2 \tilde{P}^n}{\partial\xi\partial\xi'} + \frac{1}{4}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\xi'^2}\right)\tilde{D}^n\right. \\
&\qquad \left. + \frac{\gamma}{2}\left(\frac{\partial\tilde{D}^n}{\partial\xi'} + 2\frac{\partial\tilde{P}^n}{\partial\xi}\right)\right]\cdot(-x_0 Z(x_n)) \\
T_{yz}^n &= -\frac{1}{2}\left[\frac{\partial\tilde{D}^n}{\partial\xi'} + 2\frac{\partial\tilde{P}^n}{\partial\xi}\right]\cdot\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right) \\
T_{zx}^n &= i\frac{\partial\tilde{S}^n}{\partial\xi'}\cdot\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right) \\
T_{zy}^n &= \left[-\frac{1}{2}\left(\frac{\partial\tilde{D}^n}{\partial\xi} + 2\frac{\partial\tilde{P}^n}{\partial\xi'}\right) + 2\gamma\tilde{P}^n\right]\cdot\left(\frac{x_0^\alpha}{2}Z'(x_n^\alpha)\right) \\
T_{zz}^n &= \tilde{P}^n\cdot\left[x_0^\alpha x_n^\alpha Z'(x_n^\alpha)\right]
\end{aligned} \tag{46}$$

This form of the constitutive relation is identical with the one used by Sauter and Vaclavik [20] to solve numerically the wave equations in the electrostatic limit. Moreover, Eq. (44) is in the form considered by Berk and Pfirsch [21], who have developed a

WKB approach for its asymptotic solution. Thus, in spite of the apparent complication of the conductivity Kernel, Maxwell equations with this constitutive relation could be used as starting point for the investigation of IBW propagation in non homogeneous plasmas.

References

[1] Berstein I.B. Phys. Rev. **109** (1958) 10.

[2] Jacquinot J., McVey B.D., Scharer J.E., Phys. Rev. Lett. **39** (1977) 88.

[3] Puri S., Phys. Fluids **22** (1979) 1716.

[4] Ono M., Stix T.H., Wong K.L., Horton R., Proc. 2d Gernoble–Varenna Int. Symp. on Heating in Toroidal Plasmas, Como 1980 (EUR 7424 EN), Vol. 1 p. 593.

[5] Moody J.D., Porkolab M., Fiore C.L., Dermott F.S., et al., Phys. Rev. Lett. **60** (1988) 298.

[6] Ono M., Beiersdorf P., Bell R., Bernabei S., et al., Phys. Rev. Lett. **60** (1988) 294.

[7] Swanson D.G. Phys. Fluids **28** (1986) 2645, and references therein.

[8] Colestock P.L., Kashuba R.J., Nucl. Fus. **23** (1983) 763.

[9] Ottaviani M., Lett. Nuovo Cimento **42** (1985) 385.

[10] Martin T., Vaclavik J., Helv. Phys. Acta **60** (1987) 471.

[11] Sy W. N.-C., Amano T., Ando R., Fukuyama A., Watari T., Nucl. Fus. **25** (1985) 785.

[12] Brambilla M., Nucl. Fus. **28** (1988) 549.

[13] Stix T.H., The thory of plasma waves, McGraw Hill (N.Y.) 1961.

[14] Cho S., Swanson D.G., Proc. 7th topical Conf. on Applications of RF Power to Plasmas, Kissimmee (Florida) 1987, p. 191.

[15] Porkolab M., Phys. Rev. Lett. **54** (1985) 434.

[16] Brambilla M., unpublished.

[17] Brambilla M., Nucl. Fus. **16** (1976) 47.

[18] Karney C.F.F., Phys. Fluids **21** (1978) 1584.

[19] Yasseen F., Vaclavik J., Phys. Fluids **26** (1983) 468.

[20] Sauter O., Vaclavik J., Proc. 15th Europ. Conf. on Contr. Fusion and Plasma Heating, Dubrovnik 1988, Vol. 2, p. 758.

[21] Berk H.L., Pfirsch D. J. Math. Phys **21** (1980) 2054.

[22] Gordeev G.V., J. Exptl. Theoret. Phys. USSR **6** (1952) 660.

[23] Lashmore–Davies C.N., Dendy R.O., Report CLM–P849, Culham 1988 (submitted to Phys. Rev. Lett.).

Appendix A

The explicit form of the matrix Π_{ij}^n is

$$\Pi_{xx}^n = \left[\frac{n}{\mu}\cos\psi\, J_n(\mu w) + iw\sin\psi\, J_n'(\mu w)\right]\cdot$$
$$\left[\left(\frac{n}{\mu'}\cos\psi' - \gamma\sin\theta\right) J_n(\mu' w) - iw\sin\psi'\, J_n'(\mu' w)\right]$$

$$\Pi_{xy}^n = \left[\frac{n}{\mu}\cos\psi\, J_n(\mu w) + iw\sin\psi\, J_n'(\mu w)\right]\cdot$$
$$\left[iw\cos\psi'\, J_n'(\mu' w) + \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta\right) J_n(\mu' w)\right]$$

$$\Pi_{xz}^n = \left[\frac{n}{\mu}\cos\psi\, J_n(\mu w) + iw\sin\psi\, J_n'(\mu w)\right]\cdot J_n(\mu' w)u$$

$$\Pi_{yx}^n = \left[-iw\cos\psi\, J_n'(\mu w) + \frac{n}{\mu}\sin\psi\, J_n(\mu w)\right]\cdot$$
$$\left[\left(\frac{n}{\mu'}\cos\psi' - \gamma\sin\theta\right) J_n(\mu' w) - iw\sin\psi'\, J_n'(\mu' w)\right]$$

$$\Pi_{yy}^n = \left[-iw\cos\psi\, J_n'(\mu w) + \frac{n}{\mu}\sin\psi\, J_n(\mu w)\right]\cdot$$
$$\left[iw\cos\psi'\, J_n'(\mu' w) - \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta\right) J_n(\mu' w)\right]$$

$$\Pi_{yz}^n = \left[-iw\cos\psi\, J_n'(\mu w) + \frac{n}{\mu}\sin\psi\, J_n(\mu w)\right]\cdot J_n(\mu' w)u$$

$$\Pi_{zx}^n = J_n(\mu w)u\cdot\left[\left(\frac{n}{\mu'}\cos\psi' - \gamma\sin\theta\right) J_n(\mu' w) - iw\sin\psi'\, J_n'(\mu' w)\right]$$

$$\Pi_{zy}^n = J_n(\mu w)u\cdot\left[iw\cos\psi'\, J_n'(\mu' w) + \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta\right) J_n(\mu' w)\right]$$

$$\Pi_{zz}^n = J_n(\mu w)u\cdot J_n(\mu' w)u$$

The explicit form of $\hat{\sigma}^n_{ij}$ is

$$\hat{\sigma}^n_{xx} = \left\{ \frac{n}{\mu}\cos\psi \left[\left(\frac{n}{\mu'}\cos\psi' - \gamma\sin\theta \right) S^n(\mu,\mu') - i\sin\psi' D^n(\mu,\mu') \right] + i\sin\psi \left[\left(\frac{n}{\mu'}\cos\psi' - \gamma\sin\theta \right) D^n(\mu',\mu) - i\sin\psi' T^n(\mu,\mu') \right] \right\} \cdot \left(-x_0^\alpha Z(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{xy} = \left\{ \frac{n}{\mu}\cos\psi \left[i\cos\psi' D^n(\mu,\mu') + \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta \right) S^n(\mu,\mu') \right] + i\sin\psi \left[i\cos\psi' T^n(\mu,\mu') + \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta \right) D^n(\mu',\mu) \right] \right\} \cdot \left(-x_0^\alpha Z(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{xz} = \left\{ \frac{n}{\mu}\cos\psi\, S^n(\mu,\mu') + i\sin\psi\, D^n(\mu',\mu) \right\} \cdot \left(\frac{x_0^\alpha}{2} Z'(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{yx} = \left\{ -i\cos\psi \left[\left(\frac{n}{\mu'}\cos\psi' - \gamma\sin\theta \right) D^n(\mu',\mu) - i\sin\psi' T^n(\mu,\mu') \right] + \frac{n}{\mu}\sin\psi \left[\left(\frac{n}{\mu'}\cos\psi' - \gamma\sin\theta \right) S^n(\mu,\mu') - i\sin\psi' D^n(\mu,\mu') \right] \right\} \cdot \left(-x_0^\alpha Z(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{yy} = \left\{ -i\cos\psi \left[i\cos\psi' T^n(\mu,\mu') - \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta \right) D^n(\mu',\mu) \right] + \frac{n}{\mu}\sin\psi \left[i\cos\psi' D^n(\mu,\mu') - \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta \right) S^n(\mu,\mu') \right] \right\} \cdot \left(-x_0^\alpha Z(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{yz} = \left\{ -i\cos\psi D^n(\mu',\mu) + \frac{n}{\mu'}\sin\psi S^n(\mu,\mu') \right\} \cdot \left(\frac{x_0^\alpha}{2} Z'(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{zx} = \left\{ \left(\frac{n}{\mu}\cos\psi' - \gamma\sin\theta \right) S^n(\mu,\mu') - i\sin\psi' D^n(\mu,\mu') \right\} \cdot \left(\frac{x_0^\alpha}{2} Z'(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{zy} = \left\{ i\cos\psi' D^n(\mu,\mu') + \left(\frac{n}{\mu'}\sin\psi' + \gamma\cos\theta \right) S^n(\mu,\mu') \right\} \cdot \left(\frac{x_0^\alpha}{2} Z'(x_n^\alpha) \right)$$

$$\hat{\sigma}^n_{zz} = S^n(\mu,\mu') \cdot \left(x_0^\alpha x_n^\alpha Z'(x_n^\alpha) \right)$$

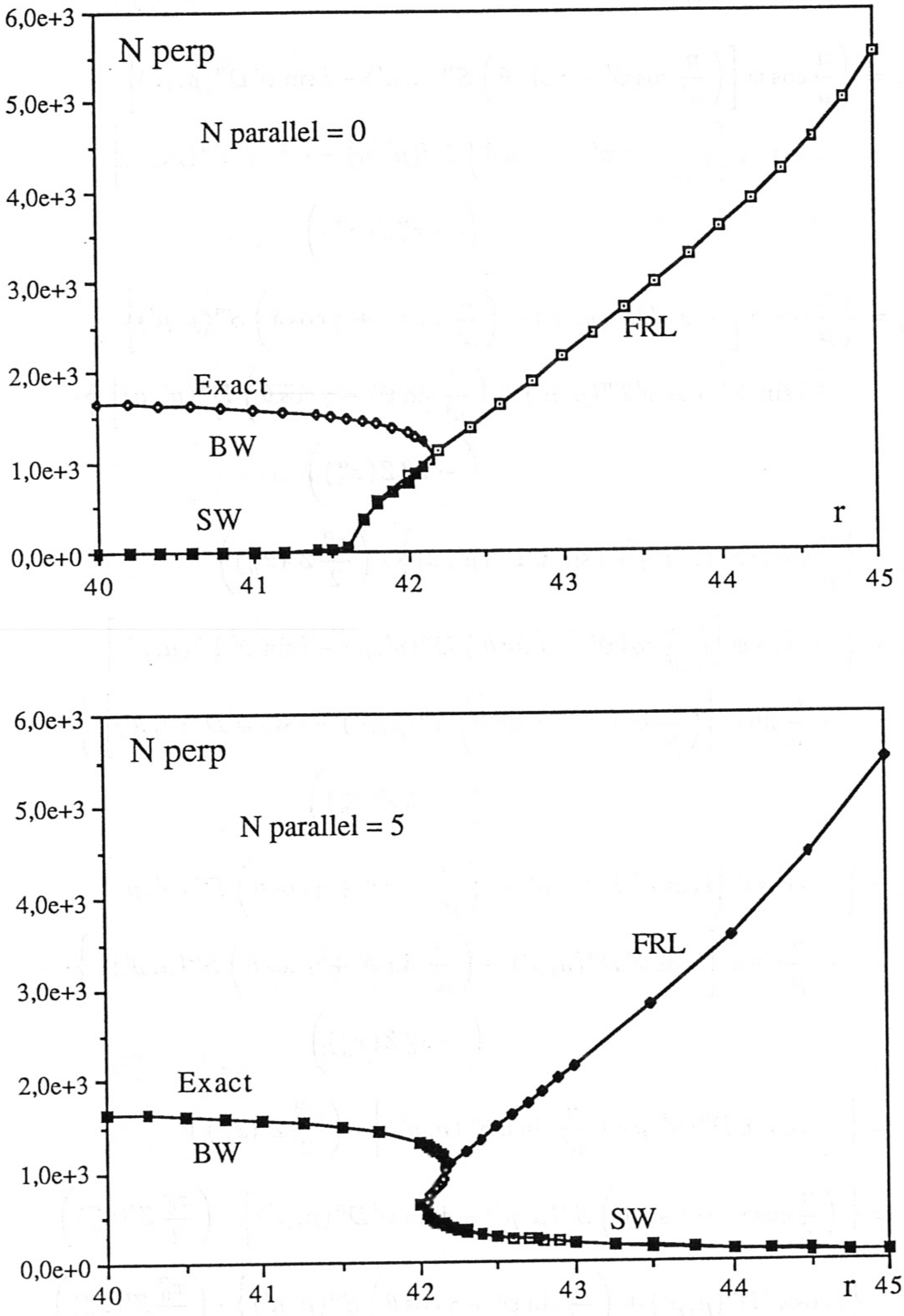

Fig. 1 - Dispersion curves in the scrape-off of a Tokamak (B = 8 T at r=0, f = 450 Ghz; at the limiter, r=40 cm, n = 10^13 cm-3, with e-folding length of 2 cm; Te = Ti = 100 eV, with e-folding length of 5 cm.

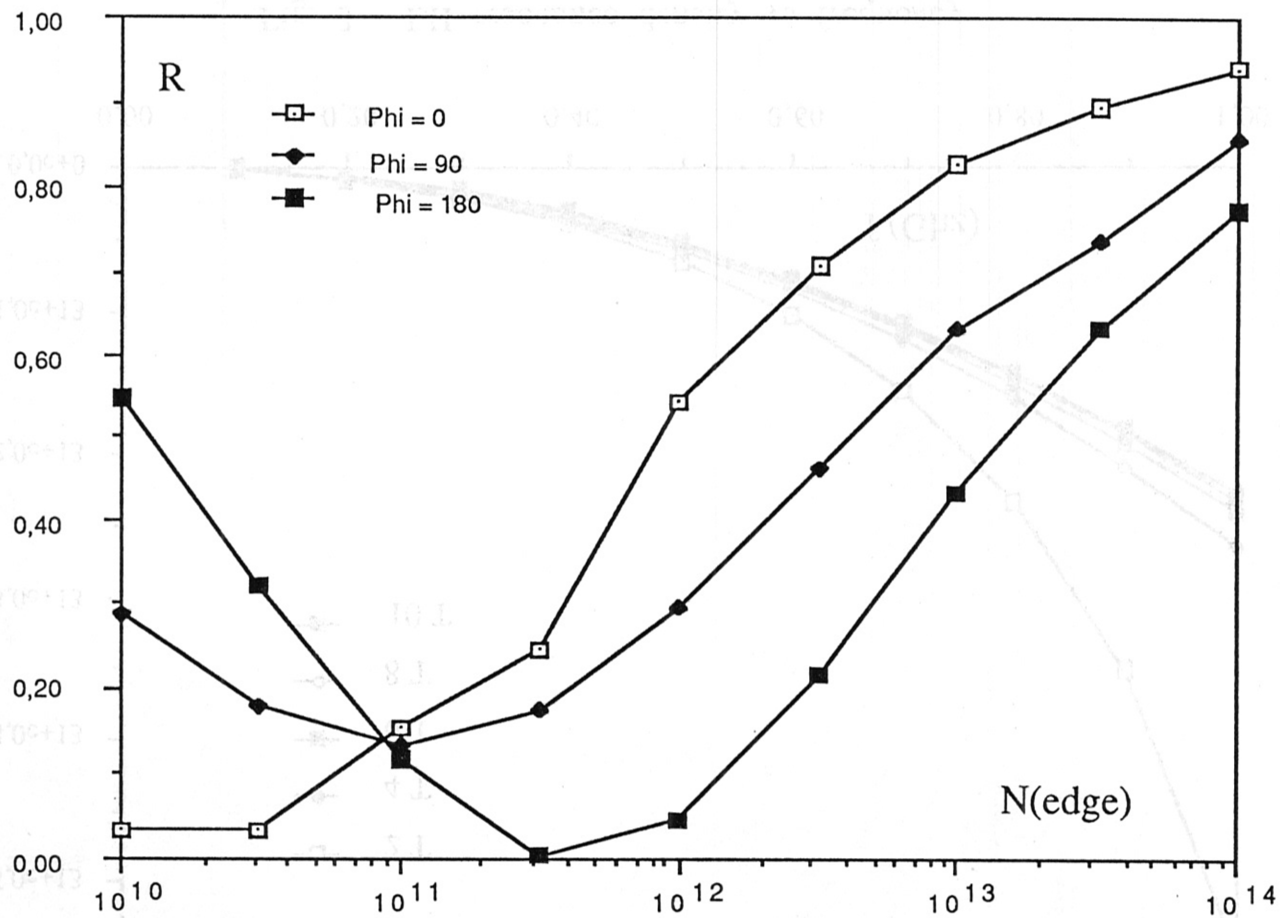

Fig. 2 - Reflection coefficient in a two-guide Grill vs edge density (dn/dx in cm-4 is taken numerically equal to n(edge) in cm-3) at three relative phases.

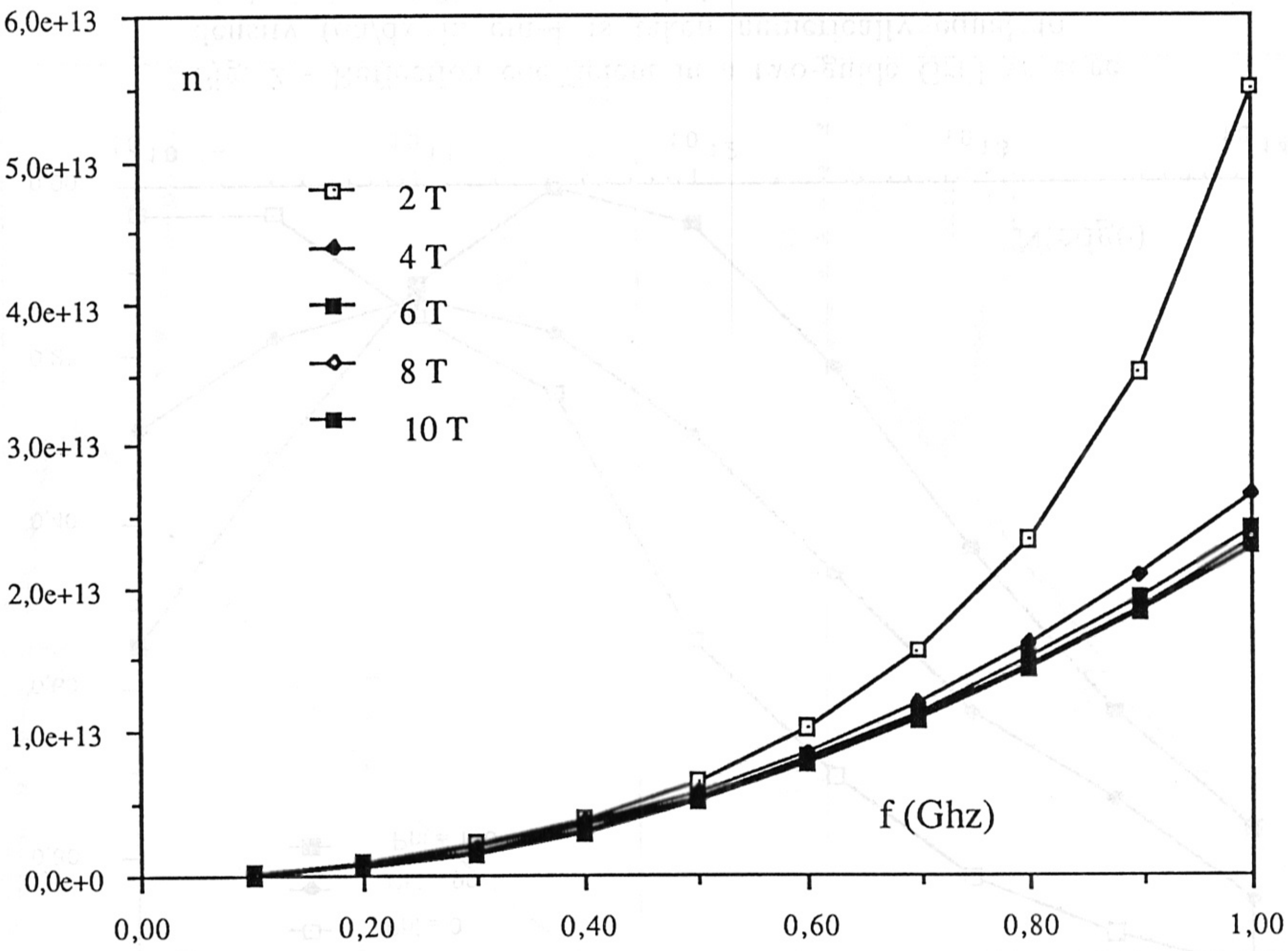

Fig. 3 - LH resonance density vs frequency