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Quasilinear space–dependent diffusion

in the lower hybrid waves regime

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Abstract

The quasilinear diffusion coefficient for the inherently space–dependent diffusion in lower hybrid heating and current drive experiments is derived; it coincides with the usual space–independent approximation where the RF field amplitude is almost constant. The form of the distribution function in the 'plateau' region and for large velocities (where the quasilinear diffusion coefficient seems to lead to unphysical results) is obtained. The position and width of the 'plateau' (which are essential quantities in the theoretical description of LH experiments) are derived and their dependence on the RF field amplitude gradient is stressed. Finally, the ponderomotive effect (which plays a substantial role in LH experiments by causing large gradients of the electrostatic field amplitude) has been derived, allowance being made for distortions of the distribution from a Maxwellian.

Introduction

In lower hybrid heating and current drive problems it is generally considered to be an excellent approximation if the wave effect on the plasma is represented as induced diffusion in velocity space (see Ref. [1]); indeed, the LH experiments seem to afford no indication that one should distrust the quasilinear description of the plasma–wave interaction. On the other hand, the currently used quasilinear diffusion coefficient (QL DC) in LH problems is space–independent, although the LH field is well delimited in space, not only at the antenna mouth but also in the plasma, owing to the 'resonance cone' structure exibited by a LH field in the plasma. The QL DC is usually obtained by superposing the QL DCs corresponding to the $k_{\parallel}$ components of the LH field according to the given $k_{\parallel}$ spectrum of the wave amplitude. This procedure is only a good approximation in a region where the field amplitude is almost constant; however, it disregards, or at least badly describes, the effect of the field amplitude gradients, which extends the spectrum to higher $k_{\parallel}$ values, although it is the very $k_{\parallel}$ values higher than those corresponding to the flat part of the field amplitude that are necessary to explain the observed power absorption and current drive. As was shown in Ref. [2], the

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ponderomotive effect produces strong gradients of the density and electrostatic electric field amplitude. It is thus necessary to establish the form of the QL DC (if it exists) by considering also the presence of gradients of the field amplitude, and, on the other hand, to show how far the ponderomotive effect is influenced by the presence of a 'plateau' in the distribution function, which follows from the QL DC in the LH heating and current drive experiments. The width of the 'plateau' is essential for explaining the observed phenomena; usually a QL DC that is non–zero only in a limited $k_\parallel$ interval, and hence in a limited $v_\parallel$ interval, is considered, an assumption which does not correspond to the experimental situations, where a 'square profile' is rather a good approximation for the field amplitude in space. However, with any reasonable assumption for the field amplitude in space the QL DC does not decrease faster than $1/v_\parallel^3$, as it should if the effect of the field can be neglected in relation to collisions, so as to make the distribution function become Maxwellian at large velocities. The paper shows that this theoretical difficulty disappears if the space dependence of the QL DC, and hence of the distribution function, is properly taken into account. The paper is organized as follows:

In Section 1 the basic equations and a necessary condition for the existence of diffusion. coefficients (DC) are derived. The aim of Section 2 is to derive the ponderomotive effect in a plasma with a distribution function which possibly deviates from a Maxwellian in some velocity intervals whose contribution to the density is negligible (an example of such a deviation is a 'plateau' in the LH regime). An averaging process is introduced in Section 3 for situations where Larmor radius effects are important. The collision term appropriate to our problem is explicitly given and averaged. A formal expression for the quasi–linear DC is derived, and also the effect on power absorption of the sign of the QL DC (when it is smaller than the DC due to collisions, so that the total DC is positive). In Section 4 the explicit form of the QL DC and some approximating formulae are derived, both for ions and electrons, in the intervals where the QL DC are smaller

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than the DC due to collisions. A comparison is made with the usual approximation of the QL DC for a localized RF field and a relevant example is given. In the same section it is shown that when the velocity is large enough the QL DC is larger than the collision–induced DC, a property shared with the usual approximation of the QL DC; this, at first sight unreasonable, property seems to extend the width of a 'plateau' to infinity (a problem discussed in Section 6). When the QL DC is larger than the collision–induced DC the definition given in Sect. 3 is no longer valid. Expressions appropriate to this situation are derived in Section 5, where it is also shown that the new definitions give positive definite DC. Finally, in Section 6 it is shown that, although for large velocity the QL DC is larger than the collision–induced DC, the 'plateau' interval of the distribution function has a finite width, which is derived. Moreover, it is shown that after a 'plateau' the distribution function is a Maxwell–Boltzmann–like distribution, with the electric potential replaced by an integral over velocity of the QL DC.

1. Basic equations

We consider particles moving in a temporally constant and spatially uniform magnetic field $\vec{B}$ perturbed by an electrostatic field $(E_i(x_1, x_3)e^{i\omega t}+c.c.)$; let the phase space be referred to Cartesian coordinates (x_i, v_i) with x_1 along the magnetic field and, without loss of generality, the electric field in the plane (x_1, x_3). The Boltzmann equation then has the form

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + \Omega v_3 \frac{\partial f}{\partial v_2} - \Omega v_2 \frac{\partial f}{\partial v_3} + \left(A_j e^{i\omega t} + c.c.\right) \frac{\partial f}{\partial v_j} = C(f) \tag{1}$$

where $A_i \equiv (q/m)E_i$, $\Omega \equiv qB/mc$ and q, m are the charge and the mass of the particles; repeated indices have to be summed. The explicit form of the collision term is given in Section 3.

The discussion is conducted mainly in the reference system defined by the unperturbed

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particle trajectories in phase space described as functions of x_1; we denote the new variables by (ς_i, w_i), with $\varsigma_1 = x_1$ and $w_1 = v_1$. In this reference system one has

$$v_j \frac{\partial}{\partial x_j} + \Omega v_3 \frac{\partial}{\partial v_2} - \Omega v_2 \frac{\partial}{\partial v_3} \equiv w_1 \frac{\partial}{\partial \varsigma_1}, \tag{2}$$

an equivalence which can be considered as the definition of the new system. The formulae which relate the two reference systems are

$$
\begin{aligned}
w_2 &= -\text{arctg}(v_3/v_2) - x_1\Omega/v_1, & v_2 &= w_3 \cos\psi, \\
w_3^2 &= v_2^2 + v_3^2, & v_3 &= -w_3 \sin\psi \\
\varsigma_2 &= x_2 - v_3/\Omega, & x_2 &= \frac{w_3}{\Omega}\sin\psi + \varsigma_2, \\
\varsigma_3 &= x_3 + v_2/\Omega, & x_3 &= -\frac{w_3}{\Omega}\cos\psi + \varsigma_3,
\end{aligned}
\tag{3}
$$

where $\psi(\varsigma_1, w_2) \equiv (\varsigma_1\Omega/w_1) + w_2$.

The connection formulae for the derivatives in the two reference systems are

$$\frac{\partial}{\partial v_1} \rightarrow \frac{\partial}{\partial w_1} + \frac{\varsigma_1\Omega}{w_1^2}\frac{\partial}{\partial w_2},$$

$$\frac{\partial}{\partial v_2} \rightarrow \left(-\frac{\sin\psi}{w_3}\frac{\partial}{\partial w_2} + \cos\psi\frac{\partial}{\partial w_3} + \frac{1}{\Omega}\frac{\partial}{\partial \varsigma_3}\right),$$

$$\frac{\partial}{\partial v_3} \rightarrow -\left(\frac{\cos\psi}{w_3}\frac{\partial}{\partial w_2} + \sin\psi\frac{\partial}{\partial w_3} + \frac{1}{\Omega}\frac{\partial}{\partial \varsigma_2}\right). \tag{4}$$

The inverse formulae are

$$\frac{\partial}{\partial w_1} \rightarrow -\frac{x_1}{v_1^2}\left(v_2\frac{\partial}{\partial x_2} - v_3\frac{\partial}{\partial x_3} + \Omega v_3\frac{\partial}{\partial v_2} - \Omega v_2\frac{\partial}{\partial v_3}\right) + \frac{\partial}{\partial v_1},$$

$$\frac{\partial}{\partial w_2} \rightarrow \frac{1}{\Omega}\left(v_2\frac{\partial}{\partial x_2} - v_3\frac{\partial}{\partial x_3} + \Omega v_3\frac{\partial}{\partial v_2} - \Omega v_2\frac{\partial}{\partial v_3}\right),$$

$$\frac{\partial}{\partial w_3} \rightarrow \frac{1}{w_3}\left(-\frac{v_3}{\Omega}\frac{\partial}{\partial x_2} - \frac{v_2}{\Omega}\frac{\partial}{\partial x_3} + v_2\frac{\partial}{\partial v_2} + v_3\frac{\partial}{\partial v_3}\right). \tag{5}$$

In the reference system (ς_i, w_i) the Boltzmann equation takes the form

$$\frac{\partial f}{\partial t} + w_1\frac{\partial f}{\partial \varsigma_1} + \left[(A_j e^{i\omega t} + c.c.)\frac{\partial f}{\partial v_j}\right] = [C(f)], \tag{6}$$

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where the brackets indicate that the quantities they embrace have to be taken in the new reference system.

Let us now formally develop f in the following way:

$$f(x_i, v_i, t) = f_o(x_i, v_i, t) + \sum_{n \neq 0} f_n(x_i, v_i, t) e^{in\omega t}, \tag{7}$$

where the time scale of f_o is small in comparison with $1/\omega$. By neglecting f_n with n larger than 2 one can write eq. (7) in the form

$$\frac{\partial f_o}{\partial t} + w_1 \frac{\partial f_o}{\partial \varsigma_1} + \left[A_j \frac{\partial f_{-1}}{\partial v_j} \right] + \left[A_j^* \frac{\partial f_1}{\partial v_j} \right] = [C(f_o)],$$

$$(i\omega + \nu) f_1 + \frac{\partial f_1}{\partial t} + w_1 \frac{\partial f_1}{\partial \varsigma_1} + \left[A_j \frac{\partial f_o}{\partial v_j} \right] = 0, \tag{8}$$

$$(-i\omega + \nu) f_{-1} + \frac{\partial f_{-1}}{\partial t} + w_1 \frac{\partial f_{-1}}{\partial \varsigma_1} + \left[A_j^* \frac{\partial f_o}{\partial v_j} \right] = 0,$$

where $C \approx -\nu f_{\pm 1}$ has been substitued in the equations for $f_{\pm 1}$ because $\nu \ll \omega$. Note that $f_{-1} = f_1^*$; consequently, we only need to consider the equation for f_1. The time derivative will be neglected with respect to $\omega f_{\pm 1}$; thus in the reference system of the unperturbed particle trajectories the equation for f_1 is immediately integrated and the solution has the following form for $v_1 > 0$:

$$f_1 = -e^{-i(\omega - i\epsilon)\varsigma_1/w_1} \int_{-\infty}^{\varsigma_1} e^{i(\omega - i\epsilon)\sigma/w_1} \left[\frac{A_j}{v_1} \frac{\partial}{\partial v_j} f_o \right] d\sigma \tag{9}$$

(the solution for $v_1 < 0$ is obtained by putting the lower limit of the integral equal to ∞). For the sake of brevity, an operator S is introduced for the RHS of eq. (9), which is written

$$f_1 = \frac{1}{w_1} S \left\{ \left[A_j \frac{\partial}{\partial v_j} f_o \right] \right\}. \tag{10}$$

The application of eqs. (4) (note that the electric field does not depend on x_2) now yields

$$w_1 f_1 = S \left\{ [A_1] \left(\frac{\partial}{\partial w_1} - \frac{\sigma \Omega}{w_1^2} \frac{\partial}{\partial w_2} \right) f_o \right\} -$$

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$$-S\left\{[A_3]\left(\frac{\cos\psi(\sigma,w_2)}{w_3}\frac{\partial}{\partial\theta}+\sin\psi(\sigma,w_2)\frac{\partial}{\partial w_3}\right)f_o\right\}. \tag{11}$$

The process described by f_o when f_1, given by eq. (11), is inserted in the first of eqs. (8) is diffusion in velocity space if the effect of the application of the integral operator S on $\partial f_o/\partial v_j$ is to multiply $\partial f_o/\partial v_j$ by some factor; in other words, the interaction of the particles with the electric field in the interval $(-\infty,\varsigma_1)$ (for $w_1>0$) approximately has to depend on the value of $\partial f_o/\partial v_j$ at the point ς_1, and should not appreciably depend on the previous interaction in the interval $(-\infty,\varsigma_1)$. In order to determine the conditions under which f_o describes diffusion in velocity space, and the corresponding diffusion coefficients, we integrate by parts in the operators $S\{\}$ of eq. (11):

$$w_1 f_1 = S\{[A_1]\}\frac{\partial f_o}{\partial w_1} + \frac{\Omega}{w_1^2}S\{\sigma[A_1]\}\frac{\partial f_o}{\partial w_2}-$$

$$-\frac{1}{w_3}S\{[A_3]\cos\psi\}\frac{\partial f_o}{\partial w_2} - S\{[A_3]\sin\psi\}\frac{\partial f_o}{\partial w_3}$$

$$+ \text{terms containing the derivative of } f_o \text{ and of } \frac{\partial f_o}{\partial w_1} \text{ with respect to } \sigma. \tag{12}$$

Equation (12) shows that the scale of variation (in the variable ς_1) of f_o is either the scale of $A_i S\left\{\left[A_j^*\right]\right\}$ (and similar terms), or the scale due to collisions. When collisions dominate, and Larmor radius effects can be neglected, $\partial f_o/\partial v_j$ can be taken out of the operator S, because the logaritmic derivative of $\partial f_o/\partial v_j$ with respect to ς_1 is sufficiently small, the solution f_o being approximately a Maxwellian. In this case the process is diffusion in velocity space and from the first of eqs. (8) and eq. (12) one can easily derive the diffusion coefficients. They are given explicitly later, in the form which takes into account Larmor radius effects. Also the change necessary in the discussion of eq. (12) when the effect of f_1 dominates over collisions and the distribution function f_o appreciably deviates from a Maxwellian is shown later. We now derive the ponderomotive effect.

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2. Ponderomotive effect

Before we proceed with the deduction of the diffusion coefficients, we deduce the effect of the electric field on density (the so-called ponderomotive effect) for frequencies in the lower hybrid (LH) range and in the small Larmor radius limit. The last assumption is not necessary for ions; the result is also correct outside this limit provided that $(\omega/k_\perp)$ is larger than the ion thermal velocity. The essential approximation will be that for most of the particles collisions dominate over the effect of the electric field, limited deviations from a Maxwellian being allowed, e.g. a 'plateau' in the velocity distribution such as is produced in LH experiments. The ponderomotive effect produces strong gradients of the density and amplitude of the electrostatic electric field, as was shown in [1] (see also references therein). As a consequence, it modifies the $k_\parallel$-spectrum injected into the plasma and is therefore essential for the comprehension of LH experiments.

The system of equations (8) is written in the reference system (ς_i, w_i) for f_o and for $f_{\pm 1}$; however, for the discussion of the present problem it is better, after integrating the equation for f_1 in the system (ς_i, w_i), to introduce cylindrical coordinates (v_1, w_3, ϕ) in velocity space. By using eqs. (4) and (5) and by noting that

$$-v_3 \frac{\partial}{\partial v_2} + v_2 \frac{\partial}{\partial v_3} \quad \rightarrow \quad -\frac{\partial}{\partial \phi},$$

$$v_2 \frac{\partial}{\partial v_2} + v_3 \frac{\partial}{\partial v_3} \quad \rightarrow \quad w_3 \frac{\partial}{\partial w_3} \tag{13}$$

one can write eq. (12) as

$$v_1 f_1 = \frac{\partial f_o}{\partial v_1} \left(S\{[A_1]\} \right) -$$

$$-\left(\frac{\cos\phi}{\Omega} \frac{\partial}{\partial x_2} + \frac{\sin\phi}{\Omega} \frac{\partial}{\partial x_3} + \frac{1}{w_3} \frac{\partial}{\partial \phi} \right) \cdot \left(f_o S\{\cos\psi(\sigma - x_1, \phi)[A_3]\} \right) -$$

$$-\frac{1}{w_3} \left(\frac{\sin\phi}{\Omega} \frac{\partial}{\partial x_2} - \frac{\cos\phi}{\Omega} \frac{\partial}{\partial x_3} + \frac{\partial}{\partial w_3} \right) \cdot \left(w_3 f_o S\{\sin\psi(\sigma - x_1, \phi)[A_3]\} \right). \tag{14}$$

By noting that nothing depends on x_2 and by collecting the terms proportional to the x_3 derivative one gets

$$v_1 f_1 = \frac{\partial f_o}{\partial v_1} \left(S\{[A_1]\} \right) - \frac{1}{w_3} \frac{\partial}{\partial \phi} \left(f_o S\{\cos\psi(\sigma - x_1, \phi)[A_3]\} \right) -$$

$$-\frac{1}{w_3} \frac{\partial}{\partial w_3} \left(w_3 f_o S\{\sin\psi(\sigma - x_1, \phi)[A_3]\} \right) + \frac{1}{\Omega} \frac{\partial}{\partial x_3} \left(f_o S\{\sin\psi(\sigma - x_1, 0)[A_3]\} \right). \quad (15)$$

Since $[A_i]$ do not depend on ϕ when Larmor radius effects are neglected and the distribution function f_o is approximately a Maxwellian, except possibly in velocity intervals whose contribution to the density is negligible, one gets

$$v_1 f_1 = \frac{\partial f_o}{\partial v_1} S\{[A_1]\} - \frac{\partial}{\partial w_3} \left(f_o S\{\sin\psi(\sigma - x_1, \phi)[A_3]\} \right) +$$

$$+ \frac{1}{\Omega} \frac{\partial}{\partial x_3} \left(f_o S\{\sin\psi(\sigma - x_1, 0)[A_3]\} \right). \quad (16)$$

Note that all the terms proportional to (w_3/Ω) which appear owing to the x_3 derivative have to be neglected; otherwise the derivative of ς_3 with respect to w_2, which is identically zero, and which in the new coordinates is given by

$$\left(\frac{\sin\phi}{\Omega} \frac{\partial}{\partial x_3} + \frac{1}{w_3} \frac{\partial}{\partial \phi} \right) \left(x_3 + \frac{w_3}{\Omega} \cos\phi \right),$$

would be different from zero.

The acceleration due to the electric field in the electrostatic limit with charge neutrality can be written as

$$A_1 \equiv e^{-ik_\parallel x_1} e^{-ik_\perp x_3} \alpha((x_1 - (k_\perp/k_\parallel)x_3)/l), \quad (17)$$

where $k_\perp/k_\parallel$ is approximately given by $\omega_{pe}/\omega\sqrt{\epsilon_{11}}$, with $\epsilon_{11} \equiv 1 + (\omega_{pe}^2/\Omega_e^2) - (\omega_{pi}^2/\omega^2)$; α is a real, bell–shaped function suitable for describing the 'resonance cone' structure of the LH field in the plasma; A_3 is equal to $A_1 k_\perp/k_\parallel$.

Since we are looking for a solution with a time scale smaller than that characterizing the

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energy deposition, the equation for f_o (the first of eqs. (8)) once written in cylindrical coordinates in velocity space takes the form

$$\left(v_1 + \frac{k_\perp}{k_\|} v_3\right) \frac{\partial f_o}{\partial x_1} + \left(\frac{\partial}{\partial v_1} + \frac{k_\perp}{k_\|} \frac{\partial}{\partial v_3}\right) (A_1^* f_1 + c.c.) = 0, \tag{18}$$

the collision term being identically zero for a Maxwellian. In order to deduce the ponderomotive effect we have to integrate over the velocity; hence the terms of eq. (16) proportional to the w_3 derivative of f_o disappear because they are proportional to $w_3 \cos\phi$ and to $w_3 \sin\phi$. The contribution of $(A_1^* f_1 + c.c.)$ to the ponderomotive effect is therefore

$$(A_1^* f_1 + c.c.) = -2\frac{\alpha}{v_1} \frac{\partial f_M}{\partial v_1} \int\limits_{-\infty}^{x_1} \cos(\Delta(\sigma - x_1)) \alpha((\sigma - (k_\perp/k_\|)x_3)/l) \, d\sigma -$$

$$-\frac{2\alpha}{\Omega v_1} \frac{k_\perp}{k_\|} f_M \frac{\partial}{\partial x_3} \int\limits_{-\infty}^{x_1} \cos(\Delta(\sigma - x_1)) \sin\psi(\sigma - x_1, 0)) \alpha((\sigma - (k_\perp/k_\|)x_3)/l) \, d\sigma, \tag{19}$$

where $\Delta \equiv (\omega/v_1) - k_\|$; thus $\Delta = 0$ gives the Cerenkov wave–particle resonance condition. Since the ponderomotive effect is a global effect in velocity space and not a resonance effect, for the frequencies under consideration the quantities $l\Delta$ e $l(\Delta \pm (\Omega/v_1))$ are much larger than one, both for ions and electrons; thus integration by parts of the integrals of eq. (19) yields

$$(A_1^* f_1 + c.c.) \approx -\frac{\partial \alpha^2}{\partial x_1} \left(\frac{v_1}{\omega^2} \frac{\partial f_M}{\partial v_1} + \frac{k_\perp^2}{k_\|^2} \frac{f_M}{\omega^2 - \Omega^2}\right). \tag{20}$$

Before introducing this expression in eq. (18), let us note that the solution to eq. (18) cannot be a Maxwellian in the velocity region $v_1 + \frac{k_\perp}{k_\|}v_3 \ll v_t$, whose contribution, however, can be neglected, since the ponderomotive effect is a global effect in velocity space. Outside this region one gets, by dividing the equation by v_1 and then by integrating over velocity and over x_1,

$$\frac{n}{n_o} = -\frac{\alpha^2}{v_t^2} \left(\frac{1}{\omega^2} + \frac{k_\perp^2}{k_\|^2} \frac{1}{\omega^2 - \Omega^2}\right) + \frac{2qV}{mv_t^2} \equiv N + \frac{2qV}{mv_t^2}, \tag{21}$$

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where the ambipolar potential has been added as last term. This equation is now written for ions and electrons, and V is determined by means of the neutrality condition; in this way one gets

$$\frac{n_e}{n_o} = \frac{T_e N_e + T_i N_i}{T_e + T_i}.$$ (22)

It follows that

$$-\frac{n_e}{n_o} = \frac{1}{T_e + T_i}(eE_1)^2 \left(\frac{1}{m_e\omega^2} - \frac{k_\perp^2}{k_\parallel^2}\frac{1}{m_e\Omega_e^2} + \frac{1}{m_i\omega^2} + \frac{k_\perp^2}{k_\parallel^2}\frac{1}{m_i\omega^2} \right).$$ (23)

The definition of $k_\perp/k_\parallel$ (see eq. (17)) now yields

$$-\frac{n_e}{n_o} = \frac{1}{m_e\omega^2\epsilon_{11}} \left(\epsilon_{11} - \frac{\omega_{pe}^2}{\Omega_e^2} + \frac{m_e}{m_i}\frac{\omega_{pe}^2}{\omega^2} + \frac{m_e}{m_i} \right)$$

$$\rightarrow \quad \frac{1}{\omega^2\epsilon_{11}} \left(\frac{1}{m_e} + \frac{1}{m_i} \right),$$ (24)

which is the classical result (see Ref. [3].) With our rather general derivation we have shown that the ponderomotive effect and a possible 'plateau' in velocity space, as is produced in LH experiments, can coexist.

3. Introduction of the diffusion coefficients

Let us come back to the deduction of the diffusion coefficients. When Larmor radius effects are not negligible, in particular for the ions, the scale of variation of f_o in ς_1, being of the order of the Larmor radius, is much smaller than the scale of variation of $[A_i]$. It is then usual to introduce some kind of average which cancels small–scale effects and allows introduction of diffusion coefficients; we take the mean over w_2. Let us introduce the function

$$g_o \equiv \frac{1}{2\pi} \int_0^{2\pi} f_o \, dw_2 \equiv \langle f_o \rangle. \tag{25}$$

Then from the first of eqs. (8) one gets the following equation for g_o when the terms which do not appear explicitly in eq. (12) can be neglected:

$$\frac{\partial g_o}{\partial t} + w_1 \frac{\partial g_o}{\partial \varsigma_1} + \frac{\partial}{\partial w_1} \langle [A_1^*] f_1 + c.c. \rangle -$$

$$-\left\langle \left(\frac{\cos \psi(\varsigma_1, w_2)}{w_3} \frac{\partial}{\partial \theta} + \sin \psi(\varsigma_1, w_2) \frac{\partial}{\partial w_3} \right) ([A_3^*] f_1 + c.c.) \right\rangle = \langle [C] \rangle, \tag{26}$$

which can also be written as

$$\frac{\partial g_o}{\partial t} + w_1 \frac{\partial g_o}{\partial \varsigma_1} + \frac{\partial}{\partial w_1} \langle [A_1^*] f_1 + c.c. \rangle -$$

$$- \frac{1}{w_3} \frac{\partial}{\partial w_3} \langle w_3 \sin \psi(\varsigma_1, w_2) ([A_3^*] f_1 + c.c.) \rangle = \langle [C] \rangle. \tag{27}$$

One can now substitute g_o for f_o in $\langle [A_i^*] f_1 \rangle$ if the mean value (over w_2) of the deviation of f_o from g_o is negligible, as we assume. Then eq. (12) simplifies to

$$w_1 f_1 \equiv S\{[A_1]\} \frac{\partial g_o}{\partial w_1} - S\{[A_3] \sin \psi\} \frac{\partial g_o}{\partial w_3} \tag{28}$$

and eq. (27) becomes

$$\frac{\partial g_o}{\partial t} + w_1 \frac{\partial g_o}{\partial \varsigma_1} + \frac{\partial}{\partial w_1} \langle [A_1^* f_1] \rangle - \frac{1}{w_3} \frac{\partial}{\partial w_3} (w_3 \langle [A_3^* f_1] \sin \psi + c.c. \rangle) = \langle [C] \rangle. \tag{29}$$

The appropriate approximation of the collision term in the electron equation, in the regime of LH electron current drive we consider here, is the linearized, high velocit limit (see, for example, Vedenov [4]) :

$$C(f) \equiv \nu v_t^3 \frac{\partial}{\partial v_i} \left[\frac{1}{|v|^3} \left(2v_i f + \left(|v|^2 \delta_{jk} - v_j v_k - v_t^2 \frac{|v|^2 \delta_{jk} - 3v_j v_k}{|v|^2} \right) \frac{\partial f}{\partial v_k} \right) \right] \qquad (30)$$

where $\nu = \omega_p^4 \ln \Lambda / 8\pi n v_t^3$. The term proportional to $(|v|^2 \delta_{jk} - v_j v_k)$ is the contribution of the electron–ion collisions. By averaging eq. (30) one gets

$$\langle [C] \rangle = \nu v_t^3 \frac{\partial}{\partial w_1} \left[\frac{1}{|w|^3} \left(2w_1 g_o + (w_3^2 - (v_t^2/2|w|^2)(|w|^2 - 3w_1^2)) \frac{\partial g_o}{\partial w_1} \right. \right. -$$

$$\left. \left. - w_1 w_3 (1 - (3v_t^2/2|w|^2) \frac{\partial g_o}{\partial w_3} \right) \right] +$$

$$+ \frac{1}{w_3} \frac{\partial}{\partial w_3} \left[\frac{1}{|w|^3} \left(2w_3^2 g_o - w_3^2 w_1 (1 - (3v_t^2/2|w|^2)) \frac{\partial g_o}{\partial w_1} + \right. \right.$$

$$\left. \left. + w_3 |w|^2 (1 - (v_t^2/2|w|^2)) \frac{\partial g_o}{\partial w_3} - w_3^3 (1 - (3v_t^2/2|w|^2)) \frac{\partial g_o}{\partial w_3} \right) \right], \qquad (31)$$

where a term proportional to the second derivative of g_o with respect to ς_3 has been neglected, because it is smaller than the others for a factor $v_t / l\Omega$. With the notations

$$D_{\nu i} \equiv \nu \frac{v_t^3}{|w|^3} \left(|w|^2 - w_i^2 - (v_t^2/2|w|^2)(|w|^2 - 3w_i^2) \right) \qquad (32)$$

and

$$D_{\nu m} \equiv \nu \frac{v_t^3}{|w|^3} w_1 w_3 (1 - (3v_t^2/2|w|^2) \qquad (33)$$

the averaged electron collision term can be written

$$\langle [C] \rangle = \frac{\partial}{\partial w_1} \left[D_{\nu 1} \frac{\partial g_o}{\partial w_1} + \frac{2\nu v_t^3}{|w|^3} w_1 g_o - D_{\nu m} \frac{\partial g_o}{\partial w_3} \right] +$$

$$+ \frac{1}{w_3} \frac{\partial}{\partial w_3} \left[w_3 D_{\nu 3} \frac{\partial g_o}{\partial w_3} + \frac{2\nu v_t^3}{|w|^3} w_3^2 g_o - w_3 D_{\nu m} \frac{\partial g_o}{\partial w_1} \right] \qquad (34)$$

The e-i collisions are the origin of the term proportional to $(|w|^2 - w_i^2)$ in $D_{\nu i}$ and of the term proportional to $w_1 w_3$ in $D_{\nu m}$.

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For the collision term in the ion equation we only need to keep the self–collision part of the linearized, high velocity approximation. It has therefore the same form as the electron term, where the e-i contributions are neglected.

The definition of the diffusion coefficients follows from eqs. (28) and (29); one gets:

$$-w_1 D_1 \equiv \langle ([A_1^*] S\{[A_1]\} + c.c.) \rangle,$$

$$w_1 D_3 \equiv \langle (\sin \psi(\varsigma_1, w_2) [A_3^*] S\{\sin \psi(\sigma, w_2) [A_3]\} + c.c.) \rangle, \tag{35}$$

$$w_1 D_{13} = w_3 \langle [A_1^*] S\{[A_3] \sin \psi\} \rangle,$$

$$w_1 D_{31} = w_3 \langle [A_3^*] \sin \psi S\{[A_1]\} \rangle.$$

If Larmor radius effects are neglected, the quantities $[A_i]$ do not depend on w_2 and thus D_{13} and D_{31} are zero.

The equation for g_o can now be written

$$\frac{\partial g_o}{\partial t} + w_1 \frac{\partial g_o}{\partial \varsigma_1} = \frac{\partial}{\partial w_1} \left[(D_1 + D_{\nu 1}) \frac{\partial g_o}{\partial w_1} + \frac{2\nu v_t^2}{|w|^3} w_1 g_o + (D_{13} - D_{\nu m}) \frac{\partial g_o}{\partial w_3} \right] +$$

$$+ \frac{1}{w_3} \frac{\partial}{\partial w_3} \left[w_3 (D_3 + D_{\nu 3}) \frac{\partial g_o}{\partial w_3} + \frac{2\nu v_t^2}{|w|^3} w_3^2 g_o + w_3 (D_{13} - D_{\nu m}) \frac{\partial g_o}{\partial w_1} \right]. \tag{36}$$

In accordance with what was stated when deducing eq. (12), these expressions are correct when $|D_i| \ll D_{\nu i}$. On the other hand, from their definitions it immediately follows that the diffusion coefficients are proportional to $1/w_1$ when $|w_1|$ goes to infinity, and that therefore for large velocity they no longer satisfy the condition $|D_i| \ll D_{\nu i}$, under which they were derived. The first two of equations (35) can also be written in the form

$$-w_1 D_1 \equiv \frac{1}{2\pi} \frac{\partial}{\partial \varsigma_1} \int_0^{2\pi} |S\{[A_1]\}|^2 \, dw_2,$$

$$w_1 D_3 \equiv \frac{k_\perp^2}{k_\parallel^2} \frac{1}{2\pi} \frac{\partial}{\partial \varsigma_1} \int_0^{2\pi} |S\{\sin \psi(\sigma, w_2) [A_1]_\sigma\}|^2 \, dw_2. \tag{37}$$

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Equations (35) and (37) show that the D_i are not positive definite. An unreasonable consequence, at first sight, of this property is that the absorbed power, which in this case $(g_o \approx f_M)$ is proportional to

$$\int \left(w_1^2 D_1 + w_3^2 D_3 \right) f_M \, dw_1 \, w_3 \, dw_3 , \tag{38}$$

can be negative. However, by using eqs. (37) it is easy to see that the integral over ς_1 from $-(|w_1|/w_1)\infty$ to ς_1 is always positive. The oscillations of the diffusion coefficients only cause oscillations of the absorbed power around a value which depends on ς_1, but is positive.

4. Explicit form of the diffusion coefficients

We give a more explicit form to the diffusion coefficients in the LH case. By using eqs. (3) one gets from eq. (17)

$$[A_1] \equiv \exp\left\{-ik_\parallel \varsigma_1\right\} \exp\left\{i(k_\perp w_3/\Omega)\cos(\varsigma_1 \Omega/w_1 + w_2)\right\} \alpha(\varsigma_1/l, \varsigma_3/l), \tag{39}$$

where it has been assumed that $w_3/\Omega \ll l$. First of all let us discuss the equation for D_1. By expanding the exponentials in eq. (39) in series of Bessel functions with argument $(k_\perp w_3/\Omega)$ one gets

$$e^{i\omega(\sigma-\varsigma_1)}\langle[A_1][A_1^*]\rangle = \alpha^2 e^{i\omega(\sigma-\varsigma_1)} \sum \int_0^{2\pi} (-1)^n i^{n+p} J_n J_p e^{i\Delta(\sigma-\varsigma_1)} e^{i(p\sigma-n\varsigma_1)\Omega/w_1} e^{i(p-n)w_2} \, d\theta$$

$$= \alpha^2 e^{i\omega(\sigma-\varsigma_1)} \sum_n e^{i\Delta_n(\sigma-\varsigma_1)} J_n^2(k_\perp w_3/\Omega) \qquad , \tag{40}$$

where $\Delta_n \equiv \Delta - n\Omega/w_1$; the condition $\Delta_n = 0$ gives the wave–particle resonance condition with Doppler effect.

With the definition of the operator S given by eq. (10) the first of eqs. (35) becomes

$$w_1 D_1 = 2\alpha \sum J_n^2 \int_{-\infty}^{\varsigma_1} \alpha \cos(\Delta_n(\sigma - \varsigma_1)) \, d\sigma. \tag{41}$$

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The limit $(k_\perp w_3/\Omega) = 0$ yields the onedimensional case, because $J_n = 0$ for $n \neq 0$ and $J_o(0) = 1$.

Asymptotic forms of D_1 in different intervals can easily be deduced. When $\varsigma_1 \to -\infty$ one has $\alpha'/\alpha \gg \Delta_n$, and thus integration by parts yields

$$w_1 D_1 \approx 2\alpha \sum_n J_n^2 \int_{-\infty}^{\varsigma_1} \alpha \, d\sigma \approx 2\alpha \int_{-\infty}^{\varsigma_1} \alpha \, d\sigma. \tag{42}$$

When $(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3) \ll l$ one gets

$$w_1 D_1 \approx \alpha(0) \sum_n J_n^2 \tilde{\alpha}(\Delta_n), \tag{43}$$

where $\tilde{\alpha}$ is the Fourier transform of $\alpha(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3)/l)$.

When $|\varsigma_1/l| \gg 1$, and if r is the value of n such that $l\Delta_r \ll 1$ while $l\Delta_n \gg 1$ for $n \neq r$, by partial integration of the terms $n \neq r$ one gets (note that $|\omega - \Omega|/w_1$ is in any case larger than ν/w_1)

$$w_1 D_1 = 2\alpha \left(\alpha' \sum_{n \neq r} \frac{J_n^2}{\Delta_n^2} + J_r^2 \int_{-\infty}^{\varsigma_1} \alpha \, d\sigma \right) \tag{44}$$

and, finally, for $\varsigma_1 \to \infty$ one has

$$w_1 D_1 \approx 2\alpha \sum_n J_n^2 \tilde{\alpha}(l\Delta_n) \cos(\Delta_n \varsigma_1). \tag{45}$$

An explicit form can be given to the first of eqs. (37) by introducing the quantity

$$H(\varsigma_1, \Delta_n) \equiv \int_{-\infty}^{\varsigma_1} \alpha e^{i\Delta_n \sigma} \, d\sigma, \tag{46}$$

whereby one can write

$$4w_1 D_1 = \sum_n J_n^2 \frac{\partial}{\partial \varsigma_1} |H(\varsigma_1, \Delta_n)|^2. \tag{47}$$

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This equation shows how D_1 is modified by the gradient of the electric field amplitude, a crucial point for the LH wave–plasma interaction picture; let α^2 be proportional to $1/l$, so that $\int_{-\infty}^{\infty} \alpha^2 \, d\sigma$ remains constant when l varies; then $|H|^2$ has the form $lF(\varsigma_1/l, l\Delta_n)$ and D_1 depends on the gradient through $l\Delta_n$.

Before discussing eq. (41), let us recall that when w_1 is sufficiently large D_1 is proportional to $1/w_1$; it would follow that the elettric field effect dominates over collisions when w_1 is sufficiently large, because collisions are proportional to $1/|w|^3$. This unacceptable result is due to the fact that for large velocity it is not permissible to neglect the ς_1 derivative of g_o. This problem will be discussed later. The same problem is also posed by the usual diffusion coefficient for a non–homogeneous plasma derived from that for a homogeneous plasma. Indeed, let α be constant; then from eq. (41) it follows that

$$D_1 = 2\pi\alpha^2 \sum J_n^2 \delta(w_1\Delta_n) \tag{48}$$

(for the usual derivation see, for example, [5]). With $\tilde{\alpha}(lk)$ for the Fourier transform of α, the diffusion coefficient is usually defined as the linear superposition of the diffusion coefficients corresponding to the different k–components of the electric field, although the diffusion coefficient (48) is quadratic in the field amplitude:

$$D_u = \frac{2\pi}{l} \sum J_n^2 \int |\tilde{\alpha}(k)|^2 \delta(w_1\Delta_n) \, dk. \tag{49}$$

Since $|\tilde{\alpha}(k)|^2$ in general is not such that the limit for w_1 going to infinity of $w_1^2|\tilde{\alpha}(\omega/w_1)|^2$ is zero, one can only deduce the lower limit v_m of the interval of w_1 where $D_u \gg D_{\nu 1}$ (i.e. the interval where the distribution function g_o has a 'plateau' in w_1, as we shall see). This difficulty can be avoided by choosing a particular form of $\tilde{\alpha}(k)$; a spectrum different from zero only in a finite $k_{\parallel}$ –interval (not containing the origin) would do it, but would otherwise correspond to an unsatisfactory electric field in real space. The spectrum corresponding to experimental situations is in fact the Fourier transform of

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a square profile in ς_1–space ('grill') rather than a square profile in $k_\parallel$–space. Let us consider the example

$$\alpha \equiv \frac{\bar{\alpha}}{\sqrt{l}}\exp\left\{-(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3)^2/l^2\right\},\tag{50}$$

which gives a gradient–independent (i.e. independent of l) total electrostatic energy. We get

$$w_1 D_u = 2\pi\bar{\alpha}^2 \sum_n J_n^2 e^{-(l\Delta_n/2)^2}\tag{51}$$

and for the diffusion coefficient given by eq. (41)

$$w_1 D_1 = \bar{\alpha}^2 \exp\left\{-2(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3)^2/l^2\right\} \sum_n J_n^2 Im\left\{Z((l\Delta_n/2) - i(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3)/l)\right\},\tag{52}$$

where Z is the 'plasma dispersion function'. Where $(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3) \ll l$, from the known properties of the Z function and eq. (49) we have

$$D_1 = D_u/2\sqrt{\pi}.\tag{53}$$

The gradient of the electric field amplitude modifies D_1 through $l\Delta_n$ at the exponent; an increase of the gradient moves v_m towards smaller velocity, thereby increasing the number of particles which interact with the electric field. The value of v_m for $(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3) \ll l$ is easily deduced from eq. (43) in the case of a sufficiently rapidly varying function α; in this case v_m is given by $l\Delta \approx 1$, i.e.

$$v_m \approx \frac{l\omega}{1 + lk}\tag{54}$$

Moreover, l also modifies the quotient $D_1/D_{\nu 1}$, i.e. it changes the relative importance of quasi–linear diffusion and collisions. Let us consider it for the example (52). When lk is smaller than $\sqrt{2}$ (large gradient) the diffusion coefficient is larger than D_1 at all $v > v_m$. When lk increases (small gradient, in particular $\alpha = constant$) D_1 is

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larger than $D_{\nu 1}$ in a region whose relative amplitude $(v_M - v_m)/v_m$ is proportional to $1/lk$; it then becomes smaller than $D_{\nu 1}$, and finally larger than $D_{\nu 1}$ when v is larger than approximately $\exp\left[l^2 k^2\right]$.

When $(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3) \leq l$, D_1 is very different from D_u; indeed, if one considers the case $n = 0$, which is the approximation of choice for the electrons, one gets, by assuming $l\Delta \gg 2$ in order to use the asymptotic formulae,

$$w_1 D_1 \approx \bar{\alpha}^2 \exp\left\{-2(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3)^2/l^2\right\} \frac{2(\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3)/l}{(l\Delta/2)^2 + (\varsigma_1 - (k_\perp/k_\parallel)\varsigma_3)^2/l^2}. \tag{55}$$

It follows that D_1 can be much larger than D_u, and thus v_m for D_1 can be smaller than the value of v_m for D_u. Leaving the resonance condition ($\Delta = 0$) has a much less drastic effect on D_1 than on D_u. Whereas for electrons $n = 0$ is the correct approximation, for ions one also has to consider values of n different from zero. Let r be the value of n which minimizes the difference $\omega - n\Omega$ (and hence minimizes Δ_n, which is zero for wave–particle resonance); then $v_m \equiv l(\omega - r\Omega)/(1 + lk)$ is the smallest of all v_m and is the required value. Note that $J_r^2(k_\perp w_3/\Omega)$ ($r \neq 0$) is proportional to $(k_\perp w_3/\Omega)^{2r}$ when the argument is much less than r, rapidly reaches the maximum ($\approx 1/r$) when $(k_\perp w_3/\Omega) \approx r$, and then decreases as $\Omega/k_\perp w_3$.

We now consider the equation for D_3; a Bessel functions expansion gives

$$w_1 D_3 = -\frac{k_\perp^2}{k_\parallel^2} \frac{1}{8\pi} \int_0^{2\pi} (-1)^n i^{n+p} J_n J_p e^{i\Delta(\sigma - \varsigma_1)} e^{i(p\sigma - n\varsigma_1)\Omega/w_1} e^{i(p-n)w_2}.$$

$$\cdot\left(e^{i(\sigma + \varsigma_1)\Omega/w_1} e^{2iw_2} + e^{-i(\sigma + \varsigma_1)\Omega/w_1} e^{-2iw_2} - e^{i(\varsigma_1 - \sigma)\Omega/w_1} - e^{-i(\varsigma_1 - \sigma)\Omega/w_1}\right) dw_2 + c.c. \tag{56}$$

After integration over w_2 only the terms $n - p - 2 = 0$, $n - p + 2 = 0$ and $n - p = 0$ remain and eq. (56) gives

$$4w_1 D_3 = \frac{k_\perp^2}{k_\parallel^2} 2\alpha \sum J_n(J_n + J_{n+2}) \int_{-\infty}^{\varsigma_1} \alpha \cos(\Delta_{n+1}(\sigma - \varsigma_1))\, d\sigma +$$

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$$+\frac{k_\perp^2}{k_\parallel^2}2\alpha\sum J_n(J_n+J_{n-2})\int_{-\infty}^{\varsigma_1}\alpha\cos(\Delta_{n-1}(\sigma-\varsigma_1)))\,d\sigma. \tag{57}$$

Let us put $r\equiv n+1$ in the first series and $r\equiv n-1$ in the second one; the r–terms of the two series can be summed up to give $(J_{r-1}+J_{r+1})^2\cos(\Delta_r(\sigma-\varsigma_1))$, so that eq. (57) becomes

$$w_1 D_3=\frac{\Omega^2}{k_\parallel^2 w_3^2}2\alpha\sum_n n^2 J_n^2\int_{-\infty}^{\varsigma_1}\alpha\cos(\Delta_n(\sigma-\varsigma_1)))\,d\sigma. \tag{58}$$

From now on the discussion is similar to that for D_1. In particular, eq. (47) is replaced by

$$w_1 D_3=2\frac{\Omega^2}{k_\parallel^2 w_3^2}\sum_n n^2 J_n^2\frac{\partial}{\partial\varsigma_1}|H(\varsigma_1,\Delta_n)|^2. \tag{59}$$

The conclusions are however different; in the case of the electrons the values of n of importance are now $n=\pm1$ (owing to the presence of the functions nJ_n instead of the functions J_n), for which v_m is a factor Ω/ω larger than the values given in eq. (55); hence the effect of D_3 on the electrons can be neglected. On the other hand, for the ions the presence of the factor $k_\perp^2/k_\parallel^2$ makes the effect of D_1 negligible in relation to the effect of D_3.

It is not difficult to see that eq. (35) gives $D_{13}=D_{31}$, and that

$$w_1 D_{13}=2\frac{\Omega}{k_\parallel w_3}\alpha\sum_n n J_n^2\int_{-\infty}^{\varsigma_1}\alpha\cos(\Delta_n(\sigma-\varsigma_1)))\,d\sigma, \tag{60}$$

which is zero when $(k_\perp w_3/\Omega)$ is zero.

5. Quasi–linear diffusion coefficients dominate over collisions

Let us consider the intervals of w_i and ς_1 where the effect of g_1 dominates over collisions and thus the distribution function g_o deviates from a Maxwellian. Let us first of all consider the case of an almost Maxwellian distribution function in the w_3 variable, whilst $D_1 \gg D_{\nu 1}$ (the electron case with $n = 0$). The equation for g_o (see eq. (36)) then reduces to

$$
w_1 \frac{\partial g_o}{\partial \varsigma_1} - \frac{\partial}{\partial w_1} \left[(D_1 + D_{\nu 1}) \frac{\partial g_o}{\partial w_1} + \frac{2 \nu v_t^3}{|w|^3} w_1 g_o - D_{\nu m} \frac{\partial g_o}{\partial w_3} \right] =
$$
$$
= \frac{1}{w_3} \frac{\partial}{\partial w_3} \left[w_3 (D_3 + D_{\nu 3}) \frac{\partial g_o}{\partial w_3} + \frac{2 \nu v_t^3}{|w|^3} w_3^2 g_o - w_3 D_{\nu m} \frac{\partial g_o}{\partial w_1} \right], \qquad (61)
$$

D_{13} being zero for $n = 0$; here the w_3–derivative of g_o has to be replaced by $-(2 w_3 / v_t^2) g_o$. Since the RHS is negligible, and if the ς_1–derivative can be neglected, one immediately gets

$$
\ln g_o \approx -\frac{w_3^2}{v_t^2} - 2 \int_0^{} \frac{(w_1 \nu v_t^3 / |w|^3) + D_{\nu m} w_3 / v_t^2}{D_1 + D_{\nu 1} w_1} \, dw_1 , \qquad (62)
$$

which, by taking into account the definition of $D_{\nu m}$, can also be written

$$
\ln g_o \approx -\frac{w_3^2}{v_t^2} - \frac{2}{v_t^2} \int_0^{} \frac{D_{\nu 1} w_1}{D_1 + D_{\nu 1} w_1} \, dw_1 . \qquad (63)
$$

This equation exhibits a dependence of g_o on w_1 weaker than that of a Maxwellian, the so–called 'plateau', for w_1 larger than v_m. The condition for the ς_1–derivative of g_o to be negligible (a necessary condition in order that the terms not explicitly written in eq. (12) be negligible, i.e. in order that diffusion coefficients can be defined) is

$$
\left| w_1 \frac{\partial g_o}{\partial \varsigma_1} \right| \ll \left| \frac{\partial}{\partial w_1} \left(g_o 2 D_{\nu 1} w_1 / v_t^2 \right) \right| \approx \frac{4 D_{\nu 1}}{v_t^2} . \qquad (64)
$$

Since from eq. (62) it follows that the value of the ς_1–derivative increases with velocity, the restriction imposed by eq. (64) is such that eq. (63) describes, strictly speaking,

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only the beginning of the 'plateau '.

The 'plateau' in w_3, the ion case, is analogous to the previous one. The approximate distribution function is given by

$$\ln g_o \approx -\frac{w_1^2}{v_t^2} - \frac{2}{v_t^2} \int\limits_{w_m} \frac{D_{\nu 3}w_3 - D_{13}w_1}{D_3 + D_{\nu 3}w_3} \, dw_3 . \tag{65}$$

The beginning of the 'plateau' is determined by $D_3 \approx D_{\nu 3}$, because $D_{13}w_1/w_3 \ll D_{\nu 3}$.

Let us see in detail the consequence of eqs. (62) and (65) on the derivation of the diffusion coefficients. Equation (62) shows that the logarithmic derivative of $\partial g_o / \partial w_1$ with respect to ς_1, which is $\left(\frac{\partial g_o / \partial \varsigma_1}{g_o} - \frac{D_1'}{D_1} \right)$, is of the order of D_1'/D_1, i.e. not much smaller than , but of the order of the logarithmic derivative of $|A_1|$ (i.e. of the order of $1/l$). Hence we cannot neglect the terms of eq. (12) which have not been explicitly written.

On the other hand, where $D_1 \gg D_{\nu 1}$ the logarithmic derivative of $D_1 \partial g_o / \partial w_1$ with respect to ς_1 is of the order of $\partial \ln g_o / \partial \varsigma_1$, which is much smaller than $1/l$, as we shall see at the end of the next section. We conclude that where $D_1 \gg D_{\nu 1}$ the quantity which varies slowly with respect to $|A_1|$ is $(D_1 + D_{\nu 1}w_1)(\partial g_o / \partial w_1)$ and not $(\partial g_o / \partial w_1)$. Since a similar argument is valid also for D_3, instead of eq. (35) we can write for every value of D_i (we do not write the equation for D_{13})

$$-w_1 D_1(\varsigma_1) \equiv (D_1(\varsigma_1) + D_{\nu 1}) \frac{1}{2\pi} \int\limits_0^{2\pi} \left([A_1^*] \, S \left\{ \frac{1}{D_1(\sigma) + D_{\nu 1}} [A_1] \right\} + c.c. \right) dw_2,$$

$$w_1 D_3(\varsigma_1) \equiv (D_3(\varsigma_1) + D_{\nu 3}) \frac{k_\perp^2}{k_\parallel^2} \cdot$$

$$\cdot \frac{1}{2\pi} \int\limits_0^{2\pi} \left(\sin \psi(\varsigma_1, w_2) [A_1^*] \, S \left\{ \frac{1}{D_3(\sigma) + D_{\nu 3}} \sin \psi(\sigma, w_2) [A_1] \right\} + c.c. \right) dw_2 . \tag{66}$$

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With eqs. (40) and (57), eqs. (66) can be written

$$w_1 D_1(\varsigma_1) = 2(D_1(\varsigma_1) + D_{\nu 1})\alpha(\varsigma_1) \sum_n J_n^2 \int_{-\infty}^{\varsigma_1} \frac{\alpha(\sigma)}{D_1(\sigma) + D_{\nu 1}} \cos(\Delta_n(\sigma - \varsigma_1))\, d\sigma,$$

$$w_1 D_3(\varsigma_1) = \frac{\Omega^2}{k_\parallel^2 w_3^2} 2(D_3(\varsigma_1) + D_{\nu 3}\alpha(\varsigma_1)) \sum_n n^2 J_n^2 \int_{-\infty}^{\varsigma_1} \frac{\alpha(\sigma)}{D_3(\sigma) + D_{\nu 3}} \cos(\Delta_n(\sigma - \varsigma_1))\, d\sigma \,.$$

$$(67)$$

The latter two equations can both be written as

$$1 - h_i(\varsigma_1) = \alpha \int_{-\infty}^{\varsigma_1} \alpha(\sigma) K_i(\sigma - \varsigma_1) h_i\, d\sigma, \tag{68}$$

where $h_i \equiv D_{\nu i}/D_i + D_{\nu i}\varsigma_1$ and K_i are the corresponding kernels. In the interval where h_i varies slowly with respect to α and K_i integration by parts yields

$$1 - h_i(\varsigma_1) = \alpha h_i(\varsigma_1) \int_{-\infty}^{\varsigma_1} \alpha(\sigma)\, K_i(\sigma - \varsigma_1)\, d\sigma, \tag{69}$$

or

$$h_i(\varsigma_1) = \left[1 + \alpha \int_{-\infty}^{\varsigma_1} \alpha(\sigma) K_i(\sigma - \varsigma_1)\, d\sigma \right]^{-1}. \tag{70}$$

Owing to the definition of h_i one gets

$$\frac{D_i}{D_{\nu i}} = \alpha \int_{-\infty}^{\varsigma_1} \alpha(\sigma)\, K_i(\sigma - \varsigma_1)\, d\sigma, \tag{71}$$

which are the same formulae as eq. (41) and eq. (58).

Equation (70) shows that the variation of h_i cannot be neglected in the neighbourhood of the zeros of the denominator, i.e. in the neighbourhood of the zeros of $D_i + D_{\nu i}\varsigma_1$. The appropriate approximation of eq. (67) is then

$$1 - h_i(\varsigma_1) = \alpha \int_{-\infty}^{x_o} \alpha(\sigma) K_i(\sigma - \varsigma_1) h_i(\sigma)\, d\sigma + \alpha^2 K_i(0) \int_{x_o}^{\varsigma_1} h_i(\sigma)\, d\sigma, \tag{72}$$

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where x_o is taken as the limit of validity of eq. (70), and in the first term of the RHS one can take eq. (70) for h_i. Equation (72) is easily integrated and it is not difficult to see that if x_o is such that $1 - \alpha \int_{-\infty}^{x_o} \alpha(\sigma) K_i(\sigma - \varsigma_1) h_i(\sigma) \, d\sigma$ is larger than zero the solution to eq. (72) is positive and finite, so that the total diffusion coefficients $D_i + D_{\nu i} \varsigma_1$ are positive definite.

6. Plateau width and large velocity region

In order to deduce a better approximation to g_o for velocities larger than v_m, in the case where the distribution function is almost a Maxwellian in the w_3 variable but D_1 can become larger than $D_{\nu 1}$ (the electron case with $n = 0$), let us write the equation corresponding to eq. (61) for the logarithm of g_o. By taking into account the definition of $D_{\nu m}$ (see also the derivation of eq. (63)) and by neglecting the RHS (which is a correct approximation, as is not difficult to see) we get

$$w_1 \frac{\partial \beta}{\partial \varsigma_1} = \frac{\partial \beta}{\partial w_1} F + \frac{\partial}{\partial w_1} F, \tag{73}$$

where

$$F \equiv (D_1 + D_{\nu 1} w_1) \frac{\partial \beta}{\partial w_1} + 2 D_{\nu 1} \frac{w_1}{v_t^2}. \tag{74}$$

Equation (73) can be formally considered as a first order linear equation for F, which has the following solution for positive w_1

$$F = e^{-\beta} \int_{\infty} v \frac{\partial \beta}{\partial \varsigma_1} e^{\beta} \, dv \tag{75}$$

Since we are considering velocities much larger than thermal velocity and β is negative, eq. (75) can be approximated by

$$F \approx w_1 \frac{\partial \beta}{\partial \varsigma_1} \Big/ \frac{\partial \beta}{\partial w_1}, \tag{76}$$

a result which, together with eq. (74), yields

$$\left[(D_1 + D_{\nu 1} w_1) \frac{\partial \beta}{\partial w_1} + 2 D_\nu \frac{w_1}{v_t} \right] \frac{\partial \beta}{\partial w_1} = w_1 \frac{\partial \beta}{\partial \varsigma_1}. \tag{77}$$

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It the highest order one has $\frac{\partial \beta}{\partial w_1} = -2w_1 D_{\nu 1}/v_t^2(D_1 + D_{\nu 1}w_1)$, whereas the square bracket is equal to zero; it the next order one gets the following approximate equation for β :

$$(D_1 + D_{\nu 1}w_1)\frac{\partial \beta}{\partial w_1} + \frac{v_t^2}{2}\frac{D_1 + D_{\nu 1}w_1}{D_{\nu 1}}\frac{\partial \beta}{\partial \varsigma_1} = -2D_{\nu 1}\frac{w_1}{v_t^2} \tag{78}$$

When the ς_1–derivative can be neglected, one again gets solution (63). Equation (78) shows that at large velocity β (and therefore g_o) obeys a first–order differential equation whose characteristics are given by the equation

$$\frac{d\varsigma_1}{dw_1} = \frac{v_t^2}{2D_{\nu 1}}. \tag{79}$$

The reference system of the characteristics is then given by the equations

$$v = w_1 \quad , \qquad y = \varsigma_1 - \frac{v_t^2}{2}\int_0^{w_1}\frac{dv'}{D_{\nu 1}(v')} \tag{80}$$

with the inverse relation

$$\varsigma_1 = \frac{v_t^2}{2}\int_0^{v}\frac{dv'}{D_{\nu 1}(v')} + y. \tag{81}$$

It allows one to write the solution of eq. (78) in the form (compare with eq. (63))

$$\beta = -\frac{2}{v_t^2}\int_0^{w_1}\frac{vD_{\nu 1}(v)}{D_1(\varsigma_1 + (v_t^2/2)\int_{w_1}^{v}(1/D_{\nu 1}(v'))\,dv',\ v) + D_{\nu 1}}\,dv. \tag{82}$$

Equation (82) describes the logarithm of a Maxwellian up to the value of w_1 where $D_1 \approx D_{\nu 1}$. Then it describes a (more or less flat) 'plateau', whose width can be deduced by the following argument. We have previously remarked that $w_1 D_1(\varsigma_1, w_1)$ is approximately a constant for velocities larger than wl, and that therefore D_1 is larger than $D_{\nu 1}$ for sufficiently large velocity. In eq. (82), however, the velocity also appears in the ς_1–dependence, so that vD_1 decreases exponentially when $(v_t^2/2)\int_{v_m}^{v}(1/D_{\nu 1})\,dv'$ is larger than l. In other words, D_1 in the integrand of eq. (82) can become smaller

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than $D_{\nu 1}$, and therefore β is approximately equal to w_1^2, for two reasons: the first is a change of the order

$$\Delta v \approx \omega / l k^2 \tag{83}$$

of the w_1 variable of D_1; it is comparable with the quantity $(v_M - v_m)$ introduced in Sect. 4. The second reason is a change of the order l of the ς_1–dependence, which is induced, owing to the characteristics, by a change in velocity given by

$$\delta v \approx \frac{2l}{v_t^2} D_{\nu 1}. \tag{84}$$

In the one–dimensional case eq. (83) has the more appealing form $(\delta v / v_t) \approx (l / \lambda(v_m))$, where λ is the free path of a particle at $w_1 = v_m$. The smaller of the two quantities $(\delta v, \Delta v)$ determines the width of the 'plateau'; for $l^2 k^2 > (\omega^4 / k^3 \nu v_t^3)$, which is a quantity much larger than unity, one has $\delta v < \Delta v$. In this situation (spatial gradient of α small with respect to wavelength and mean free path, in particular $\alpha = const.$) eq. (80) is valid in (v_m, v_M), and the 'plateau' extends over the whole region (v_m, v_M). When $l^2 k^2 < (\omega^4 / k^3 \nu v_t^3)$ (spatial gradient large with respect to wavelength and mean free path), the width of the 'plateau' is smaller than the preceding one; in particular, it is smaller than $(v_M - v_m)$ (in the case where a region (v_m, v_M) exists) and is proportional to the collisions. It is recalled that when l decreases the 'plateau' moves towards lower velocity (see eq. (55)).

When D_1 is smaller than $D_{\nu 1}$, eq. (82) is approximately

$$\beta \approx -\frac{2}{v_t^2} \int\limits_0^{w_1} v \left(1 - \frac{D_1}{D_{\nu 1}} \right) dv , \tag{85}$$

which, by using eq. (79), can be written

$$\beta \approx -\frac{w_1^2}{v_t^2} + \frac{4}{v_t^4} \int\limits_{-\infty}^{\varsigma_1} v D_1 \, dx. \tag{86}$$

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Whereas for small velocity the quantity $w_1 D_1$ goes to zero (see eq. (41) and example (52)), so that g_o is a Maxwellian, for large w_1 it can be written as $w_1 D_1 = a_o(\varsigma_1, w_3) + a_1(\varsigma_1, w_3)/w_1 + \ldots$. The quantity $\int_{-\infty}^{\infty} (a_o/v_t^4)\, dx$ is much smaller than unity, because it is of the order of magnitude of $(q\Phi/mv_t^2)^2$, where Φ is the electrostatic potential associated with the amplitude of the waves. In conclusion, the function g_o is, for large velocity, a Maxwell–Boltzmann distribution with the electrostatic potential replaced by $4 \int_{-\infty}^{\infty} (v D_1/v_t^4)\, dx$. Moreover, $\ln g_o$ is such that $l\frac{\partial \beta}{\partial \varsigma_1} \ll 1$, thereby allowing the introduction of diffusion coefficients (see Sect. 5); it is obviously sufficient to show it where $D_1 \gg D_{\nu 1}$. In this case from eq. (82) it follows that

$$l\left|\frac{\partial \beta}{\partial \varsigma_1}\right| \approx \frac{2}{v_t^2} \frac{w_1 D_{\nu 1}}{D_1}\, (\delta v \text{ or } \Delta v), \tag{87}$$

which is much less than one, as can be seen by using eq. (83) or eq. (84) for δv or Δv, depending on the value of lk.

Conclusions

One of the purposes of the paper was to investigate the possibility of describing the effect of the waves on the plasma in LH heating and current drive experiments as induced diffusion in velocity space, also when the RF field is limited in space ('resonance cone') and therefore the field amplitude gradients should not be neglected (which, however, is done in the usual approximation, where the QL DC is constant in space). We established the condition which has to be satisfied in order to define a QL DC. A QL DC was then derived for the regions in phase space where it is smaller than the collision–induced DC; the usual approximation for the QL DC coincides with our result where the RF field amplitude is almost constant. When the QL DC becomes larger than the collision–induced DC, the previous definition is no longer valid, essentially owing to the space dependence of the diffusion. The correct, positive definite, form of the QL DC in such a region (where the distribution function becomes flattened to form a 'plateau')

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was derived. An interesting theoretical problem posed by the QL DC (also by the usual approximation, except when unphysical assumptions are made) is that it does not become negligible in relation to the collisions for large velocities—a fact that, at first sight, seems to extend the 'plateau' to infinity. We have shown that in general the width of the 'plateau' is determined by the space dependence of the field amplitude and the collisions, and not by the width in velocity space of the QL DC. The general expression for the distribution function in the 'plateau' region and approximations for the 'plateau' width (an essential quantity in the theoretical description of LH experiments) have been given. We have also shown that in the limit of large velocities the distribution function is a Maxwell–Boltzmann–like distribution, where the electrostatic potential is replaced by an integral over the velocity of the QL DC.

As was shown in Ref. [2], the ponderomotive effect plays a substantial role in LH experiments by causing large gradients of the electrostatic field amplitude which modify the QL DC, thereby strongly enhancing power absorption. For this reason special care has been devoted to deriving the ponderomotive effect, allowance being made for distortions of the distribution function from a Maxwellian.

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