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IPP 6/283

October 1989

*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem
Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die
Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

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October 30, 1989

Abstract

A sufficient stability condition with respect to purely growing modes is derived for resistive MHD. Though it may be, in general, violated, its ability to reduce in the appropriate limits to known necessary and sufficient stability conditions makes it instructive and conceptually useful.

In a previous note [1] the author derived a sufficient condition for the stability of purely growing modes, valid for general dissipative systems and general geometries. This condition is applied here for resistive MHD equilibria. These equilibria generally have a flow which, for simplicity, we neglect in the equation of motion, but which we keep in Ohm's law. It will be seen later that the inclusion of the flow in the momentum equation could be taken into account but leads to cumbersome contributions which vanish with resistivity. The equilibrium equations are given by

$$\mathbf{J} \times \mathbf{B} = \nabla P_0 \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta_0 \mathbf{J} \quad (3)$$

As usual \mathbf{B} is the magnetic field, $\mathbf{J} = \nabla \times \mathbf{B}$, \mathbf{E} is the curl-free electric field, \mathbf{V} is the flow velocity due to resistivity η_0 and P_0 is the pressure. The

"existence" of magnetic surfaces is assumed and the resistivity is taken as constant on these surfaces. The equations of the linearized perturbations are

$$\rho \ddot{\xi} + \nabla P_1 - \mathbf{j} \times \mathbf{B} - \mathbf{J} \times \mathbf{b} = 0 \quad (4)$$

$$\mathbf{e} + \dot{\xi} \times \mathbf{B} + \mathbf{V} \times \mathbf{b} - \eta_1 \mathbf{J} - \eta_0 \mathbf{j} = 0 \quad (5)$$

$$\nabla \times \mathbf{e} = -\dot{\mathbf{b}} \quad (6)$$

$$\nabla \cdot \mathbf{b} = 0 \quad (7)$$

$$\mathbf{j} = \nabla \times \mathbf{b} \quad (8)$$

$$\mathbf{B} \cdot \nabla \eta_1 + \mathbf{b} \cdot \nabla \eta_0 = 0 \quad (9)$$

$$P_1 = -\gamma P_0 \nabla \cdot \xi - \xi \cdot \nabla P_0 \quad (10)$$

where ρ is the mass density, P_1 , \mathbf{j} , \mathbf{b} , \mathbf{e} and η_1 are the perturbations of, respectively, pressure, current, magnetic field, electric field and resistivity. The boundary conditions are $\mathbf{n} \cdot \mathbf{b} = \mathbf{n} \cdot \xi = 0$, where \mathbf{n} is the normal to a perfectly conducting wall.

Let us express \mathbf{e} and \mathbf{b} in terms of the vector potential \mathbf{A} and take the gauge of zero scalar potential,

$$\begin{aligned} \mathbf{e} &= -\dot{\mathbf{A}} \\ \mathbf{b} &= \nabla \times \mathbf{A} \end{aligned}$$

with the boundary condition $\mathbf{n} \times \mathbf{A} = 0$. we insert \mathbf{j} from eq.(5) into eq.(4) to obtain a system written in terms of $\Psi = \begin{pmatrix} \xi \\ \mathbf{A} \end{pmatrix}$

$$N \ddot{\Psi} + P \dot{\Psi} + Q \Psi = 0 \quad (11)$$

where N , P and Q are given, respectively, by

$$\begin{aligned} N &= \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} \\ P &= \begin{pmatrix} \mathbf{B}/\eta_0 \times (\cdots \times \mathbf{B}) & (\cdots \times \mathbf{B}/\eta_0) \\ -(\cdots \times \mathbf{B}/\eta_0) & 1/\eta_0 \end{pmatrix} \end{aligned}$$

$$\text{and } Q = \begin{pmatrix} \nabla(-\gamma P_0(\nabla \cdot \dots)) & -\mathbf{J} \times (\nabla \times \dots) \\ -\nabla(\dots \cdot \nabla P_0) & -1/\eta_0 \nabla P_0 (\mathbf{B} \cdot \nabla)^{-1} (\nabla \times \dots \cdot \nabla \eta_0) \\ & + \mathbf{B}/\eta_0 \times (\mathbf{V} \times \nabla \times \dots) \\ 0 & \nabla \times \nabla \dots \\ & + \mathbf{J}/\eta_0 (\mathbf{B} \cdot \nabla)^{-1} (\nabla \times \dots \cdot \nabla \eta_0) \\ & - \mathbf{V}/\eta_0 \times \nabla \times \dots \end{pmatrix}$$

The first two matrix operators are symmetric and positive. The last operator Q is obviously not selfadjoint. For this reason we cannot find a Lyapunov functional which would lead to a necessary and sufficient condition for stability as in [2] or [3] for example.

As shown in [1] one can, however, write a sufficient condition for stability against purely growing modes in the form

$$\delta W = (\Psi, Q\Psi) \geq 0 \quad (12)$$

where the scalar product is defined with purely real quantities. Only the symmetric part Q_S of Q survives in eq.(12), but if a symmetrized form for eq.(12) is wanted, it is easy to construct Q^+ , the adjoint of Q , by integration by parts, and use $Q_S = (Q + Q^+)/2$ instead of Q in eq.(12).

Criterion (12) implies volume integrations which can be reduced to integrations in the magnetic surfaces and integrations across them. The operator $(\mathbf{B} \cdot \nabla)^{-1}$ in eq.(12), which comes from integration of eq.(9), is singular across the rational surfaces ($1/x$ singularity). This singularity is physically prohibited by the breakdown of eq.(9) due to a finite heat conduction κ ($\kappa \rightarrow \infty$ is assumed to be infinite for eq.(9)). In fact η_1 should not become infinite on the rational magnetic surfaces, but small. It is then natural to define the integrations across the surfaces in the sense of Cauchy principal parts (no delta functions) as in [3]. Note here that these singularities are not aggravated by the above-mentioned symmetrizing integrations by parts, because they occur in the surfaces.

Let us now write δW explicitly:

$$\begin{aligned} \delta W = & \int d\tau (\gamma P_0 (\nabla \cdot \xi)^2 + (\xi \cdot \nabla P_0) \nabla \cdot \xi) \\ & + \int d\tau (\nabla \times \mathbf{A})^2 - \int d\tau \xi \times \mathbf{J} \cdot \nabla \times \mathbf{A} + \end{aligned}$$

$$\begin{aligned}
& + p \cdot p \cdot \int d\tau \mathbf{J} \cdot (\mathbf{A} - \xi \times \mathbf{B}) (\mathbf{B} \cdot \nabla)^{-1} (1/\eta_0) (\nabla \eta_0 \cdot \nabla \times \mathbf{A}) \\
& - \int d\tau (\mathbf{A} - \xi \times \mathbf{B}) \cdot \mathbf{V} \times (\nabla \times \mathbf{A}) 1/\eta_0
\end{aligned} \tag{13}$$

If we choose in δW the MHD test function $\mathbf{A} = \xi \times \mathbf{B}$, then δW reduces to δW_{MHD} . In the tokamak scaling (large axial wavelength and magnetic fields) and for $\mathbf{J} = \mathbf{e}_z J$, $\eta_0 J = ct$, $\xi = \mathbf{e}_z \times \nabla U$, $\mathbf{V} = 0$, δW reduces to the necessary and sufficient condition found in [3].

It is more convenient to treat δW in Hamada-like coordinates especially for the term $(\mathbf{B} \cdot \nabla)^{-1}$, which also appears in [3]. The symmetrization of Q , if desired, can be done either analytically in the same coordinates by integration by parts or after discretization in the case of numerical evaluation by computing the adjoint matrix.

The equilibrium quantities in eq.(13) should satisfy equations (1)-(3). To determine the contribution of the last integral in eq.(13), one requires a knowledge of unavoidable [4] Pfirsch-Schlüter-like flows, which are important especially for stellarators. The flow in a tokamak can probably be neglected if the aspect ratio is large enough and the poloidal currents are weak. One can then take $\nabla \times \eta_0 \mathbf{J} \approx 0$ as in [3]. Unfortunately the positiveness of δW with respect to all

$$\Psi = \begin{pmatrix} \xi \\ \mathbf{A} \end{pmatrix}$$

of L^2 with the boundary conditions at the wall cannot be fulfilled, in general.

Finally, let us note that the addition of resistivity-driven flow terms in the equation of motion would not alter the structure of the equations, so that a sufficient condition can also be derived as explained in [1]. Contrary to the flow term in eq.(13), which does not vanish with resistivity, all other contributions to δW coming from the flow are of higher order in resistivity. Similarly the addition of the viscosity tensor to eq.(4) would not affect the sufficient condition. More sophisticated dynamics such as 2-fluid theory could also be treated according to [1].

References

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