

Radiofrequency Plasma Heating
and Current Drive

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9. RADIOFREQUENCY PLASMA HEATING and CURRENT DRIVE

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Abstract

The mechanisms underlying radiofrequency plasma heating and current drive in closed configurations are considered together with the theoretical tools describing them, with just a brief mention of the actual experiments (see the other chapters). The heating schemes of interest for thermonuclear devices span a wide range of frequencies from the Alfvén wave resonance to the ion-cyclotron, lower hybrid and electron-cyclotron resonances. Heating at frequencies below the Alfvén wave resonance (magnetic pumping in its numerous versions), which is of limited thermonuclear relevance, is presented for its fundamental aspects — more than any other heating scheme it involves properties of particle motion in the large which are specific to the magnetic traps — in order to provide a broader and more integrated view of the whole subject. The areas covered in the various sections are indicated in the table of contents. Special attention is devoted to the following:

in Sect. 9.2, the relation between power absorption, flow and source terms in the Eulerian and in the (more useful) Lagrangian description; in Sect. 9.3, the adiabatic single-particle behaviour (with application of the three adiabatic invariants to various kinds of heating) and a relativistic effect of cyclotron resonance which already fully manifests itself at low speed; in Sect. 9.4, the interplay of Doppler effect, particle trapping by a wave, and collisional diffusion in velocity space as a) the key to the general behaviour of wave-particle resonant interaction, and b) as the cause of the Fokker-Planck-like diffusive character of the slow-time-scale response of plasmas to waves when particle trapping is negligible; in Sect. 9.5, a new quantitative derivation of quasilinear velocity space diffusion and ponderomotive plasma density shaping; in Sect. 9.6, the body of basic properties and heating-relevant aspects of waves in confined plasmas; in Sect. 9.7, a global view of RF energy flow, accessibility of resonance surfaces, mode conversion and power absorption with reference to the specific heating schemes used in fusion devices; and finally, in Sect. 9.8, a general model of the behaviour of lower hybrid waves launched by finite-length antennae, which solves both the spectral gap and the density limits problems.

9.1 Introduction

In a closed magnetic configuration, there are two ways of achieving cross-field penetration of external energy for the purpose of heating the bulk plasma (as well as for current drive):

- a) *with neutral particle beams.* The neutral particles must be of sufficiently high energy that they do not ionize until well inside the plasma. Conditions have to be such that the ionized particles lose kinetic energy in Coulomb collisions with less energetic charged particles before they suffer charge exchange (with like neutral atoms) or diffuse out of the plasma. This is discussed in Chapter 8.
- b) *with electromagnetic fields.* Conditions for energy propagation and absorption within the plasma exist over a wide range of situations, but are less straightforward than in the previous case. In this chapter special attention will be devoted to describing and clarifying the physical mechanisms underlying the various heating and current drive schemes, whereas only a brief mention will be made of the experiments, which as an integral part of most thermonuclear confinement experiments are covered in other chapters.

Broadly speaking, RF heating can be considered as sufficiently well understood, with the notable exception of some essential aspects of lower hybrid wave heating and current drive, a fact which justifies the term "problem" in the heading of Sect. 8 of this chapter. What is not yet sufficiently understood is the degradation of the confinement characteristics of all auxiliary-heated plasmas, by both neutral beams and RF fields; but this is an entirely different story, which will not be considered here.

References are limited to papers providing additional information useful for a better understanding of specific topics, often irrespective of priority and historical importance.

In agreement with the cited literature gaussian units are used throughout, at variance with the other chapters.

9.2 Macroscopic Conservation Laws

When applied to plasmas in their own and in external EM-fields, Poynting's theorem

$$\frac{\partial}{\partial t} \left(\frac{E^2 + B^2}{8\pi} \right) + \text{div} \left(\frac{c}{4\pi} \vec{E} \times \vec{B} \right) = -\vec{j} \cdot \vec{E}, \quad (9.1)$$

casts energy conservation into a similar theorem

$$\frac{\partial}{\partial t} \mathcal{E}_k + \text{div} \vec{Q}_k = \vec{j} \cdot \vec{E}, \quad (9.2)$$

where \mathcal{E}_k and \vec{Q}_k are, respectively, the density and flow density of the kinetic energy (macroscopic plus thermal). Thus, when $\vec{j} \cdot \vec{E}$ is a source of EM-energy, it is a sink of kinetic energy, and vice versa.

If a plasma volume V is enclosed by a surface through which the total flux of \vec{Q}_k vanishes, (9.2) gives

$$\int_V \frac{\partial}{\partial t} \mathcal{E}_k dV = \int_V \vec{j} \cdot \vec{E} dV. \quad (9.3)$$

Accordingly, the plasma in the volume V is heated if the time average of the RHS of (9.3) is positive.

The above equations also show that the heating power deposited by an EM-field into a plasma is some fraction of the time average of the integral of $\vec{j} \cdot \vec{E}$ over the volume of the EM-field source.

It should be clear that $\vec{j} \cdot \vec{E}$ need not be equal to $\frac{\partial}{\partial t} \mathcal{E}_k$, not even on the time average. Consider, as an example, a low- β Maxwellian plasma heated by a low-frequency ($\omega \ll \omega_{ci}$, the ion gyrofrequency) extraordinary wave ($E_{\parallel} \approx 0$). It will be shown later (Sect. 4) that, either directly or through Landau damping (and the like), Coulomb collisions cause the induced current – here the magnetization current – to have a lossy (anti-Hermitian) component; in this case $-\text{rot}(\gamma(\vec{r}) \frac{\partial}{\partial t} \vec{B})$. Here γ is a small non-negative real scalar proportional to the plasma pressure $p(\vec{r})$, which is supposed

to vary smoothly on the ion gyroradius scale length. Then, by Faraday's law and with the complex conjugate being indicated by an asterix it is found that

$$\frac{1}{2} \text{Re} \left(\vec{j} \cdot \vec{E}^* \right) = \frac{c}{2} \gamma(\vec{r}) |\text{rot} \vec{E}|^2 + \text{div} \left[\frac{c}{2} \gamma(\vec{r}) \text{Re} \left(\vec{E}^* \times \text{rot} \vec{E} \right) \right], \quad (9.4)$$

which is nothing else but the time average of (9.2): the first term of the RHS, the absorbed power density, is always positive, while the second term, the heat flow induced by the EM-field (for a single particle calculation of a special case see /9.1/), can have different signs in different regions of the plasma profile. This is best seen in the case of an x-dependent Cartesian plasma slab with \vec{B}_0 along the z -axis, in which case the RHS of (9.4) is simply proportional to

$$p(x) \frac{\partial}{\partial x} E(x, z, t) - \frac{\partial}{\partial x} \left[p(x) \text{Re} \left(\vec{E}^*(x, z, t) \frac{\partial}{\partial x} E(x, z, t) \right) \right], \quad (9.5)$$

$E(x, z, t)$ being the y -component of the \vec{E} -field, which is allowed to propagate in the x - and z -directions.

In a uniform plasma region where expression (9.5) reduces to

$$-p \text{Re} \left(E^*(x, z, t) \frac{\partial^2}{\partial x^2} E(x, z, t) \right), \quad (9.6)$$

we take $E(x, z, t) = E \exp(i(k_{\parallel} z + k_{\perp} x - \omega t))$ and consider the Alfvén wave dispersion relation in the two cases,

$$\text{a) } k_{\parallel} = 0, \quad k_{\perp} = \pm(\omega/\sqrt{v_A^2 + c_S^2})(1 + i\eta_1), \quad (9.7)$$

$$\text{b) } c_S^2 \ll (\omega/k_{\parallel})^2 \ll v_A^2, \quad k_{\perp} = \pm i k_{\parallel} (1 - i\eta_2) \quad (\text{a surface wave}). \quad (9.8)$$

Here c_S and v_A are the sound and Alfvén speeds, respectively, and η_1 and η_2 are small positive quantities proportional to the plasma resistivity /9.2/. When the wave is travelling across \vec{B}_0 (case a)), the time-averaged $\vec{j} \cdot \vec{E}$, given by (9.6), is positive and the last term in (9.5) vanishes. When the wave is evanescent across \vec{B}_0 (case b)), such an average is negative and, incidentally, just equal to minus the first term

in (9.5). The different nature of the interaction in cases a) and b) is actually already apparent from eqs. (9.7) and (9.8). Equation (9.7) states that dissipation simply damps the propagating wave, the damping factor being proportional to $e^{-\omega\eta_1 x/\sqrt{v_A^2+c_S^2}}$, while (9.8) states that as a result of dissipation the surface wave gives rise to the *emission* of a travelling wave proportional to $e^{-k_{\parallel}x}e^{i(k_{\parallel}\eta_2 x-\omega t)}$. A similar result can be found when the losses are due to Landau damping (and the like).

Returning now to a plasma profile which is x -dependent, we notice that it is the regions where one has $\frac{dp}{dx} \neq 0$ that contribute positively to the RHS of (9.4).

By subtracting from (9.2) the equation for the macroscopic kinetic energy derived from the momentum and continuity equations, one gets for the internal (thermal) kinetic energy density

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{3}{2}nkT \right) + \text{div} \left(\vec{q}_k + \vec{v} \cdot \hat{\Pi} + \frac{5}{2}nkT\vec{v} \right) = \\ (\vec{j} - \epsilon\vec{v}) \cdot \left(\vec{E} + \vec{v} \times \vec{B}/c \right) + \vec{v} \cdot \text{div} \hat{\Pi} + \vec{v} \cdot \text{grad} (nkT), \end{aligned} \quad (9.9)$$

where ϵ is the net electric charge density, $\hat{\Pi}$ the traceless stress tensor, and \vec{q}_k the flow density of the internal kinetic energy in a coordinate system in which the plasma is locally at rest.

From this equation, with the continuity equation, one gets for the entropy per particle, defined as $s \equiv \ln \left((nkT)^{\frac{3}{2}}/n^{\frac{5}{2}} \right)$,

$$\begin{aligned} \frac{\partial}{\partial t} (ns) + \text{div} \left(ns\vec{v} + \frac{\vec{q}_k}{kT} + \frac{\hat{\Pi}}{kT} \cdot \vec{v} \right) = \frac{1}{kT} (\vec{j} - \epsilon\vec{v}) \cdot \left(\vec{E} + \vec{v} \times \vec{B}/c \right) - \\ \frac{\vec{q}_k}{kT} \cdot \text{grad} (\ln kT) - \left(\frac{\hat{\Pi}}{kT} \cdot \text{grad} \right) \cdot \vec{v}. \end{aligned} \quad (9.10)$$

The first term of the RHS of eqs. (9.9 and 10) involves the work performed by the electric field in the co-moving coordinate system on the conduction current density induced in the plasma. The two remaining work terms on the RHS's, when considered

as quadratic terms in the EM-field amplitude (by proper use of the linearized versions of the various equations of the system, including (9.9) and (9.10) themselves), represent the two other possible irreversible plasma heat sources: relaxation of temperature gradients by heat conduction and relaxation of velocity shear by viscosity /9.3/.

Equation (9.10) immediately shows that when time and space scales are such that the plasma is frozen to the \vec{B} -lines of force, and the ohmic, viscous and heat flow terms are negligible, the process induced by the EM-field is adiabatic: $\frac{d}{dt} \left(kT/n^{\frac{2}{3}} \right) = 0$.

In general, however, eqs.(9.9 and 10) are of little use for evaluating the absorbed power density. A straightforward way to do this is found by abandoning the Eulerian viewpoint for the Lagrangian one. The rate of change of the *single-particle* energy (species α) due to an oscillating EM-field, \dot{K}_α , is averaged over velocity space. Then, after complex quantities are introduced, the time mean value of the absorbed power density is

$$P_\alpha = \frac{1}{2} Re \left[\int \dot{K}_\alpha f_\alpha^* d^3\vec{v} \right], \quad (9.11)$$

where f_α is the α -particle distribution function, which in first order in the EM-field amplitude obeys the Boltzmann equation

$$\frac{d}{dt} f_\alpha - \dot{K}_\alpha \frac{1}{kT_\alpha} f_{\alpha M} = C_\alpha \approx -\nu_\alpha f_\alpha \quad ; \quad (9.12)$$

here $\frac{d}{dt}$ is the total time derivative following the unperturbed particle trajectory in phase space, $C_\alpha \equiv \sum_\beta C_{\alpha\beta}(f_\alpha, f_\beta)$ is the collision term and ν_α the collision frequency for the species α (which, in general, depends on all species); for simplicity, a Maxwellian unperturbed distribution function has been assumed. The point that is important to stress here is that the very presence of a dissipative term provides f_α with a component in phase with \dot{K}_α , \hat{f}_α :

$$\hat{f}_\alpha \sim f_{\alpha M} \dot{K}_\alpha / kT_\alpha, \quad (9.13)$$

which makes a positive contribution to (9.11), while the presence of the total derivative, besides contributing to the value of the positive proportionality factor in (9.13), introduces an f_α^* component 90° out of phase with \dot{K}_α which does not contribute to (9.11).

It is (9.11) that will be used in this chapter whenever the absorbed power density is needed, with the obvious exception of the cases where conditions are such that the actual shape of $f_\alpha(\vec{v})$ is immaterial. Indeed, when the EM-field is not an oscillating function of space, e.g. when $\vec{B}(t)$ is a uniform axial field pulsating with $\omega \ll \omega_{ci}$, there is no Doppler effect and it is simpler to consider just a single, say thermal, particle /9.4/. Such situations are investigated in the next section.

9.3 Single-particle Aspects

There are few exact solutions of the equations of motion of a charged particle which are also directly relevant to the study of wave-particle interactions for the purpose of plasma heating and current drive in magnetic traps. The most important one, uniform circular motion orthogonal to \vec{B} (Larmor gyration) and uniform straight motion parallel to \vec{B} in the case of a temporally constant and spatially uniform \vec{B} -field (in the presence of a constant \vec{E}_\perp -field, the above holds in a Galilean $\vec{E} \times \vec{B}$ -convected frame where \vec{E}_\perp vanishes), plays, of course, a fundamental role in the adiabatic case of EM-fields which change only a little on the temporal and spatial scales of Larmor motion. The following is a brief discussion emphasizing some basic aspects of Alfvén's adiabatic theory on which usually both elementary and advanced treatments are insufficiently explicit.

While accelerated along \vec{B} by E_\parallel and by the changes of \vec{B} along \vec{B} , a charged particle drifts across \vec{B} with a velocity vector \vec{v}_D which depends on the changes of \vec{B} along a direction orthogonal to both \vec{B} and \vec{v}_D . In the Galilean frame of reference, which moves with the instantaneous velocity vector $\vec{v}_{GC} = v_\parallel(t) \frac{\vec{B}}{B} + \vec{v}_D$, the particle performs a gyratory motion orthogonal to \vec{B} around a Guiding Centre (GC). There are

field configurations for which the orbit of such gyrotory motion is along a closed path; in general, however, such an essentially planar orbit is an open spiral (if $\frac{d}{dt}(\vec{v} - \vec{v}_{GC})^2 \neq 0$). The arc of the spiral which is left behind by the particle in performing one complete rotation, starting from one phase value, can always be ideally closed by joining the two ends of the arc by means of a (small) segment taken along the radius vector. The remarkable fact is that on this virtual particle displacement the work of the \vec{E} -field (which is solenoidal) is negligible. This is exemplified in the two special cases,

- 1) $\vec{B} = \vec{e}_z B(t)$ with $\vec{E}(\vec{r}, t) = -B(t)\vec{e}_z \times \vec{r}/2c$, where \vec{r} refers to the GC position, and
- 2) constant, divergent, axisymmetric $\vec{B}(\vec{r}, t) \equiv (\alpha(z)r, 0, B_z(z))$ in local cylindrical coordinates with axis tangential to the \vec{B} -line of force which carries the particle GC, and with origin ($r=0, z=0$) at the GC position. In such a Galilean frame the Lorentz force, $\frac{e}{c}(\vec{v} - \vec{v}_{GC}) \times \vec{B}$, is again essentially radial.

As the gyro-phase is a physically ignorable variable in an adiabatic situation, one can conveniently get rid of it, whenever expressions involving the gyro-components of position and velocity occur in the equations of motion, by taking their time average over the gyro-period T along the artificially closed orbit. Such a procedure has a far-reaching consequence owing to the use of Stokes' theorem. For example, if S is the surface enclosed by the orbit, and \vec{n} is the unit vector perpendicular to dS , with Faraday's law one has

$$\int_T \vec{E} \cdot \vec{v} dt = \oint \vec{E} \cdot d\vec{l} = \int_S \text{rot} \vec{E} \cdot \vec{n} dS = -\frac{1}{c} \int_S \dot{\vec{B}} \cdot \vec{n} dS. \quad (9.14)$$

By means of the mean-value theorem the last term on the RHS of (9.14) can be written essentially as $-\pi \rho_L^2 \dot{B}/c$. Considerations of this kind naturally lead to the introduction of a magnetic moment

$$\mu = \frac{1}{2} m (\vec{v} - \vec{v}_{GC})^2 / B, \quad (9.15)$$

which is thus proportional to the magnetic flux enclosed by the gyration orbit closed ad hoc (there is no \vec{B} -flux leakage through the closure) as viewed from the Galilean frame of reference in which the GC is instantaneously at rest.

The remarkable fact is that since such a magnetic moment is actually an action integral $\oint \vec{p} \cdot d\vec{q}$ in the Hamiltonian sense, it is an adiabatic invariant of motion (something differing from a strict constant by an *exponentially* small function of the rate of change of the EM-field). The immediate benefit of the (adiabatic) invariance of μ is that it provides an explicit knowledge of $(\vec{v} - \vec{v}_{GC})^2$.

There are adiabatic situations where what has been said so far can, in principle, be extended to particle motion in the large. For this purpose we consider a fictitious toroidal trap whose toroidal and poloidal \vec{B} -field components satisfy condition $q \equiv rB_T/RB_P \ll 1$ (r and R are radii taken along, respectively, the minor and major radii of the torus). The GC of a passing particle /9.5/ gyrates around the magnetic axis, only slowly drifting across the \vec{B} -field lines (almost toroidally in this specific case). Connecting the ends of the GC trajectory arc after a complete gyration around the magnetic axis by a short segment along \vec{v}_D results in a virtually closed trajectory, thus allowing the introduction of a second action integral. This is the so-called longitudinal adiabatic invariant associated with the \vec{B} -flux through the virtually closed GC trajectory (being essentially the toroidal \vec{B} -field flux in the present example). Finally, because of \vec{v}_D , a passing particle also encircles the main (vertical) axis of the toroidal trap. Connecting the ends of the average GC trajectory arc after a full rotation about the vertical axis, by a short segment along the $\vec{B} \times \vec{v}_D$ - direction, again results in a virtually closed trajectory, thus allowing the introduction of a third action integral. This is the so-called flux adiabatic invariant essentially associated in the present example with the poloidal \vec{B} -field flux enclosed by the magnetic axis.

Analogous considerations could be developed for other configurations, including

tokamaks.

Both 2nd and 3rd adiabatic invariants, because of the implied loss of information concerning the actual particle position along the relevant trajectories, are of less general practical interest than μ , their usefulness being limited to situations like those evoked at the end of the previous section, which we shall now consider in some detail.

We first discuss gyrorelaxation /9.4/, which is a direct consequence of the invariance of μ of non-colliding particles in a pulsating magnetic configuration with $\omega \ll \omega_c$. We consider a pulsating uniform, axial $\vec{B}(t)$ (case 1 above). By taking appropriate moments of Boltzmann's equation (9.12) we can simply write in this case

$$\frac{d}{dt}(\mu B + \epsilon_{\parallel}) = \mu \frac{d}{dt} B, \quad (9.16)$$

$$\frac{d}{dt}(\mu B - 2\epsilon_{\parallel}) = \mu \frac{d}{dt} B - \nu(\mu B - 2\epsilon_{\parallel}), \quad (9.17)$$

where μB and ϵ_{\parallel} are, respectively, the averages of the perpendicular and parallel kinetic energies of a thermal particle, and use is made of the fact that the temperature anisotropy relaxes in a scattering time for the velocity vector pitch-angle of the thermal particles, which thereby do not appreciably change their kinetic energy. Eliminating ϵ_{\parallel} from (9.16) and (9.17) results in the second-order linear differential equation

$$\frac{d^2}{dt^2} \mu + \left(\nu + \frac{1}{B} \frac{d}{dt} B \right) \frac{d}{dt} \mu + \frac{\nu}{3} \mu \frac{1}{B} \frac{d}{dt} B = 0. \quad (9.18)$$

For $\frac{1}{B} \frac{d}{dt} B = \omega A \cos \omega t$, with $A \ll 1$, (9.18) is of the Hill type and has a solution proportional to $e^{\gamma t}(1 + A M(t))$, where M is a periodic function and γ , the heating rate, is approximately equal to $\nu(\omega A/3)^2/(\nu^2 + \omega^2)$, which is maximum for $\omega = \nu$.

In a tokamak, time modulations of B_V , the vertical component of \vec{B} , result in time modulations of the plasma major radius R and, consequently, of μB (proportional to $1/R$) and ϵ_{\parallel} (proportional to $1/R^2$), the latter as a consequence of the longitudinal adiabatic invariant of non-colliding particles. The possibility of plasma

heating by collisional relaxation of such oscillations of the temperature anisotropy was considered by ARTSIMOVICH /9.6/. Paradoxical as it may sound, heating by magnetic pumping does not require $\frac{dB}{dt} \neq 0$. In a tokamak one could imagine modulating the toroidal magnetic field B_T in time to put it 180° out of phase with B_V so that the plasma experiences a constant B-value during oscillation of its major axis. Owing to the longitudinal adiabatic invariant, the pump is now working only on the parallel motion $\frac{d}{dt}\epsilon_{\parallel} = -2\epsilon_{\parallel}\frac{d}{dt}\ln R$.

It is left to the reader to convince himself that there are situations (e.g. for $q \ll 1$) where both B and R remain constant but heating by magnetic pumping can still occur on the drift motion as a result of the third adiabatic invariant.

Before leaving the domain of adiabatic approximations, we shall briefly comment on the issue of the time average of the component of the Lorentz force, \bar{F}_{\parallel} , in the direction of motion of the particle GC, on the assumption that the EM-field amplitude varies adiabatically along such a direction but not necessarily in the other directions. In the case of the non-uniform, time-independent \vec{B}_o -field considered so far (a case which is adiabatic in all directions), \bar{F}_{\parallel} is the parallel gradient of $(-\mu B_o)$. In the presence of a time-oscillating *irrotational* electric field $\vec{E}(s)\sin\omega t$, with ω not restricted to much less than the gyrofrequency ω_c (here s is the distance along a \vec{B}_o -line of force) a first-order Taylor expansion of the EM-field around the GC that uses the expressions of the forced oscillations of the particle at s results in a quadratic expression for \bar{F}_{\parallel} in the form of the gradient of the so-called ponderomotive potential

$$- \left[\frac{e^2}{4m} \left(\frac{E_{\perp}^2(s)}{\omega^2 - \omega_c^2(s)} + \frac{E_{\parallel}^2(s)}{\omega^2} \right) + \mu B_o(s) \right]. \quad (9.19)$$

It is interesting to recall that also in this case the gyratory motion is simply described by $\mu = const.$, because $rot \vec{E} = 0$. This ponderomotive effect can be applied to particle acceleration or confinement /9.7/.

In general, it is useful to visualize the unperturbed (i.e. in the absence of the EM-waves) motion of a (non-colliding) particle confined in a toroidal configuration as the superposition of time averaged (almost uniform) motion of a GC along certain trajectories and of oscillatory motion about the GC. The former includes average motion along \vec{B}_0 as well as the almost vertical (in the tokamak approximation) \vec{B}_0 -curvature and $grad B_0$ drifts. The latter includes primarily the gyrotory motion, but also the oscillations along \vec{B}_0 in the $B_0 \sim \frac{1}{R}$ well.

In the presence of an EM-field such that $\vec{E}(\vec{r}, t) \cdot \vec{v}(t)$ does not identically vanish during the particle motion, a resonance occurs when the wave frequency perceived in a Galilean frame performing the time averaged GC motion with velocity \vec{v}_{av} either vanishes or is commensurable with the eigenfrequency ω_e of the corresponding oscillatory motion:

$$M(\omega - \vec{k} \cdot \vec{v}_{av}) = N\omega_e. \quad (9.20)$$

Obviously, not all integers M and N are equally important. The harmonics of the Doppler-shifted applied frequency are unimportant if the particle displacements *driven by the pump* are small in relation to the local wavelength of the EM-field, while the harmonics of the eigenfrequencies are unimportant if the elongation of the *unperturbed* single-particle oscillations is small in relation to the wavelength. These statements can be verified by using the Bessel function identity

$$e^{i\lambda \cos x} = \sum_n J_n(\lambda) e^{inx} \quad (9.21)$$

applied to the space dependence of the EM-pump: $e^{ikz} = e^{i(k\bar{z} + kz_{osc})}$. In the former case one has $z_{osc} \sim \cos(\omega t - k\bar{z})$, in the latter $z_{osc} \sim \cos \omega_e t$.

For $N = 0$, (9.20) is the Cerenkov condition, which is exploited in Landau (and like) damping. With $\vec{k} \cdot \vec{v}_{av} \approx k_{\parallel} v_{\parallel av}$ the second adiabatic invariant is no longer a constant, while with $\vec{k} \cdot \vec{v}_{av} \approx \vec{k}_{\perp} \cdot \vec{v}_{\perp av}$ the third adiabatic invariant is destroyed. If

ω_e is the gyrofrequency, waves satisfying condition (9.20) can destroy the invariance of μ and are used in cyclotron heating. It should be noted that N can be positive or negative. The latter case is referred to as the anomalous Doppler effect: the absorption of a wave by a particle is accompanied by an increase in the longitudinal (parallel to \vec{B}_0) energy of the particle and by a decrease in its gyroenergy (nearly elastic scattering).

Considerable insight into the gyroresonant interaction for $M=N=1$ is provided by studying the motion of a charged particle in a constant magnetic field and a circularly polarized, plane EM-wave propagating along the field:

$$\vec{B} = B_0 \vec{e}_z + nE [\vec{e}_x \cos(\omega t - kz) + \vec{e}_y \sin(\omega t - kz)],$$

$$\vec{E} = E [\vec{e}_x \sin(\omega t - kz) - \vec{e}_y \cos(\omega t - kz)],$$

with $k = \omega n/c$. In such a field, both the non-relativistic and the relativistic equations of motion can be solved exactly [9.8]. Let us consider, for simplicity, a low-amplitude wave and a particle initially at rest and in resonance with it: $\omega = qB_0/m_0 c$ (q is the electric charge and m_0 the rest mass of the particle)

$$E/(B_0 n^2) \ll 1 \quad (\text{for the non-relativistic equations}).$$

$$E/(B_0 |n^2 - 1|) \ll 1 \quad (\text{for the relativistic equations})$$

In the former case the ratio of the kinetic energy of the particle to its rest energy is found to oscillate between 0 and $2(E/(B_0 n^2))^{2/3}$ with period T so that $\omega T \approx 4.86(B_0/(nE))^{2/3}$; in the latter it oscillates between 0 and $2(E/(B_0 |n^2 - 1|))^{2/3}$, with $\omega T \approx 4.86(B_0/E)^{2/3}/|n^2 - 1|^{1/3}$, and when $n^2 = 1$ the kinetic energy increases indefinitely with time, the Doppler and mass effects just cancelling each other to maintain resonance throughout the particle motion. In the classical limit there is no mass effect and so resonance is maintained only when there is no Doppler effect ($n = 0$). By introducing the ratio R between the upper bounds of the kinetic energy excursions in the

two cases, $R = |n^2 - 1|/n^2$, one sees that for any finite n^2 -value different from 1/2, the relativistic and the classical results can substantially differ, even though in both cases the particle speed remains arbitrarily small all the time (for arbitrarily small E -values). This is due to a secular effect in the phase slip $\Psi(t)$ between the perpendicular particle velocity vector and \vec{E} ; indeed, in the classical limit one finds

$$\frac{d\Psi}{dt} \approx -\omega(nv/2c)^2 \left(\frac{5}{2} - \Theta(\cos \Psi) \right), \quad (9.22)$$

while relativistically

$$\frac{d\Psi}{dt} \approx \omega(1 - n^2)(v/2c)^2 \left(\frac{5}{2} - \Theta(\cos \Psi) \right), \quad (9.23)$$

where $\Theta(x)$ is the step function.

Thus, the function $|\Psi|$, which can be obtained from (9.22) for $n^2 > 0$ and from (9.23) for $n^2 \neq 1$, monotonically increases with time. This is the secular effect. The particle gains energy from the wave if $0 \leq |\Psi| < \pi/2$, and loses energy to the wave if $\pi/2 < |\Psi| < \pi$, and so on. The point is that $|\Psi|$ will always (sooner or later) go through all these intervals, no matter how small E/B_0 , and hence v/c is. The difference between the relativistic and the classical results is due to the different factors which precede $(v/c)^2$ in the two equations (9.22) and (9.23) for the time rate of change of Ψ .

This discussion should suffice to correct the current misconception that relativistic effects need high speeds to become appreciable.

9.4 Velocity Space Aspects

Energy flows unidirectionally from an EM-wave to a plasma if there are more plasma particles absorbing energy from the wave than particles giving energy to the wave. We discuss the case of a travelling monochromatic wave with a \vec{B} -component parallel to \vec{B}_0 , $B_{1\parallel} = -bB_0 \cos(kz - \omega t)$, which we suppose to be of small but finite amplitude and of low frequency ($\omega \ll \omega_c$ for the considered particle species). As, in the absence of collisions, one has $\mu = \text{const.}$, the wave traps particles in the low $|B_0 + B_{1\parallel}|$ -regions if $|\tilde{v}_{\parallel} - \omega/k|^2 < 2b\tilde{v}_{\perp}^2$, where the tilde indicates values at the points where $|B_0 + B_{1\parallel}|$ is minimum. In a coordinate system moving with the wave the trapped particle trajectories on the phase plane (Fig. 9.1) appear as nested closed orbits, while the untrapped particle trajectories appear as long wobbly lines. The border between trapped and untrapped trajectories is called the separatrix. The phase-plane region enclosed by the separatrix is called island. In the absence of collisions there is a steady-state f which is constant along particle trajectories. In this case, the distribution of particles along v_{\parallel} is symmetric with respect to each resonant v_{\parallel} -value given by (9.20). There is thus no flow of energy from or to the wave. There is an energy flow when Coulomb collisions, no matter how few they are, are included. Coulomb collisions tend to smear out any deviation of f from a Maxwellian. As the dominant contribution to scattering in velocity space is made by the small-angle distant encounters rather than by close encounters which completely change the particle velocity vector but are much less frequent, the collision operator C_{α} in (9.12) has the Fokker-Planck form describing diffusion and drag in velocity space /9.3/ as

$$C_{\alpha} = \text{div}_{\vec{v}} \left\{ \hat{D}_{coll} \cdot \text{grad}_{\vec{v}} f - \left(\frac{d}{dt} \vec{v} \right)_{coll} f \right\}. \quad (9.24)$$

Here $|\hat{D}_{coll}| \approx \nu_c(\vec{v})v_t^2/2$ and $\left(\frac{d}{dt} \vec{v} \right)_{coll} \approx \nu_c(\vec{v})\vec{v}$, where $\nu_c(\vec{v})$ is the frequency of a 90° deflection resulting from long-range Coulomb encounters and $v_t^2 = 2kT/m$. The

expression in curly brackets is linear in f if f deviates but slightly from a Maxwellian. From (9.24) one can see that the time scale for restoring a Maxwellian slope over the velocity range where particles are trapped by the wave, $\Delta v_{\parallel} \approx v_t \sqrt{(2b)}$, is the reciprocal of an effective collision frequency ν_{eff} given approximately by $\nu(v_t/\Delta v_{\parallel})^2 \approx \nu/2b$, where $\nu \equiv \nu_c(v_t)$. On the other hand, the time scale for plateau formation in f over Δv_{\parallel} is given by the bounce time τ_B of the trapped particles: $\tau_B \approx 2\pi/|\omega - k_{\parallel}v_{\parallel}|_{max} \approx 2\pi/k_{\parallel}v_t\sqrt{(2b)}$. If one has $\nu_{eff}\tau_B < 1$, i.e. in our example $\nu < k_{\parallel}v_t(2b)^{\frac{3}{2}}/2\pi \equiv \nu^*$, the average of f over times longer than ω^{-1} , \bar{f} , is distorted with respect to the Maxwellian in a parallel velocity range $\Delta v_{\parallel} \approx v_t\sqrt{(2b)}$ and the difference between the number of particles in this range which are slower than the wave and the number of particles in this range which are faster than the wave — difference which determines the flow of energy between wave and particles — is positive and, for $\nu \ll \nu^*$, proportional to (ν/ν^*) . The quantitative treatment of the $\nu \ll \nu^*$ case was carried out by ZAKHAROV and KARPMAN in /9.9/ in the case of an electrostatic wave (for the present case see /9.10/).

If $\nu \gg \nu^*$, the distortions of \bar{f} with respect to the Maxwellian f_M are smeared out so that $\bar{f} \approx f_M$. As the wave amplitude is taken to be small, $(f - \bar{f})$ is a small quantity changing rapidly compared to $1/\nu$ (as long as $\omega \gg \nu$), thus obeying a formally collisionless linear equation. As a matter of fact, collisions simply replace, successively, a particle leaving resonance conditions by a new indistinguishable resonant particle at a rate larger than the oscillation frequency $1/\tau_B$, thus leaving unchanged the number of particles which strongly interact with the wave. Thus, although $\nu \neq 0$, the plasma heating rate is virtually independent of ν when $\nu^* \ll \nu \ll \omega$. As is well known, in this case the quantitative treatment, originally due to Landau (for electrostatic waves), is based on (9.12).

We consider a pump wave such that

$$\dot{K} = e\vec{v}_o \cdot \vec{E} + \mu \frac{\partial}{\partial t} B_1 \equiv \dot{K}_s \sin(\vec{k} \cdot \vec{r} - \omega t), \quad (9.25)$$

where \vec{v}_o is the unperturbed GC velocity, which, for simplicity, is assumed to be independent of time and space, and where it is assumed that either $\omega \ll \omega_c$ or $\vec{E} \times \vec{B}_o = 0$, so that the gyrophase is not involved. Then, if one writes

$$f_1 = f_c \cos(\vec{k} \cdot \vec{r} - \omega t) + f_s \sin(\vec{k} \cdot \vec{r} - \omega t), \quad (9.26)$$

equation (9.12) gives

$$\begin{aligned} \nu f_s - (\vec{k} \cdot \vec{v}_o - \omega) f_c &= \dot{K}_s f_M / kT \\ \nu f_c + (\vec{k} \cdot \vec{v}_o - \omega) f_s &= 0. \end{aligned} \quad (9.27)$$

Inserting f_1 from eqs.(9.26 and 27) into (9.11) with $d^3\vec{v} = 2\pi dv_{\parallel} d(v_{\perp}^2/2)$ results in an integrand containing the function $\nu/(\nu^2 + (\vec{k} \cdot \vec{v}_o - \omega)^2)$. If one has $\nu \geq \omega$, the collisional limit is recovered where the Doppler effect is ineffective, this being the situation briefly discussed in the previous section. If, on the contrary, one has $\nu \ll \omega$, such a function is a narrow, bell-shaped function tending to $\pi\delta(\vec{k} \cdot \vec{v}_o - \omega)$ for $\nu/\omega \rightarrow 0$. In this case power dissipation is called Landau damping. Usually, the literature distinguishes between Landau damping and Transit Time Magnetic Pumping (TTMP), depending on whether it is the first term or the second term on the RHS of (9.25) which really matters.

Let us now further discuss ion TTMP. If the parallel wave number of a wave with $\omega \ll \omega_{ci}$ is chosen so as to satisfy the transit time resonance condition for thermal ions: $k_{\parallel} v_{ti} \approx \omega$, then in a low- β plasma one has $|k_{\parallel} v_A| \gg \omega$. Equation (9.8) thus gives $k_{\perp}^2 \approx -k_{\parallel}^2$: the wave is radially evanescent, but with an acceptably long evanescence length if $Im\{k_r a\} \approx na/R \leq 1$. Ion TTMP has been proposed as a plasma heating

method in various versions, depending primarily on the nature of the driving term in \dot{K}_i , which for a tokamak reads

$$\dot{K}_i \approx ev_{\parallel}E_{1\parallel} + e(v_{\parallel}^2 + v_{\perp}^2/2)E_{1z}/R\omega_{ci} + \mu \frac{\partial}{\partial t} B_1, \quad (9.28)$$

where z is along the vertical (symmetry) axis of the tokamak. The last term on the right-hand side of (9.28) is the driving pump of the original TTMP version — the *compressional* version — produced by ordinary $m=0$ azimuthal coils. The irrotational $E_{1\parallel}$ -component ensures charge neutrality in spite of the preferential action of the pump on one of the plasma components when $\omega \approx \vec{k} \cdot \vec{v}_{oi} \approx k_{\parallel}v_{\parallel i}$. For a report on successful experiments in a tokamak see /9.11/. TTMP can also occur with a *torsional* pump $\vec{B}_1 \cdot \vec{B}_0 = 0$ characterized by a solenoidal E_{1z} component, essentially constant along z , sinusoidal in the toroidal coordinate, and weakly dependent on R /9.12/. It can perform work on the particles owing to the existence of the vertical, *unidirectional* drift velocity (again in the presence of an electrostatic $E_{1\parallel}$ -component).

The characteristic frequency of these TTMP versions is between a few tens of kHz and about 200 kHz. Other TTMP versions /9.13/ involve $\omega \approx \vec{k}_{\perp} \cdot \vec{v}_{o\perp i}$ and $k_{\parallel} \rightarrow 0$, corresponding to operating frequencies from a few kHz to a few tens of kHz.

In the various cases the heating rate is roughly of the form $\gamma_H = |\vec{k} \cdot \vec{v}_{oi}| \cdot |B_1/B_0|^2$ (with \vec{v}_{oi} taken for $|\vec{v}| \approx v_{ti}$) so that in order of magnitude the power density absorbed is $\beta |\frac{\partial}{\partial t} (E^2 + B^2) / 8\pi|$. In a low- β plasma this is a small fraction of the available reactive power density. However, the thermonuclear prospects of ion TTMP (in all its versions) are poor for an even more practical reason: the first wall of a thermonuclear device is opaque to EM fields with $f \geq 100Hz$. Thus, by the Faraday law, since the line integral of \vec{E}_1 all the way round any closed path on such a conducting shell vanishes, the instantaneous RF \vec{B} -flux through any poloidal or equatorial cross-section of the vacuum vessel has to vanish. As a result, if the instantaneous RF \vec{B} -flux created by the RF coils through the poloidal or equatorial plasma cross-section does not vanish,

as here with waves which do not oscillate in space (see (9.8)), a large fraction of the \vec{B}_1 -field energy has to be between the RF coils and the first wall, where eddy currents have to flow. Then the heating efficiency is the result of a compromise between two contradictory requirements: diminishing the image currents in the wall while keeping the plasma cross-section as large as possible. One realizes that the efficiency of any ion TTMP version will remain disappointingly low even in large thermonuclear devices. The conclusion is that the lowest practicable frequencies for efficient RF heating in toroidal plasmas are the Alfvén frequencies $\omega \geq |\vec{k} \cdot \vec{v}_A|$ at which at least on some radial extent within the plasma column one has $k_{\perp}^2 > 0$. This is considered in Sect. 9.7.

The general issue of the frequency dependence of the wave-particle resonance interaction is best illustrated by plotting the heating rate γ_H , which is proportional to P (see (9.11)), as a function of ν (Fig. 9.2). The upper curve in Fig. 9.2 exhibits the existence of three different regimes. It is somehow paradoxical that γ_H depends on ν when collisions are rare (but we know, as discussed at the beginning of the section, that dissipation implies at least *some* collisions), and it is certainly unfortunate that the intermediate regime, where ν , although small compared with ω , is sufficiently large to prevent local distortion of the velocity distribution function \bar{f} with respect to f_M , is generally referred to as collisionless /9.14/. For comparison the case of gyrorelaxation is also plotted in Fig. 9.2 (lower curve) with its two regimes. The difference, as we have seen, is brought about by the Doppler effect.

The following proof of the formal similarity of Landau and cyclotron damping in the $\nu > \nu^*$ collision regimes should suffice to convince the reader that the above qualitative discussion actually applies to both damping cases /9.10/. The simplest case of cyclotron damping occurs with a circularly polarized wave propagating along a uniform \vec{B}_0 -field: $\vec{E}_1(r, t) = E_1(\vec{e}_x \cos(kz - \omega t) + \vec{e}_y \sin(kz - \omega t))$, where we use Cartesian coordinates

with z taken along \vec{B}_0 . By writing $v_x = v_\perp \sin \phi$ and $v_y = v_\perp \cos \phi$, where $\frac{d}{dt} \phi = \omega_c$, for the velocity components, and the corresponding equations for the particle coordinates x and y , we obtain

$$\dot{K} = ev_\perp E_1 \sin(kz + \phi - \omega t). \quad (9.29)$$

The calculation then proceeds as in the Landau case with $f_1 = f_c \cos(\dots) + f_s \sin(\dots)$ and with

$$\begin{aligned} \nu f_s - (k_z v_z + \omega_c - \omega) f_c &= \dot{K} f_M / kT \\ \nu f_c + (k_z v_z + \omega_c - \omega) f_s &= 0, \end{aligned} \quad (9.30)$$

instead of (9.27).

Harmonic cyclotron damping occurs if the wave vector \vec{k} has a component along, say, the x -direction. In this case, the identity (9.21) can be used to write instead of (9.30)

$$\begin{aligned} \nu f_s - (k_z v_z + n\omega_c - \omega) f_c &= \dot{K} |J_{n-1}(k_x v_\perp / \omega_c)| f_M / kT |J_n(k_x v_\perp / \omega_c)| \\ \nu f_c + (k_z v_z + n\omega_c - \omega) f_s &= 0, \end{aligned} \quad (9.31)$$

where \dot{K} can again be given by (9.29). When inserting f_1 as given by these equations into (9.11) one has of course to set $d^3 \vec{v} = d\phi dv_\parallel d(v_\perp^2/2)$. Notice, in conclusion, that eqs. (9.25, 28 and 29) give the kinetic energy excursions of a non-colliding single particle, but that the plasma temperature increase may well be much larger than these excursions: the latter are just the basic steps of the random walk process in energy space to which power absorption is ultimately due (even though this process is not explicitly exploited in the actual calculation of P).

Finally, as an introduction to the quantitative treatment of the next section, we present qualitative physical reasons why, when wave trapping is negligible, the average

of f over the wave period evolves according to a diffusion process of the Fokker–Planck type, where the so-called quasilinear diffusion coefficient, D_{QL} , is proportional to the square of the amplitude of the EM-wave. First of all, the particles experience a non-vanishing force only on those parts of their unperturbed trajectories where the wave phase they see slowly varies (or is stationary), the particles with the longest resonance duration being those whose velocity relative to the wave, Δv , is such that $k_{\parallel}\Delta v \approx \nu_{eff}$, where ν_{eff} is the effective collision frequency for scattering particles out of the Δv range, $\nu_{eff} \approx \nu(v_t/\Delta v)^2$. Secondly, collisions not only contribute to controlling the resonance duration (which in a toroidal plasma primarily depends on the presence of the rotational transform /9.15/) but also ensure that the particles forget the wave phase when leaving resonance.

If $\Delta\tau$ is the range within which τ_T , the time elapsing between two successive resonances of a particle, varies as a result of Coulomb scattering, requiring that the particles forget the wave phase, this just says that $\omega_{res}\Delta\tau \gg 1$. This randomization criterion ensures that the particle receives incoherent velocity increments when going through successive resonances, and this is why the EM-field acts through a diffusion term. The quadratic dependence of D_{QL} on the field amplitude has the same origin as the quadratic dependence on the length of the basic step of the diffusion coefficient describing any random walk process.

A noteworthy result is that diffusion of the type just discussed also describes ion heating by a coherent lower-hybrid wave ($\omega \gg \omega_{ci}$) propagating perpendicularly to a uniform magnetic field (if the amplitude of the wave exceeds a threshold /9.16/). On a time scale between the wave period and the cyclotron period, ions behave as in a virtually vanishing magnetic field so that the wave-particle resonance condition is essentially $\omega = \vec{k} \cdot \vec{v}$. If it is assumed that $\vec{k} = \vec{k}_y$, since $0 < |v_y| < v_{\perp}$, only ions with $v_{\perp} > \omega/k_y$ (i.e. with sufficiently high perpendicular energy) pass through

the resonance $v_y = \omega/k_y$ (twice per cyclotron orbit). The kick they receive here can be approximated by a δ -function (the resonance duration is much shorter than $1/\omega_{ci}$). Any mechanism capable of decorrelating the ions and the wave at least once per cyclotron period causes the wave to be ion-Landau-damped (note, incidentally, that here the resonant particles are much more numerous than those satisfying the Landau condition along y in the absence of \vec{B}_0 , because the magnetic field sweeps the vector \vec{v}_\perp through all angles). Collisions are insufficient to destroy phase coherence at these high frequencies. Instead, it has been found [9.16] that phase coherence is destroyed when the electric field is such that the kick received by an ion on one transit through resonance is sufficient to change the phase that it sees when next in resonance, by at least $\pi/2$ on the average. This is so because the magnitude of the kicks received at resonance is a sensitive function of the phase at the beginning of the resonance. If we look at the phase space, we discover that at such field amplitudes (common in heating experiments) a particle orbit wanders over most of phase space, spending roughly equal lengths of time in equal areas (i.e. the particle orbit is approximately ergodic). The phase space is no longer characterized by a single island centred around one resonant point, but by the presence of an infinity of higher-order islands centred around the points defined by (9.20). For fields above threshold, these islands overlap, thus allowing almost unrestricted motion.

9.5 Slow-time-scale Plasma Response

We now proceed to prove formally that, under the influence of an EM-field, the average of f over the wave period, \bar{f} , evolves according to a Fokker-Planck-like diffusion term

$$\text{div}_{\vec{v}} \left\{ \hat{D}_{QL}(\vec{x}, \vec{v}) \cdot \text{grad}_{\vec{v}} \bar{f} + \frac{q}{m} \vec{E}_{DC} \bar{f} \right\}, \quad (9.32)$$

where \hat{D}_{QL} is quadratic in the amplitude of the AC part of the electric field. The DC term will only be considered when the ponderomotive change of the plasma density is discussed later in this section. Here a one-dimensional situation in both real and velocity space is assumed in order to avoid unnecessary complication. This implies integrating over the perpendicular velocity components under the simplifying assumption that these are Maxwellian-distributed. In spite of this limitation the following treatment should be more general and more directly relevant to heating and current drive situations than the usual (see /9.17/) microinstability-oriented approaches.

As a starting point we consider the electrostatic situation

$$qE(x, t)/m = a(x, t)e^{i\omega t} + \sum_{n \geq 2} a_n(x, t)e^{in\omega t} + c.c. \quad (9.33)$$

$$f(x, v, t) = \frac{1}{2}g(x, v, t) + h(x, v, t)e^{i\omega t} + \sum_{n \geq 2} h_n(x, v, t)e^{in\omega t} + c.c. \quad (9.34)$$

where the t -dependence indicated in the parentheses on the RHS of the equations is slow on the ω^{-1} time scale, while the space dependence is unrestricted. The terms with $n \geq 2$ are neglected since, within the plasma, the amplitude $a(x, t)$ used in heating and current drive experiments is small (in the sense of the discussion following (9.20)). Then, by indicating complex conjugate (c.c.) by a star, the Boltzmann equation gives

$$\frac{\partial}{\partial t}g + v \frac{\partial}{\partial x}g + \frac{\partial}{\partial v}(ah^* + a^*h) = C(g) \quad (9.35)$$

$$(i\omega + \nu)h + v \frac{\partial}{\partial x}h + a \frac{\partial}{\partial v}g = 0 \quad (9.36)$$

where in (9.36) we have taken $C(h) \approx -\nu h$ as $\nu \ll \omega$. By solving (9.36) for h and substituting it in (9.35) one obtains

$$\frac{\partial}{\partial t}g + v \frac{\partial}{\partial x}g - \frac{\partial}{\partial v} \left\{ \frac{a(x)}{v} e^{(i\omega - \nu)x/v} \int_{-\frac{v}{|\nu|}\infty}^x a^*(y) e^{-(i\omega - \nu)y/v} \frac{\partial}{\partial v}g dy + c.c. \right\} = C(g). \quad (9.37)$$

In the special case where $a(x) = ae^{-ik_0x}$ (so that $|a(x)|^2 = \text{const.}$) there is no reason to keep the space dependence of g and (9.37) becomes simply

$$\frac{\partial}{\partial t}g + \frac{\partial}{\partial v} \left\{ \frac{2\nu|a|^2 \frac{\partial}{\partial v}g}{(\omega - k_0v)^2 + \nu^2} \right\} = C(g), \quad (9.38)$$

where the driving term has precisely the form of (9.32) with the diffusion coefficient

$$D_{QL} \sim \frac{\nu}{(\omega - k_0v)^2 + \nu^2}. \quad (9.39)$$

In general, however, the x -dependence of $|a(x)|^2$ induces a space dependence of g . We consider in some detail the case

$$a(x) \equiv \frac{\alpha(x/l)}{\sqrt{l}} e^{-ik_0x}, \quad (9.40)$$

where $\alpha(x/l)$ is a real, bell-shaped function, so that the Fourier k -spectrum of $a(x)$ is centred around $k = k_0$ — a situation of obvious relevance to experiments. The normalization ensures that the amount of electric energy available in the plasma is l -independent.

In the limit $\nu/\omega \rightarrow 0$ the expression in brackets in (9.37) can now be written

$$2 \frac{\alpha(x/l)}{v} \int_{-\frac{v}{|\nu|}\infty}^0 \alpha((x-y)/l) \frac{\partial}{\partial v}g(x-y, v) \cos(\Delta y) dy, \quad (9.41)$$

where $\Delta \equiv (k_0 - \omega/v)$. According to the discussion of the previous section expression (9.41) should take the form

$$D_{QL}(x, v) \frac{\partial}{\partial v}g. \quad (9.42)$$

Let us first derive some consequences of (9.42). We take for $C(g)$ the high- v approximation /9.3/

$$C(g) \approx -\nu \frac{\partial}{\partial v} \left[\frac{v_t^2}{2v^3} \frac{\partial}{\partial v} g + \frac{g}{v^2} \right], \quad (9.43)$$

Then, by neglecting for a moment the derivatives of g with respect to t and x , one gets

$$g(v) = g_\circ \exp \left\{ -2 \int_{-\infty}^v u D^{-1}(x, u) du \right\}, \quad (9.44)$$

where g_\circ is a constant determined by particle conservation, and $D(x, v) \equiv v_t^2 + 2v^3 D_{QL}(x, v)/\nu$. As long as one has $v^2 \frac{\partial}{\partial x} D \ll D^2$ (D_{QL} is not yet determined), solution (9.44) is correct. It is essentially a Maxwellian where $v^3 D_{QL} \ll \nu v_t^2/2$, and has a “plateau” over the v regions where $v^3 D_{QL} \gg \nu v_t^2/2$. Expression (9.41) can be put in the form (9.42), as expected, only if $\frac{\partial}{\partial v} g(x-y, v) \approx \frac{\partial}{\partial v} g(x, v)$, i.e. if $\min\{l, 1/\Delta\} \frac{\partial}{\partial x} \frac{\partial}{\partial v} g \ll \frac{\partial}{\partial v} g$. From (9.44) it can easily be seen that g does not have this property, but $D \frac{\partial}{\partial v} g$ does. Equation (9.41) is thus written in the form

$$2 \frac{\alpha(x/l)}{v} \int_{-\frac{v}{|\Delta|} \infty}^0 \frac{\alpha((x-y)/l)}{D(x-y, v)} D(x-y, v) \frac{\partial}{\partial v} g(x-y, v) \cos(\Delta y) dy, \quad (9.45)$$

$D \frac{\partial}{\partial v} g$ is expanded around $y = 0$, and it is concluded that, in order to put expression (9.45) in a diffusion-like form, one must have

$$D_{QL}(x, v) \equiv 2 \frac{\alpha(x/l)}{v} D(x, v) \int_{-\frac{v}{|\Delta|} \infty}^x \frac{\alpha(y/l)}{D(y, v)} \cos(\Delta(x-y)) dy. \quad (9.46)$$

When one has $v^3 D_{QL} \ll \nu v_t^2/2$ the approximation for D_{QL} follows directly from (9.46)

$$D_{QL}(x, v) \approx 2 \frac{\alpha(x/l)}{lv} \int_{-\frac{v}{|\Delta|} \infty}^x \alpha(y/l) \cos(\Delta(x-y)) dy. \quad (9.47)$$

In the general case (9.46) can easily be transformed in a second-order differential equation for D_{QL} (in the variable x , v being a parameter). Here we write the approximation valid when $v^3 D_{QL} \gg \nu v_t^2/2$:

$$\frac{\partial}{\partial x} \left(\frac{2\alpha}{v D_{QL}} \right) = \frac{\Delta^2}{\alpha} + \frac{\partial^2}{\partial x^2} \left(\frac{1}{\alpha} \right), \quad (9.48)$$

from which it follows that

$$\frac{2\alpha}{v D_{QL}} \approx \int_{x_0}^x \left[\frac{\Delta^2}{\alpha} + \frac{\partial^2}{\partial x^2} \left(\frac{1}{\alpha} \right) \right] dy + 4v^2 \frac{\alpha(x_0)}{\nu v_t^2}, \quad (9.49)$$

x_0 being the x -value where $v^3 D_{QL}(x, v) = \nu v_t^2/2$. Note that D_{QL} is finite (and positive) because $\frac{\partial^2}{\partial x^2}(1/\alpha) > 0$.

From this general form (9.39) can easily be recovered by taking the limits $l \rightarrow \infty$ and $|x_0| \rightarrow \infty$.

For future applications it is important to consider the l -dependence of the lower end of the velocity region, where $v^3 D_{QL} = \nu v_t^2/2$, because it gives the dependence of the width of the plateau in velocity space on the spatial gradient of $a(x/l)$. As it is required that in this process $|a|$ remain constant, x_0/l is considered as (almost) constant. Then, from (9.47) written for $x = x_0$, it follows that also the product $l\Delta$ must remain constant, i.e. the lower end of the plateau interval displaces towards lower v -values as the spatial gradient of $|a(x/l)|$ increases:

$$D_{QL} \approx \frac{2}{v} \alpha(x_0/l) \int_{-\frac{v}{|v|} \infty}^{x_0/l} \alpha(\tau) \cos(l\Delta(\frac{x_0}{l} - \tau)) d\tau. \quad (9.50)$$

Let us next consider the effect of the driving term on the x -dependence of the plasma density, in the case of field (9.40) with $\alpha \equiv \alpha_0 e^{-(x/l)^2}$ and to the first order in $|a|^2$, the lowest significant order. The expression for the driving term to this order is obtained simply by taking for g in the curly brackets in (9.37) a (spatially) uniform Maxwell

distribution. In the present discussion the DC term in (9.32) is the gradient of the ambipolar potential $V(x)$ which ensures charge neutrality (see below). Dividing both sides of (9.37) by v/v_t and integrating over v/v_t from $-\infty$ to ∞ , one gets

$$\frac{dn}{dx} + \frac{2n_o}{v_t^2} \frac{q}{m} \frac{dV}{dx} - \frac{4n_o l |a|^2}{\sqrt{\pi} v_t^3} \cdot i \int_0^\infty \frac{1}{u} \frac{\partial}{\partial u} \left\{ u e^{-u^2} (\omega l - (k_o l + 2ix/l) u v_t)^{-1} - c.c. \right\} du = 0, \quad (9.51)$$

where the $\nu/\omega \rightarrow 0$ limit is taken and both the time derivative and the collisional terms are neglected on the ground that, even to order $|a|^2$, $g(v)$ differs only slightly from f_M . If one has $|\omega/k_o v_t (1 + 2ix/k_o l^2)| \gg 1$, one finds

$$\frac{dn}{dx} + \frac{2n_o}{v_t^2} \left(\frac{q}{m} \frac{dV}{dx} - \frac{4|a|^2 x}{\omega^2 l^2} \right) = 0, \quad (9.52)$$

where the last term in parentheses can also be written $\frac{d}{dx}(|a(x)|^2/\omega^2)$. Considering (9.52) for both ions and electrons, under the assumption that they move along \vec{B}_o only ($\omega_{pe}/\omega_{ce} \rightarrow 0$, $\omega_{pi}/\omega \rightarrow 0$), and imposing charge neutrality determines both $V(x)$ and the density, $n(x) = n_o(1 - \gamma|a(x)|^2)$, where $\gamma|a|^2$ is the ponderomotive effect (see later on for the actual γ -value in the case of lower hybrid waves; for a general review of ponderomotive effects see /9.18/).

Finally, let us consider one of the most important applications of the quasilinear (QL) theory of the EM-field plasma interaction: the treatment of current drive.

The Fokker-Planck collision term for the electrons has a noteworthy property: the collisional rate of change of the component of the electron free path along a given direction is equal to the electric current density along this same direction

$$\frac{e}{5 + Z} \int \frac{v_{||}}{\nu_e(v_{te}/|v|)^3} C_e(g) d^3v = -e \int v_{||} g dv \equiv j_{||}, \quad (9.53)$$

where $-e$ is the electric charge and Z specifies the ion charge state. Since, on the other hand, one has $\frac{\partial}{\partial t} g + \text{div}_{\vec{v}}(\vec{\Gamma}) = C_e$, where $\vec{\Gamma}$ is the flux of electrons in velocity

space due to any process (QL-diffusion, DC-electric field acceleration, etc.), it follows that

$$j_{\parallel} = -\frac{e}{(5+Z)} \int \vec{\Gamma} \cdot \frac{\partial}{\partial \vec{v}} \left(\frac{v_{\parallel}}{\nu_e(v_{te}/|v|)^3} \right) d^3v \quad (9.54).$$

This is a special case of a more general result derived by ANTONSEN and CHU /9.19/. On the other hand, the time average of the power density absorbed by the electrons is

$$P_e = \int \left(\frac{1}{2} m_e v^2 \right) \frac{\partial}{\partial t} f_e d^3v = \int \vec{\Gamma} \cdot \frac{\partial}{\partial \vec{v}} \left(\frac{1}{2} m_e v^2 \right) d^3v. \quad (9.55)$$

Thus, in the case where $\vec{\Gamma}$ is very localized in velocity space we can write for the figure of merit $-j_{\parallel}/P_e$:

$$-j_{\parallel}/P_e = \frac{e}{5+Z} \left[\left(\vec{\Gamma} \cdot \frac{\partial}{\partial \vec{v}} \right) \left(\frac{1}{2} m_e v^2 \right) \right]^{-1} \left(\vec{\Gamma} \cdot \frac{\partial}{\partial \vec{v}} \right) \frac{v_{\parallel}}{\nu_e(v_{te}/|v|)^3}, \quad (9.56)$$

an interesting result, independent of the form of the distribution function, first derived (in a different way) by FISCH and BOOZER (see the extensive review of current drive theory /9.20/). The surprise in (9.53) to (9.56) is that j_{\parallel} can be generated even by inducing a purely *perpendicular* electron flux. According to OHKAWA /9.20/ a cyclotron wave travelling in one direction along the toroidal coordinate can be used to increase the perpendicular energy of the resonant circulating electron population selectively, so that such a population becomes trapped. The result would be a deficit of current-carrying circulating electrons. At the same time, trapped particles are symmetrically detrapped by Coulomb collisions. As a result, there is a net increase in the electron toroidal angular momentum in the direction opposite to the propagation of the wave. The momentum is dissipated by the ions to generate a toroidal current. The fact that a net momentum is produced counter to the wave momentum is not surprising; for instance, in axisymmetric geometry it is the *canonical* momentum of a particle which is conserved; thus, if there is radial displacement of the particle orbit or a driven radial flow, toroidal angular momentum mRv_{ϕ} is indeed produced /9.5/

$$\frac{d}{dt}(mRv_{\phi}) = -e \frac{d}{dt}(RA_{\phi}) = ev_{dr}RB_{\theta}, \quad (9.57)$$

where A_ϕ is the toroidal component of the EM vector potential and v_{dr} is the electron drift velocity along the minor radius of a plasma embedded in concentric magnetic surfaces.

9.6 Waves in Collisionless Plasmas

This brief review of wave propagation in hot plasmas uses the customary terms a) cold and b) warm plasma to designate the following situations:

- a) the width of the particle distribution functions in velocity space is immaterial because the relevant wave velocities greatly exceed the thermal speeds;
- b) the global width of the velocity distribution functions does matter, but not their actual shape, as in the case where the kinetic effects are sufficiently well described by pressure gradients and where the wave characteristics are such that the preferential interactions with particles in selected regions of velocity space (Landau damping and the like) are negligible.

For a general view of RF heating and current drive in toroidal, maxwellian plasmas one can assume the following somewhat restrictive picture, although not always necessary, as a representative reference situation: a small-amplitude EM-wave with a single, fixed frequency and a given spectrum of the wave number components in the toroidal and poloidal directions, essentially determined by the external launching structure, propagates in a radially stratified plasma. For the present discussion we also neglect non-local effects, which are due to the fact that, strictly speaking, the current density at a given point depends on the \vec{E} -fields throughout the plasma (see below).

Wave propagation is then described by an ordinary linear differential equation of order $2N$, where the integer N gives the number of different kinds of waves present in the plasma (there are two waves of each kind differing only in their directions of propagation, which are opposite). In a uniform plasma the solubility condition for such an equation – the dispersion relation (DR) – contains the N -th power of the wave

number squared.

The number of possible wave types in a cold plasma is 2, and in a warm plasma $(2+1+i)$ if the plasma contains (besides electrons) i species of ions of different charge-to-mass ratios, these numbers corresponding to the different kinds of forces acting per unit plasma volume. A hot plasma in the same situation has 2 plus an infinite number of waves (see below).

The two forces acting in a cold plasma are the divergences of the electric and the magnetic components, each involving tension along and pressure across the field, of Maxwell's stress tensor (after subtraction of the rate of change of the EM-field momentum per unit volume). As is well known, the former component stems from the displacement current term and the latter from the $rot \vec{B}$ term in the equation $4\pi\vec{j}/c = -\dot{\vec{E}}/c + rot \vec{B}$. In a vacuum as well as in an isotropic, cold plasma ($\vec{j} \sim \dot{\vec{E}}$), the two forces coincide and, thus, the two kinds of wave degenerate into a single one.

In a warm plasma the remaining forces are of course the partial pressure gradients. If ω is so high that $\omega_{ci} \equiv (ZeB_0)/(m_i c)$ and $\omega_{pi} \equiv \sqrt{(4\pi Z^2 e^2 n_o/m_i)}$ can be neglected (infinitely heavy ions), there will be 3 waves. If ω is so low that it can be neglected in comparison with $\omega_{ce} \equiv -(eB_0)/(m_e c)$ and ω_{pe} (negligible electron inertia), there will be $(2+i)$ waves.

In a hot, collisionless, uniform plasma there is an infinity of waves because the dispersion relation involves transcendental functions of the components of the wave number vector \vec{k} , instead of only the first three powers of k^2 as in the warm plasma case; only some of them will be observable and even fewer will be relevant in practice, e.g. the so-called Bernstein modes, which are very slow, electrostatic waves propagating almost across \vec{B}_0 /9.21/. Transcendental functions enter for two reasons:

- 1) only particles whose velocity along \vec{B}_0 is sufficiently close to a resonant value (see (9.20)) are noticeably affected by a low-amplitude wave with given ω and k_{\parallel} .

Since the strength of such an interaction is controlled by the slope of the equilibrium distribution function (a Maxwellian) at the resonant velocity, the plasma dispersion function is introduced.

- 2) Because of the Larmor excursions of the particles across \vec{B}_0 , the identity (9.21) introduces series of Bessel functions of argument $\frac{1}{2}(k_\perp v_{t\alpha}/\omega_{c\alpha})^2$.

If a hot, collisionless plasma is spatially non-uniform, a Fourier analysis in space and time doesn't lead to a dispersion relation but, rather, to an integral equation. For example, in the case of a Cartesian slab where the z -coordinate is along \vec{B}_0 , the equilibrium particle distribution functions depend on x and \vec{v} according to

$$\eta_\alpha e^{-v^2/v_{t\alpha}^2}, \quad \text{with } \alpha = i, e \quad (9.58)$$

η_α being a function of the constants of motion $(x + v_y/\omega_{c\alpha})$ /9.22/. Then, if it is assumed for simplicity that $v_{te}|\frac{\partial}{\partial x}\eta_e| \ll |\omega_{ce}|\eta_e$ and that ions are infinitely heavy, one finds

$$4\pi\vec{j}(\vec{k}, \omega) \approx i\omega \left[\vec{E}(\vec{k}, \omega) - \hat{\epsilon}(\vec{k}, \omega) \int_{-\infty}^{\infty} \eta(k_x - k'_x) \vec{E}(\vec{k}', \omega) dk'_x \right], \quad (9.59)$$

where $\hat{\epsilon}$ is the dielectric tensor for the corresponding hot, collisionless, *uniform* plasma. In the case of waves in the electrostatic approximation ($\vec{E} = -grad \Phi$) the final result is remarkably simple

$$\Phi(\vec{k}, \omega) = \left(A(\vec{k}, \omega) - 1 \right) \int_{-\infty}^{\infty} \eta(k_x - k'_x) \Phi(k'_x, k_y, k_z, \omega) dk'_x \quad (9.60)$$

where, for $\vec{k} = (k_x, 0, k_z)$,

$$A(\vec{k}, \omega) \equiv k_x^2 \epsilon_{11}(\vec{k}, \omega) + 2k_x k_z \epsilon_{13}(\vec{k}, \omega) + k_z^2 \epsilon_{33}(\vec{k}, \omega). \quad (9.61)$$

$A(\vec{k}, \omega) = 0$ is the dispersion relation in the uniform case (from $0 = div(\hat{\epsilon} \cdot \vec{E}) = (\vec{k} \cdot \hat{\epsilon} \cdot \vec{k})\Phi$).

An important consequence of (9.60) is that the “polarization” i.e. the ratio E_x/E_z , is no longer given by the k_x/k_z -ratio (as in the uniform case). Such a non-uniformity effect can have far reaching consequences, especially on linear wave mode conversion, as it will be mentioned later on in relation to Bernstein waves and their coupling to vacuum radiation.

In the geometric-optics limit $k^{-1} \ll L$, the scale length of the plasma inhomogeneity, a successful method of dealing with wave propagation problems in realistic hot plasma configurations is to evaluate the wave group-velocity vector field (ray tracing) by integrating (in general numerically) the equations: $d\vec{r}/dt = -(\partial D/\partial \vec{k})/(\partial D/\partial \omega) \equiv \vec{v}_g$, $d\vec{k}/dt = (\partial D/\partial \vec{r})/(\partial D/\partial \omega)$, where $D(\omega(\vec{k}, \vec{r}), \vec{k}, \vec{r}) = 0$ is the local dispersion relation /9.23/.

The cold (inhomogeneous) plasma case is now considered in some detail. Taking wave amplitudes proportional to $e^{-i\omega t}$ and normalizing lengths to c/ω yields

$$\text{rot } \vec{E} = i\vec{B}, \quad (9.62)$$

$$\text{rot } \vec{B} = -i \hat{\epsilon} \vec{E}. \quad (9.63)$$

With orthogonal coordinates (x_1, x_2, x_3) having x_3 along \vec{B}_0 , the non-vanishing components of the dielectric tensor $\hat{\epsilon}$ are /9.24/

$$\begin{aligned} \epsilon_{11} = \epsilon_{22} &\equiv \epsilon_1 = 1 - \sum_{\alpha} \omega_{p\alpha}^2 / (\omega^2 - \omega_{c\alpha}^2), \\ \epsilon_{12} = -\epsilon_{21} &\equiv \epsilon_2 = -i \sum_{\alpha} \omega_{c\alpha} \omega_{p\alpha}^2 / \omega (\omega^2 - \omega_{c\alpha}^2), \\ \epsilon_{33} &\equiv \epsilon_3 = 1 - \sum_{\alpha} \omega_{p\alpha}^2 / \omega^2. \end{aligned} \quad (9.64)$$

(neglecting unity in ϵ_1 and ϵ_3 is equivalent to neglecting the displacement current).

In a uniform plasma with assumed wave amplitude proportional to $e^{in_{\perp} x_1 + in_{\parallel} x_3}$ the DR is

$$\epsilon_1 n_{\perp}^4 - [(\epsilon_1 + \epsilon_3)(\epsilon_1 - n_{\parallel}^2) + \epsilon_2^2] n_{\perp}^2 + \epsilon_3 [(\epsilon_1 - n_{\parallel}^2)^2 + \epsilon_2^2] = 0. \quad (9.65)$$

Solutions of (9.65) not involving square roots exist for purely perpendicular and purely parallel propagation as well as in the $\vec{B}_o = 0$ case ($\epsilon_1 = \epsilon_3$, $\epsilon_2 = 0$). Useful approximate solutions, which hold strictly if $n_{\parallel} = 0$, or $\epsilon_3 \rightarrow -\infty$ (which is the case when electron inertia is negligible, i.e. when $\omega \ll \omega_{LH}$, see (9.78)), are

$$(\epsilon_1 - n_{\parallel}^2) n_{\perp x}^2 = [(\epsilon_1 - n_{\parallel}^2) + i\epsilon_2] [(\epsilon_1 - n_{\parallel}^2) - i\epsilon_2] \quad (9.66)$$

(extraordinary waves), and

$$\epsilon_1 n_{\perp o}^2 \approx \epsilon_3 (\epsilon_1 - n_{\parallel}^2), \quad (9.67)$$

(ordinary waves).

When $\epsilon_3 \rightarrow -\infty$, (9.67) reduces to $n_{\parallel}^2 = \epsilon_1$. Approximations (9.66) and (9.67) hold as long as $n_{\perp o}^2 \gg n_{\perp x}^2$; in particular, propagation is parallel when the RHS of (9.66) vanishes and when $\epsilon_3 = 0$ (see (9.65)). From eqs. (9.62 to 64) one can deduce the following polarization properties:

$$\text{for X-waves: } E_1/E_{\parallel} \approx -n_{\parallel} n_{\perp} / (\epsilon_1 - n_{\parallel}^2 + \epsilon_2^2 / (\epsilon_1 - n_{\parallel}^2)) \rightarrow 0,$$

$$\text{for O-waves: } B_2/B_{\parallel} \approx n_{\parallel} \epsilon_2 / n_{\perp} (n_{\parallel}^2 - \epsilon_1 + \epsilon_1 n_{\perp}^2 / \epsilon_3) \rightarrow 0.$$

As far as the polarization of the transverse electric field is concerned, one has in all cases

$$E_1/E_2 = [\epsilon_1 - (n_{\perp}^2 + n_{\parallel}^2)] / \epsilon_2. \quad (9.68)$$

If besides electron inertia also the displacement current can be neglected ($v_A \ll c$), then both equations (9.66) and (9.67) take a particularly simple form in a single ion species plasma: $n_{\parallel}^2 = \epsilon_1$ becomes

$$(k_{\parallel} v_A / \omega)^2 = \omega_{ci}^2 / (\omega_{ci}^2 - \omega^2) \equiv A, \quad (9.69)$$

and (9.66) becomes /9.25/

$$(k_{\perp} v_A / \omega)^2 (A - (k_{\parallel} v_A / \omega)^2) = \left(A - (k_{\parallel} v_A / \omega)^2 + (A^2 - A)^{\frac{1}{2}} \right).$$

$$\cdot \left(A - (k_{\parallel} v_A / \omega)^2 - (A^2 - A)^{\frac{1}{2}} \right). \quad (9.70)$$

A plot of $(k_{\perp} v_A / \omega)^2$ versus $(k_{\parallel} v_A / \omega)^2$ for the two cases $\omega < \omega_{ci}$ ($A > 1$) and $\omega > \omega_{ci}$ ($A < 0$) is shown in Fig. 9.3.

The MHD limit, $\omega \ll \omega_{ci}$, corresponds to $A \rightarrow 1$. From eqs. (9.67 and 70) it is seen that only one kind of wave can propagate across \vec{B}_0 . This is a consequence of the fact that in the $\epsilon_3 \rightarrow -\infty$ limit, since the parallel component of the displacement current is negligible, the radial components of the electric part of Maxwell's tensor do not depend on the radius.

The fact that, under the specified conditions, X-waves have $E_{\parallel} \approx 0$ while O-waves have $B_{\parallel} \approx 0$ can be exploited to derive decoupled differential equations for the X and O-waves in a moderately inhomogeneous plasma, when the WKB approximation can be used along \vec{B}_0 :

$$\vec{B}_0 \cdot \text{grad } f \approx i n_{\parallel} B_0 f. \quad (9.71)$$

From $E_{\parallel} = 0$ and the x_1 and x_2 -components of (9.62) for the X-wave we obtain

$$B_1 = -n_{\parallel} E_2; \quad B_2 = n_{\parallel} E_1. \quad (9.72)$$

The x_1 and x_2 -components of (9.63) then give

$$\begin{aligned} \left[(\epsilon_1 - n_{\parallel}^2)^2 + \epsilon_2^2 \right] E_1 &= i \left[-\epsilon_2 \frac{\partial}{\partial x_1} + (\epsilon_1 - n_{\parallel}^2) \frac{\partial}{\partial x_2} \right] B_{\parallel}, \\ \left[(\epsilon_1 - n_{\parallel}^2)^2 + \epsilon_2^2 \right] E_2 &= -i \left[(\epsilon_1 - n_{\parallel}^2) \frac{\partial}{\partial x_1} - \epsilon_2 \frac{\partial}{\partial x_2} \right] B_{\parallel}. \end{aligned} \quad (9.73)$$

By disregarding the space derivatives of $\hat{\epsilon}$ except for those of $(\epsilon_1 - n_{\parallel}^2)$ which can vanish, expressions (9.73) inserted into the x_3 -component of (9.62) yield an equation which we put into the intrinsic form

$$\text{div}_{\perp} \left[(\epsilon_1 - n_{\parallel}^2) \text{grad}_{\perp} B_{\parallel} \right] + \left[(\epsilon_1 - n_{\parallel}^2)^2 + \epsilon_2^2 \right] B_{\parallel} = 0, \quad (9.74)$$

since in a uniform isotropic plasma ($\epsilon_2 = 0$) the equation has to be of the Helmholtz type.

From $B_{\parallel} = 0$, from the x_1 and x_2 -components of (9.63) and from the x_1 and x_2 -components of (9.62) we obtain for the O-wave

$$(\epsilon_1 - n_{\parallel}^2)E_1 = in_{\parallel} \frac{\partial}{\partial x_1} E_{\parallel} - \epsilon_2 E_2, \quad (9.75)$$

$$(\epsilon_1 - n_{\parallel}^2)E_2 = in_{\parallel} \frac{\partial}{\partial x_2} E_{\parallel} + \epsilon_2 E_1. \quad (9.76)$$

When inserted into the x_3 -component of (9.63) these expressions by disregarding the $x_{1,2}$ derivatives of ϵ_2 and $(\epsilon_1 - n_{\parallel}^2)$ and again using $B_{\parallel} = 0$, yield an equation that we put into the intrinsic form

$$\text{div}_{\perp}(\epsilon_1 \text{grad}_{\perp} E_{\parallel}) + \epsilon_3(\epsilon_1 - n_{\parallel}^2)E_{\parallel} = 0, \quad (9.77)$$

since for $n_{\parallel}^2 \gg |\epsilon_1|$ the equation has to be the equation for an electrostatic wave. According to a previous remark, (9.77) holds when $n_{\parallel}^2 \rightarrow \epsilon_1$ only if $\epsilon_3 \rightarrow -\infty$.

When the WKB approximation is also made along the other coordinate - say x_2 - on the magnetic surface, eqs. (9.74 and 77) become ordinary differential equations in the radial direction. They can always be put into the canonical form $d^2y/dx^2 + k^2(x)y = 0$, where $y(x)$ is the product of an appropriately chosen function of x and of the field component under consideration.

Points where one has $k^2(x) = 0$ separate a region of wave propagation ($k^2 > 0$) from a region of evanescence ($k^2 < 0$). They are called cut-offs (C-points) since they produce at least partial reflection of the wave energy flux. In a *nonuniform* plasma no general simple statement can be made about the number and position of the C-points, since they depend upon the behaviour of the ϵ 's as functions of x . In a uniform plasma the C-points are immediately found by setting $n_{\perp} = 0$ in (9.65). Energy transmission through an evanescence region of finite width is called tunnelling /9.26/.

The physical significance of the points where the coefficient in front of the B_{\parallel} - and E_{\parallel} -derivatives in eqs. (9.74 and 77) goes to infinity is simply a statement of wave polarization: in space such points are usually quite close to C-points.

Points where one has $k^2 \rightarrow \infty$ are called resonances (R-points): they are the only places where energy can be absorbed in the limit of zero collision frequency (see next section). In the case of perpendicular propagation only the X-wave has R-points, these being given by $\epsilon_1 = 0$. When $n_{\parallel} \neq 0$, it is the O-wave which has R-points when $\epsilon_1 = 0$, while (9.74) now has R-points when $\epsilon_1 = n_{\parallel}^2 \neq 0$. However, (9.74) is correct when $\epsilon_1 \rightarrow n_{\parallel}^2 \neq 0$ only in the limit $\epsilon_3 \rightarrow -\infty$, and it is only in this limit, i.e. when ω is so low as to make electron inertia negligible, that we can speak of a resonance at $\epsilon_1 = n_{\parallel}^2 \neq 0$. In all other cases, the points where one has $\epsilon_1 = n_{\parallel}^2$ are close to the points where $\epsilon_1 = 0$, and there are no longer two decoupled waves, because the two phase velocities have comparable values. These physically significant points are called linear mode conversion or turning points (T-points) since the RF energy can pass from one wave to the other. Discussion of the T-points is held over till the next section.

The resonances occurring for $\epsilon_1 = 0$ are generally called hybrid resonances. There is one hybrid resonance frequency above ω_{ce} and one between every two consecutive gyrofrequencies of the various particle species of the plasma. Thus, in the important case of a two-ion plasma (e.g. D-T or H-D) there are three hybrid resonance frequencies:

- 1) the upper hybrid (UH) resonance frequency

$$\omega^2 = \omega_{UH}^2 = \omega_{pe}^2 + \omega_{ce}^2 + O\left(\frac{m_e}{m_i}\right), \quad (9.78)$$

- 2) the lower hybrid (LH) resonance frequency

$$\omega^2 = \omega_{LH}^2 = \sum_i \omega_{pi}^2 / (1 + \omega_{pe}^2 / \omega_{ce}^2) + O\left(\frac{m_e}{m_1}\right), \quad (9.79)$$

where it is assumed that $\omega_{ci}^2 \ll \omega_{pi}^2$, and, with the same assumption,

3) the ion-ion hybrid (IIH) resonance frequency

$$\omega^2 = \omega_{IIH} = \omega_{c1}\omega_{c2} \frac{n_1 m_1 + n_2 m_2}{n_1 m_2 + n_2 m_1} + O\left(\frac{m_e}{m_i}\right), \quad (9.80)$$

where $n_i (m_i)$ is the number density (mass) of the ion of species $i=1,2$. Equation (9.80) depends on the ratio n_1/n_2 , but is independent of the plasma density.

In a single-ion-species plasma the $\epsilon_1 = n_{\parallel}^2 \neq 0$ resonance of the X-wave occurs, in the limit of (9.70), at the frequency

$$\omega^2 = \omega_{ci}^2 / (1 + (\omega_{pi}/ck_{\parallel})^2). \quad (9.81)$$

This is called (rather improperly) either the shear Alfvén resonance or the perpendicular ion-cyclotron resonance /9.21/. In a two-ion-species plasma there are two solutions to the equation $\epsilon_1 = n_{\parallel}^2 \neq 0$ in the ionic frequency domain. If $[\omega_{c1}^2, \omega_{c2}^2] \ll \sum_i \omega_{pi}^2 \ll (ck_{\parallel})^2$, they are close to ω_{c1}^2 and ω_{c2}^2 . If $\sum_i \omega_{pi}^2 \gg [(ck_{\parallel})^2, \omega_{c1}^2, \omega_{c2}^2]$, one solution corresponds to a resonance at the frequency

$$\omega^2 \approx (k_{\parallel} c \omega_{c1} \omega_{c2})^2 / (\omega_{c1} \omega_{p2}^2 + \omega_{c2} \omega_{p1}^2), \quad (9.82)$$

which is related to (9.81). The other, very close to ω_{IIH}^2 , corresponds to a T-point.

We conclude this section by considering wave propagation in a warm plasma in the MHD limit ($\omega \ll \omega_{ci}$) in the case of a diffuse pinch. In a Cartesian case one has for the equilibrium configuration

$$\vec{B}_o = (0, B_{oy}(x), B_{oz}(x)); \quad \frac{d}{dx}(p_o(x) + |\vec{B}_o(x)|^2/8\pi) = 0. \quad (9.83)$$

By introducing the radial plasma displacement $\xi(\vec{r}, t) = \xi(x)e^{i(k_y y + k_z z - \omega t)}$ with $\frac{\partial}{\partial t} \xi = v_x$, and a similar dependence for the perturbation of the total pressure so that

$$P_1(x) = p_1(x) + \vec{B}_o(x) \cdot \vec{B}_1(x)/4\pi, \quad (9.84)$$

one finds

$$A(x)\xi(x) = \frac{d}{dx}P_1(x),$$

$$P_1(x) = [A(x)R(x)/C(x)] \frac{d}{dx}\xi(x), \quad (9.85)$$

where

$$A(x) = 4\pi m_i n_o(x) - (\vec{k} \cdot \vec{B}_o(x))^2,$$

$$R(x) = v_A^2(x) + c_S^2(x) \left[1 - (\vec{k} \cdot \vec{v}_A(x)/\omega)^2 \right],$$

$$C(x) = (k_y^2 + k_z^2)R(x) - \omega^2. \quad (9.86)$$

These equations show that, while there are now 3 kinds of waves propagating along \vec{B}_o , there is still only one type of wave propagating across \vec{B}_o . This is because the kinetic pressure only appears in the combination (9.84). Notice that $R(x) = 0$ gives R-points for $\xi(x)$, while $C(x) = 0$ gives C-points for $\xi(x)$. The equation $A(x) = 0$ gives resonances for $\xi(x)$. These would not appear, however, in a WKB treatment, while the $R(x) = 0$ resonance would. Notice that equation $A(x) = 0$ gives the cut-offs for $P_1(x)$. Notice also that the incompressible limit of eqs. (9.85 and 86) is obtained by setting $(R(x)/C(x)) \rightarrow 1$ (from $k^2(v_A^2 + c_S^2) \gg \omega^2$).

The cylindrical equivalent of eqs. (9.83-86) has been derived in /9.27/.

9.7 Resonances and RF Energy Flow

By definition, the wave resonance frequencies are the frequencies at which one has $k \equiv |\vec{k}| \rightarrow \infty$. From the Maxwell equations it follows that k tends to infinity as $\vec{k} \cdot (\vec{B}_0 \times \vec{j}_1)/k (\vec{B}_0 \cdot \vec{B}_1)$ does. At the resonances (9.78) to (9.82) (at which $k_\perp \rightarrow \infty$) it is $(\vec{B}_0 \cdot \vec{B}_1)$ which tends to zero, while for $\vec{k} \times \vec{B}_0 = 0$ the dispersion relation $(\epsilon_1 - n_\parallel^2)^2 + \epsilon_2^2 = 0$ (which is the condition for having $B_{\parallel 1} \neq 0$) gives $|k_\parallel| \rightarrow \infty$ as a result of $|j_\perp| \rightarrow \infty$. This is the only case of non vanishing plasma density where the cold-plasma wave resonance frequencies coincide with the particle gyro-frequencies. If one has $\vec{k} \times \vec{B}_0 \neq 0$, the preferential action of the wave on one of the particle species — depending on the $(\omega - \omega_{c\alpha})$ values — gives rise to an ambipolar \vec{E} -field, which minimizes charge separation by coupling the collective motion of the various particle species. That the $|k_x/k_\parallel| \rightarrow \infty$ resonance frequencies have to differ from the particle gyro-frequencies is seen by comparing the limit $j_x \rightarrow i\omega E_x/4\pi$ given by the Maxwell equations for $(k_x/k_\parallel) \rightarrow \infty$ with the limit $j_x \rightarrow e_{\bar{\alpha}} n_{o\bar{\alpha}} v_{\bar{\alpha}x}$ valid in a cold plasma (to first order in \vec{E}) when $\omega \rightarrow \omega_{c\bar{\alpha}}$. As in this case $|v_{\bar{\alpha}x}/E_x| \rightarrow \infty$ (nonrelativistically), the two limiting j_x values are incompatible.

Further insight in the nature of the various kind of resonances is gained by considering the ratio of $E^+ \equiv E_x + iE_y$, (the \vec{E}_\perp -component rotating in the sense of the ion gyromotion), to $E^- \equiv E_x - iE_y$ (the \vec{E}_\perp -component rotating in the sense of the electron gyromotion). In a cold plasma one has, from (9.62 and 63)

$$E^+/E^- = (n^2 - \epsilon_1 + i\epsilon_2)/(n^2 - \epsilon_1 - i\epsilon_2). \quad (9.87)$$

This condition shows that, when $n_\perp^2 \rightarrow \infty$, one has $E_y/E_x \rightarrow 0$ (linear polarization), while when $\omega \rightarrow \omega_{ci}$ one has $E^+/E^- \propto (\omega_{ci} - \omega) \rightarrow 0$ (i.e. \vec{E}_\perp rotates in the sense of the *electron* gyromotion), and when $\omega \rightarrow \omega_{ce}$ one has $E^+/E^- \propto (\omega_{ce} - \omega)^{-1} \rightarrow \infty$ (i.e. \vec{E}_\perp rotates in the sense of the *ion* gyromotion). The two last results play

a fundamental role in the mechanism underlying ion and electron cyclotron heating in toroidal devices (see below). The frequencies given by eqs. (9.78 to 82) are the eigenfrequencies of the collective oscillations of the cold plasma *that do not propagate energy in the x -direction* (at resonance, either the group velocity of these oscillations is parallel to the resonance surface or it vanishes). For instance, when one has $\omega \ll \omega_{ci}$, the group velocity is equal to the Alfvén velocity vector $\vec{B}_0/\sqrt{4\pi m_i n_0}$: in this limit the collective oscillations of the plasma are essentially those of a collection of elastic strings stretched along \vec{B}_0 .

At the LH resonance (9.79) the control of charge separation is a subtle process which does not involve the main part of the electron motion — the $\vec{E} \times \vec{B}_0$ drift, which is divergence-free when $\vec{E} \times \vec{k} = 0$ — but, rather, the $m_e c^2 \vec{E}/e B_0^2$ correction term.

We now consider the problem of power absorption in the case of a wave incident on a plasma profile where it encounters an R-point. The fact that for real ω eqs. (9.74 and 77) are singular at the R-points is due to the fact that no dissipation mechanism is included in the equations. However, absorption is formally obtained from their singular solutions if, when integrating over real space, these are interpreted as generalized functions of $(\omega_R(x) - \omega)^{-1}$. The power absorbed is thus independent of ν , as in the Landau damping case. The underlying physical picture is the following. If we examine the group velocity, we discover that t_R , the time it takes the wave energy to approach the R-point when starting from any point at a finite distance from it, is infinitely long /9.24/ in the sense that $t_R = \lim_{(\nu=0)} \delta/\nu$, where $\delta \leq 1$ is a constant. The RF power density would therefore become infinite at the R-point ($|\vec{E}|$ tending to infinity) unless energy is dissipated by some mechanism. If one has $\delta \approx 1$, Coulomb collisions in the immediate neighbourhood of the R-point provide the required mechanism. If one has $\delta \ll 1$, a different effect must be looked for. This is provided by linear mode con-

version to a warm (or hot) plasma wave since an R-point of a cold plasma is always the point toward which the T-point of a warm (or hot) plasma tends when the temperature goes to zero. This is illustrated in Fig. 9.4, which is a plot of $k_x^2 \equiv k_x^2(k_y^2, k_z^2, \omega; x)$ as derived by a local dispersion relation. The important point here is that the square of the phase velocity of the warm (or hot) plasma wave steadily decreases as the distance from the T-point increases. Thus, if the phase velocity at the T-point is still too high to allow substantial wave energy dissipation there, the RF energy is diverted away from the T-point until the wave reaches a plasma region where the conditions for efficient damping are met. There is no reflection when the wave encounters the R (or T) -point but there can be some transmission beyond such a point, in the evanescence region. ZASLAVSKII et al. in /9.28/ used the theory of the solutions of differential equations of the type $\alpha \frac{d^4 y}{dx^4} + \beta(x) \frac{d^2 y}{dx^2} + \gamma(x)y = 0$ with a small parameter α preceding the highest-order derivative, in order to handle *quantitatively* the wave transformation which takes place at the T-point, $\beta^2 = 4\alpha\gamma$. A direct application of these techniques to the hybrid resonances is due to STIX /9.29/.

Let us now briefly discuss power flow and absorption when two C-points are present on a plasma profile at some distance from an R-point. Figure 9.5a is an example of a $C_1 - R - C_2$ triplet where the R-point separates propagation from evanescent regions. This situation is encountered at low frequency ($\omega^2 < \omega_{ci}^2$, see (9.70) and Fig. 9.3, and eqs.(9.74 and 85)) when one has $n_2^2 < |\epsilon_1 - n_{\parallel}^2|$. The energy flow in the case of a wave incident from the right is calculated in /9.30/, where the power absorption is found to vanish with the distance $C_1 - C_2 \sim (\omega/\omega_{ci})$. A wave first encountering the C_2 -point is to a large extent reflected there. As a result of tunnelling through the evanescent region some energy reaches the R-point and is absorbed there, and the rest is transmitted beyond this point. If $n_2^2 \gg |\epsilon_1 - n_{\parallel}^2|$, the R-point is located inside the propagation region (Fig. 9.5b and /9.13/).

Let us consider the consequences of (9.70) (Fig. 9.3) for n_2^2 small in some detail, in relation to plasma heating. The existence of k_{\parallel}^2 -intervals where $k_{\perp}^2 > 0$ makes eigenmodes within the plasma torus possible (see /9.31/ for experimental results in the ion cyclotron range of frequencies). They occur when the toroidal wave number is such that

$$k_{\phi}R = n = 0, 1, 2, \dots, \quad \text{and} \quad J_m(k_r a) \approx 0 \quad (9.88)$$

where R and a are the major and minor radius of the plasma, respectively, and J_m is a Bessel function.

If $\omega < \omega_{ci}$ there are two C-points, and one R-point given by (9.81); if $\omega > \omega_{ci}$ there is one C-point and no R-point.

In a toroidal plasma the quantity $k_{\parallel}v_A$, which is proportional to

$$\vec{k} \cdot \vec{B}_0 \equiv (nB_{\phi}/R + mB_{\theta}/r) \equiv (nq + m)B_{\theta}/r \quad (9.89)$$

(q is the usual safety factor or inverse rotational transform) is a function of r which vanishes at the MHD singular surfaces $q(r) = -m/n$. Thus it may happen that the R-point ($A = (k_{\parallel}v_A/\omega)^2$) for $\omega < \omega_{ci}$, occurs within the plasma even if at the plasma periphery one has $k_{\parallel}(a)v_A(a) \gg \omega$. Of course the R-point may occur within the plasma even if k_{\parallel} never vanishes as in the original proposal of plasma heating by a resonant Alfvén wave at $\omega < \omega_{ci}$ /9.32/. The theory, which includes finite Larmor radius effects, indicates that the R-point becomes a T-point of mode conversion to a "kinetic" Alfvén wave, $\omega^2 \approx (k_{\parallel}v_A)^2(1 + 2(k_x v_{ti}/\omega_{ci})^2)$ for $T_e \approx T_i$, which is cut-off on the lower density side of the T-point, but propagates on the higher density side. The theoretical prediction is that as long as there is one R-surface or two R-surfaces well separated in space the fraction of the available reactive power which can be absorbed (mainly via electron Landau damping) should be substantial. Experimental and theoretical results are reviewed in /9.33/, where both ω/ω_{ci} and equilibrium plasma current effects are retained to reproduce the structure of the observed eigenmodes spectra.

Plasma heating in the ion cyclotron range of frequencies (ICRH) (eqs. (9.66, 70 and 80)) requires antenna parallel wave numbers such that $|k_{\parallel}|c \ll \omega_{pi}$, the larger wave numbers being strongly evanescent between the antenna and the C-surface at the periphery of the plasma for the waves under consideration (see Fig. 9.6). As in the cold plasma limit heating can only occur at a R-surface, cold plasma heating is only possible if there are 2 (or more) ion species, at the ion-ion hybrid R-surface (9.80). As shown in Fig. 9.6, such a R-surface is only accessible to waves launched from the high B_o -field side of the torus. This overall picture remains true also in arbitrary axisymmetric configurations, although the location of resonances (9.80 to 82) is shown in /9.34/ to differ from the one in the one-dimensional model. In a finite temperature plasma there is a multifarious ICRH scenario: not only E^+ is no longer strictly zero at gyroresonance, but also harmonic cyclotron heating and selective minority species cyclotron heating become possible which, as they do not rely upon the presence of the R-surface (9.80), can be more conveniently produced by waves launched from the low B_o -field side of the torus. Moreover, as the parameter space of interest (especially in fusion devices) involves relatively high temperatures and densities, low k_{\parallel} and $\omega \approx N\omega_{ci}$, mode conversion to ion Bernstein waves becomes an ingredient of ICRH. The various possibilities, reviewed by J. ADAM together with the experimental results /9.35/, are:

- a) *fundamental ICRH*: in the parameter space of interest, E^+/E^- , although increased in comparison with the one-dimensional model when the presence of the rotational transform is accounted for /9.15/, is too small to give appreciable heating.
- b) *Second harmonic ICRH*: in spite of the usually very small value of the $k_x v_{ti}/\omega_{ci}$ ratio, the fraction of the field polarized in the ion gyration direction is large enough to allow significant heating (in a preheated plasma). If $\left(\frac{\omega_{pi}}{k_{\parallel}c}\right)^2 \left(\frac{k_{\parallel}v_{ti}}{\omega_{ci}}\right) \leq 1$ one has $E^+/E^- \approx 1/3$ /9.36/, while in the opposite case Bernstein waves play a dominant role, owing to the presence of a T-point. Harmonic ICRH has the distinctive

advantage that the RF power is directly deposited, in a fusion device, to a reacting ion species, thus leading to enhanced reactivity.

- c) *Minority ICRH*: as first indicated by TAKAHASHI /9.37/ in the case of a D-H plasma, if the density ratio is such that

$$n_H/n_D < \frac{k_{\parallel} v_{tH}}{\omega} \left((1 - (\omega_{cD}/\omega)^2)^{-1} + (k_{\parallel} c/\omega_{pD})^2 \right),$$

E^+/E^- at $\omega = \omega_{cH}$ is large enough for the main part of the RF power to be absorbed before reaching the C-layer where it would be partially reflected; the absorption is due to ion cyclotron damping of the minority component, rather than due to electron Landau damping of mode-converted Bernstein waves. These conditions lead /9.25/ to a strong distortion of $f_H(\vec{v})$ with respect to the Maxwellian with consequent heating of electrons and majority ions by collisional equipartition.

In the electron cyclotron range of frequencies ($\omega^2 \gg \omega_{ci}\omega_{ce}$) the location of the C and R points of perpendicularly propagating O and X waves on the $\{(\omega_{pe}/\omega)^2, (\omega_{ce}/\omega)\}$ plane is shown in Fig. 9.7. When $0 < n_{\parallel}^2 < 1$ (we do not consider slowed down waves as, to have substantial tunnelling in this frequency range, the launching structure would have to be placed only some mm apart from the plasma), and $n_2 = 0$ both the C-surface of the O-wave and the C-surface of the X-wave remain the same as for $n_{\parallel} = 0$, while the R-surface of the X-wave remains the same provided that the abscissa ω_p^2 be substituted by $\omega_{pe}^2/(1 - n_{\parallel}^2)$. As previously shown, in the cold plasma limit heating can only occur at a R-surface. Because of the 1/R dependence of B_o , the UH-R surface (see (9.78)) is accessible only to X-waves launched from the high field side. The X-wave is deflected towards regions of weaker B_o -amplitude up to a T-surface, where it mode-converts to an electron Bernstein wave which is eventually cyclotron damped /9.29/. However, heating tokamaks in thermonuclear regimes at $\omega = \omega_{UH} > \omega_{ce}$ is severely demanding in terms of frequency of the (high power) RF source, as $(\omega_{ce}/2\pi)$ measured in unity of 100 GHz is approximately $0.28B_o$, when

B_o is measured in Tesla. For this same reason heating at harmonics of ω_{ce} is unrealistic in fusion devices. On the other hand, in a hot plasma the E^- components of O and X-waves with $n_{\parallel}^2 \neq 0$ are not fully screened at the $\omega = \omega_{ce}$ surface, so that substantial electron cyclotron absorption is possible. While the O-waves can only penetrate up to the density where $\omega_{pe}^2 = \omega^2$ — a condition again severely demanding in terms of frequency in fusion devices: the frequency measured in 100 GHz should be larger than $0.9 n^{\frac{1}{2}}$, the density being measured in 10^{14} cm^{-3} — the X-waves, launched from the low B_o -field side in the equatorial plane with $\omega > \min\{\omega_{ce}\}$, can penetrate (see Fig. 9.7) up to the density where $\omega_{pe}^2 = (1 - n_{\parallel}^2)(\omega^2 + \omega\omega_{ce})$, which is higher than the cutoff density of the O-wave if $(1 + \omega/\omega_{ce})n_{\parallel}^2 < 1$. However, as in the corresponding ICRH case, the damping of a X-wave at $\omega = \omega_{ce}$ scales unfavourably with the density and becomes negligible in regimes of thermonuclear interest. Fortunately, as FIDONE et al. have pointed out /9.38/, already in plasmas with electron temperatures of a few keV there are enough electrons obeying the *relativistic* resonance condition, $\omega(1 - n_{\parallel}v_{\parallel}/c) = \omega_{ce}(1 - (v/c)^2)^{\frac{1}{2}}$, to produce strong absorption at the substantially down-shifted frequency $\omega \leq (\omega_{ce}^2 - \omega_{pe}^2)^{\frac{1}{2}}$ (e.g. $\omega/\omega_{ce} \sim 0.75 - 0.85$); moreover, under this condition the wave damping is relatively density independent. For a number of electron cyclotron resonance heating (ECRH) experiments see /9.39/.

Of fundamental if not of thermonuclear interest (except for possible diagnostic purposes) is the emission of high harmonics of ω_{ce} . First observed from a Penning discharge (up to about the 25th harmonics /9.40/) such an emission has been related /9.41/ (on an indication by D. PFIRSCH) to the excitation of electron Bernstein waves propagating in proximity of $\omega = N\omega_{ce}$, almost perpendicular to \vec{B}_o with a very short wavelength: $k_{\perp}v_{te} \geq N\omega_{ce}$. The energy of such electrostatic waves can be efficiently radiated into a vacuum because of the existence, in the very neighbourhood of each of the harmonics, of a T-point of k_x^2 . The result, even in a moderately non uniform plasma, is an abrupt

radial dependence of k_x^2 (for $\omega \approx N\omega_{ce}$) which is the requirement for substantial coupling to vacuum waves. Note that, as a relevant approximation of the dispersion relation would have to involve terms of the kind $k_{\perp}^{2n}/(k_{\perp}^2 + k_{\parallel}^2) [\text{const.} + k_{\perp}^{n+1/2}]$, a quantitative treatment of such a linear mode conversion problem, based on approximating *differential* equations (obtained by substitution of $\partial/\partial x$ for ik_{\perp}) would involve derivatives of a correspondingly high order and, thus, would be less appropriate than the correct *integral* equation introduced at the beginning of Sect. 6.

It remains to consider the case of the lower hybrid resonance (9.79). Although both the X- and the O-waves are resonant at this frequency — the former in the limit of strictly perpendicular propagation — only the O-wave has practical applications, since resonance (9.79) is inaccessible to an X-wave from the low-density side (Fig. 9.5c and 5d for a wave incident from the left). As a matter of fact, for the frequencies under consideration the C-point of the O-wave is at the very edge of the plasma:

$$(\omega_p(x)/\omega)^2 \approx 1 + k_y^2 / (k_{\parallel}^2 - (\omega/c)^2); \quad (9.90)$$

from here inward k_x^2 is real and positive up to the point where condition

$$|n_{\parallel}| > \omega_{pe}/\omega_{ce} + \sqrt{\{1 + (\omega_{pe}/\omega_{ce})^2 - (\omega_{pi}/\omega)^2\}} \quad (9.91)$$

is no longer satisfied (ϵ_1 , the expression in the curly brackets, is assumed to be non-negative). The condition for the accessibility of the LH-R-point is /9.42/ $n_{\parallel}^2 > 1 + (\omega_{pe}/\omega_{ce})^2$ for $\omega_{LH} = \omega$. When the accessibility condition is violated, there is an interval $T_1 < x < T_2$, $T_{1,2}$ being *T*-points (see Fig. 9.8), in which the two roots k_x^2 of the dispersion relation (9.65) are complex conjugate and the wave is (strongly) evanescent. Although a positive k_x^2 on the high-density side of the C-point (9.90) implies radial wave evanescence on the other side, this happens on an interval which can in practice be kept optically thin enough to allow efficient tunnelling from the antenna.

The most successful LH wave antenna is the grill /9.43/, a phased array of waveguides (mounted flush on the liner) with their small side in the \vec{B}_0 -direction and excited in the fundamental TE_{01} mode so as to create an electric field essentially parallel to \vec{B}_0 and to concentrate most of the RF-power in the part of the k_{\parallel} -spectrum satisfying the accessibility condition (the other part of the spectrum being trapped near the vacuum wall and eventually absorbed, resulting at best in low-grade heating). For a complete theory of the linear radiation properties of the grill see /9.44/. To a first approximation, within the plasma the waves (slowed down along \vec{B}_0) are electrostatic and obey the simple dispersion relation $\omega^2 = \omega_{LH}^2 (1 + (k_{\parallel}/k_{\perp})^2 m_i/m_e)$. As one has $\vec{k} \cdot \frac{\partial}{\partial \vec{k}} \omega = 0$, the group velocity trajectories of these waves (which, by the way, are backward waves) are independent of the wave numbers, so that the field radiated by a finite-length antenna tends to concentrate around patterns of constructive interference, called resonance cones /9.45/.

These are the main ingredients which, together with the linear mode conversion process previously introduced (when the LH-R-point is present), were the theoretical basis for the LH wave experiments and for the expectations. In particular, with the additional assumption that D_{QL}^e and D_{QL}^i are large (in the sense of the discussion in Sect. 5; in this section and the following one the usual approximation of D_{QL}^e is used, i.e. the D_{QL}^e for an single wave, given by (9.39), are superposed in accordance with a given k_{\parallel} -spectrum of waves) in the respective velocity ranges $(\omega/k_{\parallel}^u) < v_{e\parallel} < (\omega/k_{\parallel}^l)$ and $(\omega/k_{\perp}^u) < v_{i\perp} < (\omega/k_{\perp}^l)$, where the upper and lower ends are those corresponding to the extension of the *launched* k_{\parallel} -spectrum, the expectation was that P_e is controlled by the $e^{-(\omega/k_{\perp}^u v_{te})^2}$ factor (with a figure of merit for current drive proportional to $\ln(k_{\parallel}^u/k_{\parallel}^l) \left[(k_{\parallel}^l)^{-2} - (k_{\parallel}^u)^{-2} \right]^{-1}$, see (9.56)) and P_i by the $e^{-(\omega/k_{\perp}^u v_{ti})^2}$ factor.

The experiments /9.46/, with a variety of wave launchers and under different plasma conditions, have consistently shown, however, that the most substantial effects

are produced in current drive and bulk electron heating scenarios by injecting waves with considerably lower k_{\parallel}^u -values than expected, and that bulk ion heating occurs also at considerably lower plasma densities than expected (especially in deuterium plasmas). There is, in addition, strong experimental evidence of an unexpected density effect: above a certain density limit, which depends on frequency, the LH wave effects within the plasma vanish. The whole issue is briefly considered in the next section.

9.8 The Lower Hybrid Wave Problem

A most unexpected and exemplary result of the LH wave experiments is that a substantial fraction of the tokamak current can be driven with high efficiency (see (9.56)) by launching LH waves which are inadequate to interact strongly with enough particles to be fully absorbed since they are within a narrow, high phase velocity range. Apparently, as the LH waves progress towards the plasma core there is a physical mechanism which more than compensates for the progressive linear depletion of the low phase velocity part of the launched spectrum (if any) by producing out of the narrow high phase velocity range a new k_{\parallel} -spectrum which extends to the required higher $|k_{\parallel}|$ values. In the following a brief account is given of a plausible mechanism, which is described in detail in /9.47/.

In a Cartesian uniform plasma slab of period $2\pi R$ in the z -direction (which is along \vec{B}_0), and in the linear electrostatic approximation, the wave pattern consists of parallel straight rays which project the boundary values of the RF field amplitude into the plasma (resonance cone; see previous section). As the amplitude, although small, is finite, the ponderomotive effect modifies the density as $n = n_0(1 - \gamma M)$ where, in this case, one has $\gamma \equiv \omega_p^2 / 4\pi n_0 \epsilon_1 (T_i + T_e) \omega^2$ and $M \equiv |E_z|^2$, with γM being typically of the order of $10^{-2} - 10^{-3}$. The dependence of $\hat{\epsilon}$ and k_{\perp} on the density entails a dependence on M , so that one gets the following non linear differential equation for

M :

$$\epsilon_1(3M) \frac{k_{\perp}(3M)}{k_{\parallel}} \frac{\partial}{\partial x} M + \epsilon_3(3M) \frac{\partial}{\partial z} M = 0. \quad (9.92)$$

The ponderomotive effect changes the slope of the rays, which are thereby focused or defocused, depending on the sign of $\frac{\partial}{\partial z} M$, according to the implicit equation

$$M = M^{\circ}(z + x k_{\perp}(3M)/k_{\parallel}), \quad (9.93)$$

which is the formal solution of (9.92). Here $M^{\circ}(z) \equiv M(x=0, z)$ is the given value of $|E_z|^2$ at $x=0$ and $k_{\perp}^2(M)/k_{\parallel}^2$ is given by $-\epsilon_3(M)/\epsilon_1(M)$ (E_y is assumed to be zero at the grill mouth). Equation (9.93) immediately reveals the nonperiodicity of wave propagation in the (radial) x -direction, due to the existence of a secular effect which was discovered in a somewhat different context by KARNEY /9.48/. Eventually the convergent rays build up a caustic surface, i.e. the surface $(x = x_c(\tau), z = z_c(\tau); -\infty < \tau < \infty)$ where $|\text{grad } M| \rightarrow \infty$:

$$1 + \frac{\partial}{\partial \tau} (M^{\circ}(\tau)) 3\gamma \frac{x}{k_{\parallel}} \frac{\partial}{\partial n_o} k_{\perp} = 0,$$

$$z + x k_{\perp}(3M^{\circ}(\tau))/k_{\parallel} - \tau = 0. \quad (9.94)$$

Only the rays corresponding to $\tau < 0$ (in the case of right-going rays) build up a caustic surface within the plasma; this surface consists of two branches which merge in a cusp line; they correspond to the intervals $\tau < \tau_o$ and $\tau_o < \tau < 0$ respectively, where τ_o is the negative value of τ such that $\partial^2 M^{\circ}(\tau)/\partial \tau^2 = 0$. For a single-humped M° , a reasonable assumption, the caustic has the form sketched in Fig. 9.9. To within a form factor the distance of the cusp from the plasma surface, x_{min} , coincides with the self-focusing distance derived in /9.48/. With realistic M° , x_{min} is found to be a small fraction of the plasma minor radius. Figure 9.9 also shows the existence of a "downstream" region where the solution of (9.92) is multi-valued; this happens because wave absorption has been disregarded. However, when convergent rays start

building up a caustic surface, the increased gradient of $|E_z|^2$ displaces the lower velocity end of the quasilinear diffusion coefficient towards still lower velocities (see (9.50)). Although D_{QL} decreases at the same time, an increase of $\frac{\partial}{\partial x}|E_z|^2/|E_z|^2$ by a factor of 2-3 is enough for *total* absorption because of the *exponential* increase in electron Landau and/or ion Karney damping. The multi-valuedness of the solution of (9.92) is then consistently removed by including a δ -like energy sink (resulting from an anti-Hermitian part of the otherwise Hermitian dielectric tensor, which is δ -like as a result of the explosive growth of P when $|grad M| \rightarrow \infty$) centred on the surface $x = x_0(z)$ determined by the condition $|grad M| \rightarrow \infty$. This energy sink ensures that when the rays pass through it, they bend, thus being focused, and fade as the transported energy concomitantly undergoes total absorption. The x_0 surface is found from the jump conditions that the Maxwell equations imply for such a case; again it consists of two branches which extend radially to infinity and merge in a cusp line. One of the branches obeys the same differential equation as the Hermitian caustic, while the other branch is a new surface. With a proper choice of the integration constant the first branch is made to coincide with the $\tau < \tau_0$ branch of the Hermitian caustic. The second branch is described by the values $\tau_0 < \tau < \infty$, and joins the first branch at the same cusp line as in the Hermitian problem: it is a bow-wave-like surface induced by the first branch (Fig. 9.9). All the rays entering the plasma from the RF source intercept the x_0 discontinuity surface because, at variance with the Hermitian situation, $x_0(\tau)$ is within the plasma also for $\tau > 0$ (extending radially to infinity for $\tau \rightarrow \infty$). As a result, the whole RF power available is deposited along the discontinuity surface, typically on the first passage of the wave rays through the plasma.

The question of the distribution of the absorbed power between ions and electrons requires cumbersome evaluation of the quasilinear diffusion coefficients. The following is simply a general remark on the expected effect of the form of the k_{\parallel} -spectrum on the

heating of ions and electrons with reference to a frequently used argument /9.49/. In the expressions for the power densities absorbed by ions and electrons, P_i and P_e , there is an exponential dependence only on $(\omega/k_{\perp}^u v_{ti})^2$ and $(\omega/k_{\parallel}^u v_{te})^2$. The equipartition equation $P_i = P_e$, once written as

$$(k_{\perp}^u v_{ti}/k_{\parallel}^u v_{te})^2 = 1 + \ln F, \quad (9.95)$$

can thus be directly used to determine with logarithmic accuracy the density for “switch-over” from electron to ion heating. The point is that the function F depends on the various wave and plasma parameters but is such that for most experimental conditions the value of the RHS of (9.95) is between 1 and 2; moreover, the dispersion relation is approximately electrostatic, so that the LHS of (9.95) contains the equilibrium quantities and the wave frequency but not the wave numbers. Equation (9.95) can therefore provide the value of the “switch-over” density even if the actual k_{\parallel} -spectrum in the plasma has to be different (so to speak, “upshifted”) from the antenna spectrum to give substantial absorption. However, the validity of this density determination rests on the tacit assumption that the k_{\parallel} -spectrum of the LH wave within the plasma is such as to produce virtually single-humped diffusion coefficients D_{QL}^i and D_{QL}^e . The point here is that the diffusion coefficients D_{QL}^i and D_{QL}^e corresponding to a LH pump with a given range of k_{\parallel} ’s have different extents along the relevant velocity component. A rectangular k_{\parallel} -spectrum line gives a rectangular D_{QL}^e in $v_{e\parallel}$, but results in a rectangular D_{QL}^i in $v_{i\perp}$ with an additional high-velocity part which decreases as slowly as $v_{i\perp}^{-2}$ down to the value $D_{QL}^i \approx \frac{\nu_i v_{ti}^2}{2v^3}$ (see (9.44)), and hence appreciably enlarges the plateau of the distribution function. Thus, if the actual k_{\parallel} -spectrum has separate lines, as expected from the narrow-line spectra of antennae with many waveguides /9.44/, ion heating should be favoured with respect to the prediction of (9.95) because the ion distribution function would have a *plateau-like* enhancement over a broad velocity range, while over the corresponding velocity range the electron distribution function would

only have a *stair-like* enhancement.

In many experiments (see /9.46/) density limits have been observed (a few times 10^{13} cm^{-3} for $f \approx 1 \text{ GHz}$) above which it was not possible to achieve efficient plasma heating and/or current drive. Their existence is explained in the framework of the theoretical model described in /9.47/ as being due to the destruction of the discontinuous wave pattern at the x_0 surface by wave interference when the extent of that surface along the toroidal direction becomes sufficient to allow rays with different slopes (e.g. fast EM- waves) to cross. The density at which the slow and fast rays cross at some point $r = r_c$ is given by

$$\left| \frac{d}{dk_{\parallel}} k_{\perp} \right|_{\text{slow}} - \left| \frac{d}{dk_{\parallel}} k_{\perp} \right|_{\text{fast}} = 2\pi R/r_c. \quad (9.96)$$

When (9.96) has no positive solutions for (ω_p^2/ω^2) at the plasma centre ($r_c = r_p$), there is no density limit. A tabular comparison with many experiments supports the suggested theoretical model, particularly in respect of the weak m_i -scaling of the density limit, which cannot be recovered with (9.95).

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FIGURE CAPTIONS

- 9.1 Phase-space trajectories in a coordinate system moving with a low-frequency wave with $B_{1\parallel} = -bB_0 \cos(kz - \omega t)$.
- 9.2 Schematic plot of the heating rate γ_H versus the collision frequency ν in a wave-particle resonance case (upper curve) and in gyrorelaxation (lower curve).
- 9.3 Schematic plot of $(k_{\perp} v_A / \omega)^2$ versus $(k_{\parallel} v_A / \omega)^2$ in a single ion species plasma for $\omega > \omega_{ci}$ (lower curve) and $\omega < \omega_{ci}$ (upper curve).
- 9.4 Schematic plot of k_x^2 versus $x \equiv (\omega_{pe} / \omega)^2$ in the lower hybrid resonance range of frequencies for ordinary (O) and extraordinary (X)-waves with a sufficiently large $(k_{\parallel} c / \omega)^2$ -value (see (9.91)). As the plasma temperature tends to zero the broken line, which is a warm plasma asymptote, approaches the vertical $x = x_{LH}$.
- 9.5 Profiles of the square of the *radial* wave-number when there is an R-point together with two C-points (cases a and b), with one C-point (case c) and with no C-points (case d).
- 9.6 Geometrical-optics plot of the C and R-curves of the extraordinary wave (9.67) in the ion-ion hybrid resonance range of frequency in a tokamak minor cross-section. The tokamak major axis is to the left of the figure. The wave is evanescent in the shaded region.
- 9.7 The Clemmow-Mullaly-Allis (CMA) diagram /9.24/ for exact perpendicular wave propagation in the upper hybrid resonance range of frequencies. The O-wave is evanescent to the right of $(\omega_{pe} / \omega)^2 = 1$. The X-wave is evanescent in the shaded region.
- 9.8 Schematic plot of k_x^2 versus $x \equiv (\omega_{pe} / \omega)^2$ for a cold plasma in the lower hybrid resonance range of frequencies when the accessibility condition (9.91) is not fulfilled (for the accessible case see Fig. 9.4).

9.9 Schematic plot of 1) the Hermitian caustic branches $x = x_c(\tau)$ merging in a cusp at $\tau = \tau_0$ where $x_c = x_{min}$, and 2) the anti-Hermitian bow-wave-like branch $x = x_o(\tau)$ stemming from the cusp, for a single-humped $M^{(o)}(\tau)$ which vanishes at τ_1 and τ_2 , and has an inflection point at $\tau = \tau_0$. The heavy straight lines are the characteristics of (9.92) for $\gamma = 0$ (limits of the resonance cone), and the light straight lines are the characteristics of (9.92) for $\gamma \neq 0$. In the shaded region the solution of (9.92) is multi-valued. In the hatched region M vanishes.

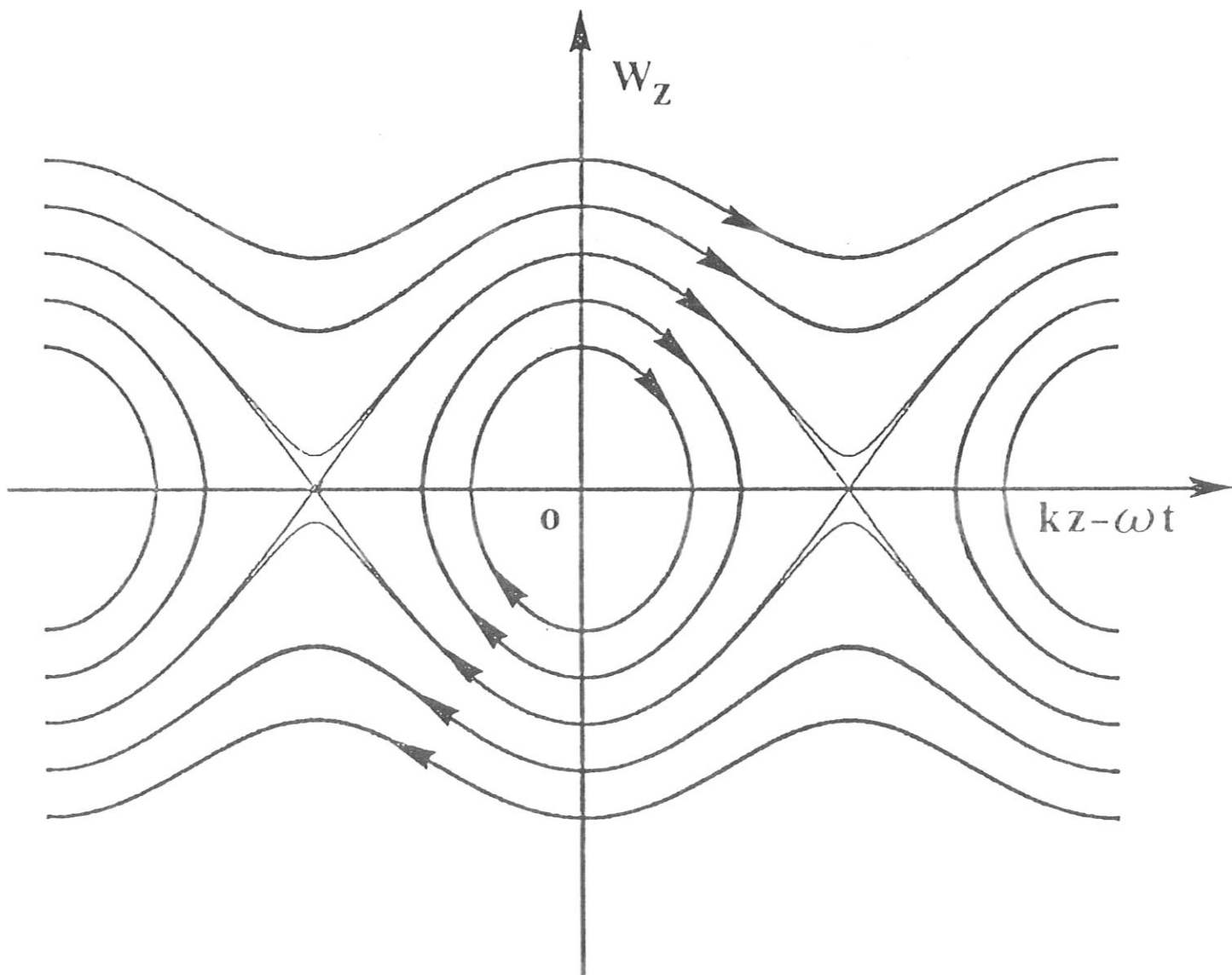


Fig. 1

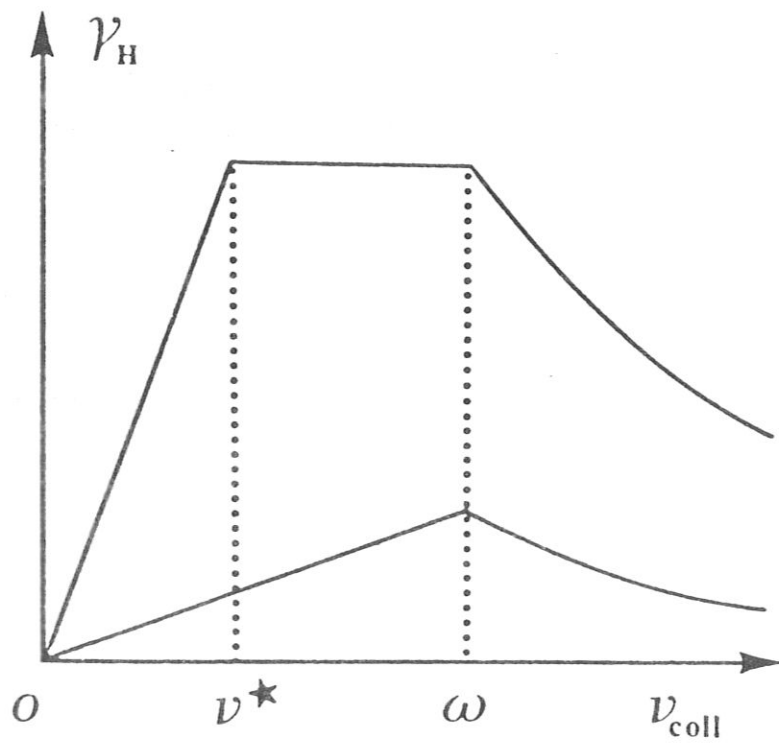


Fig. 2

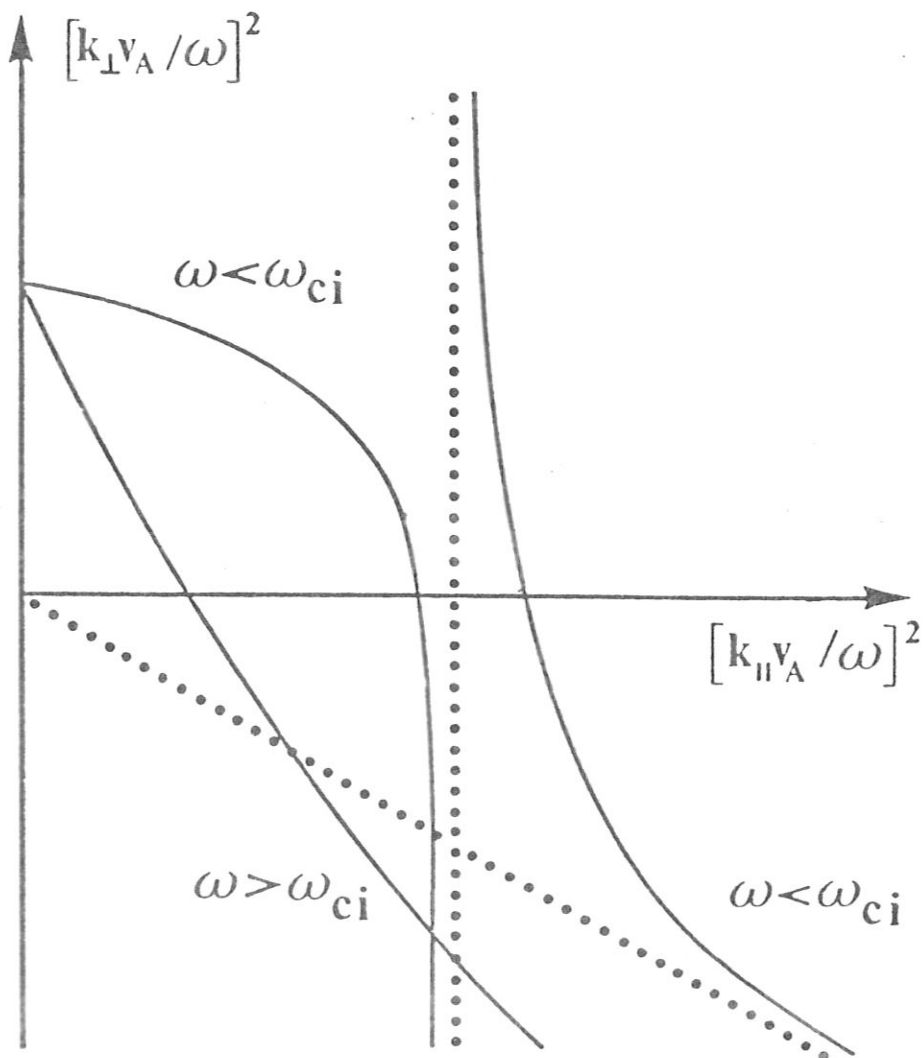


Fig. 3

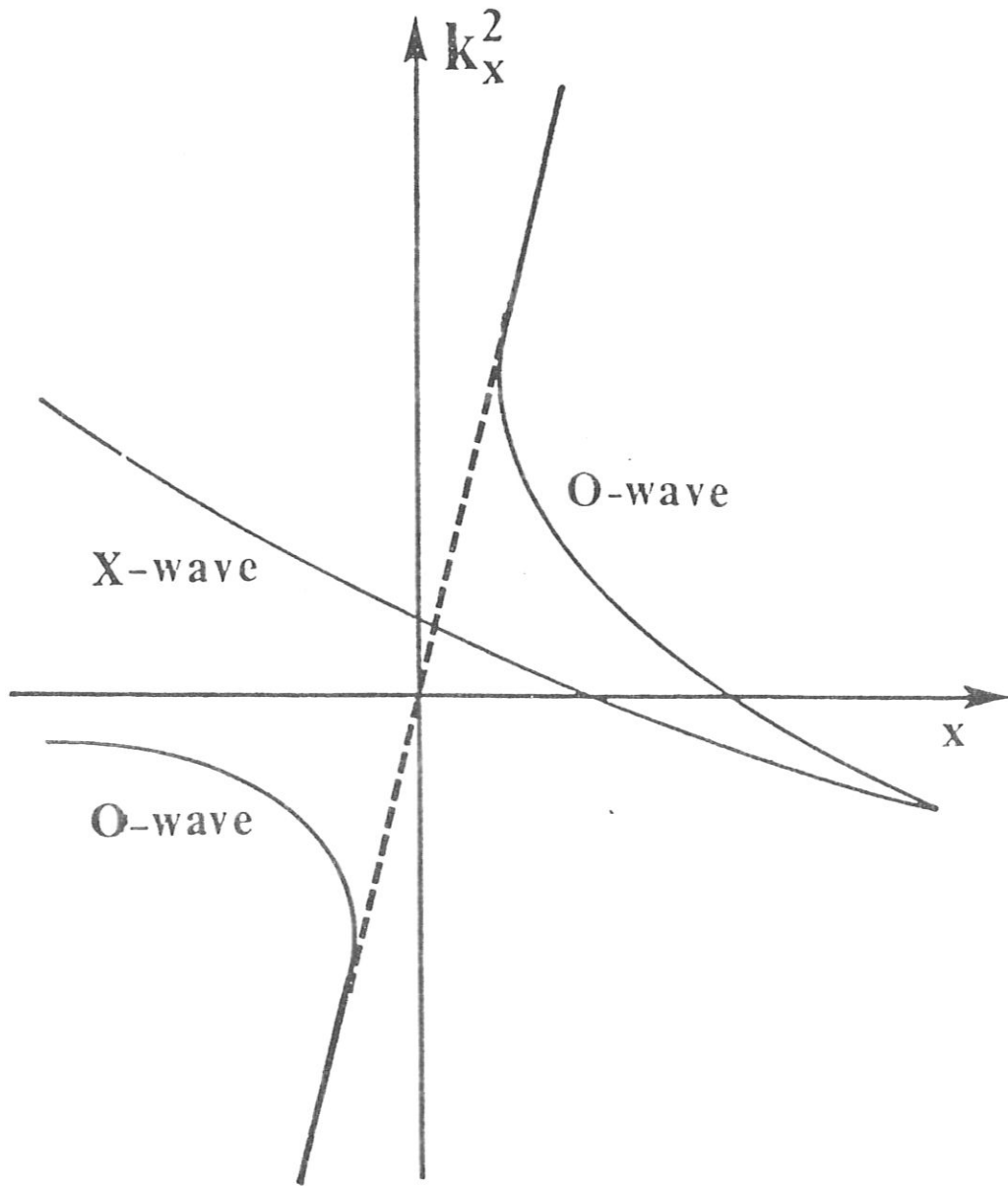


Fig. 4

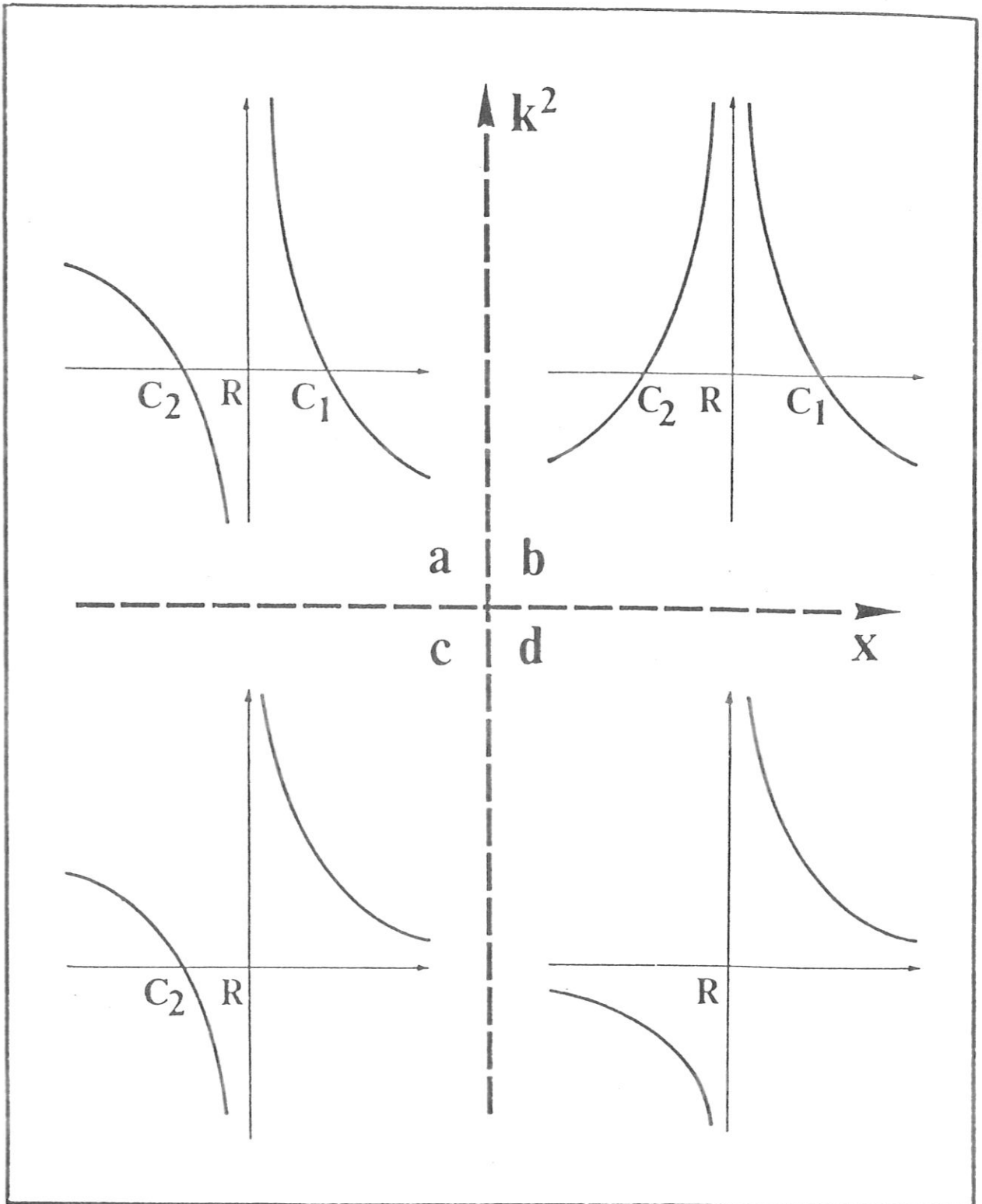


Fig. 5

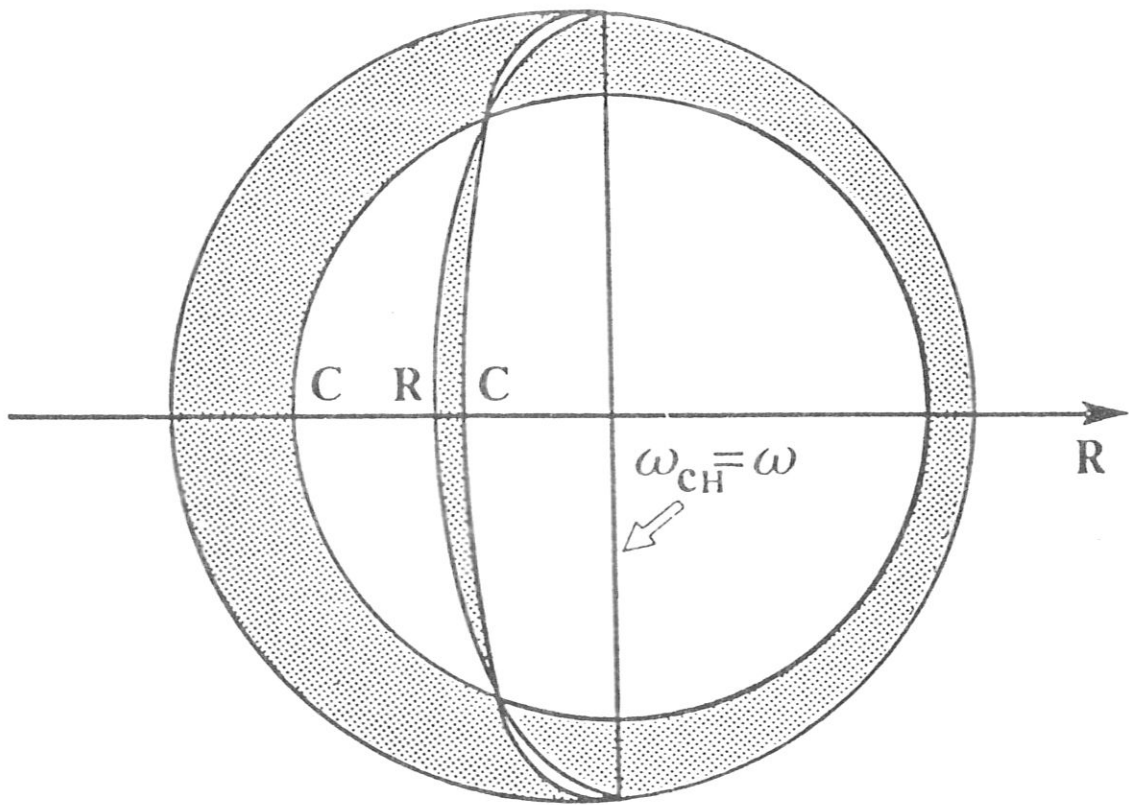


Fig. 6

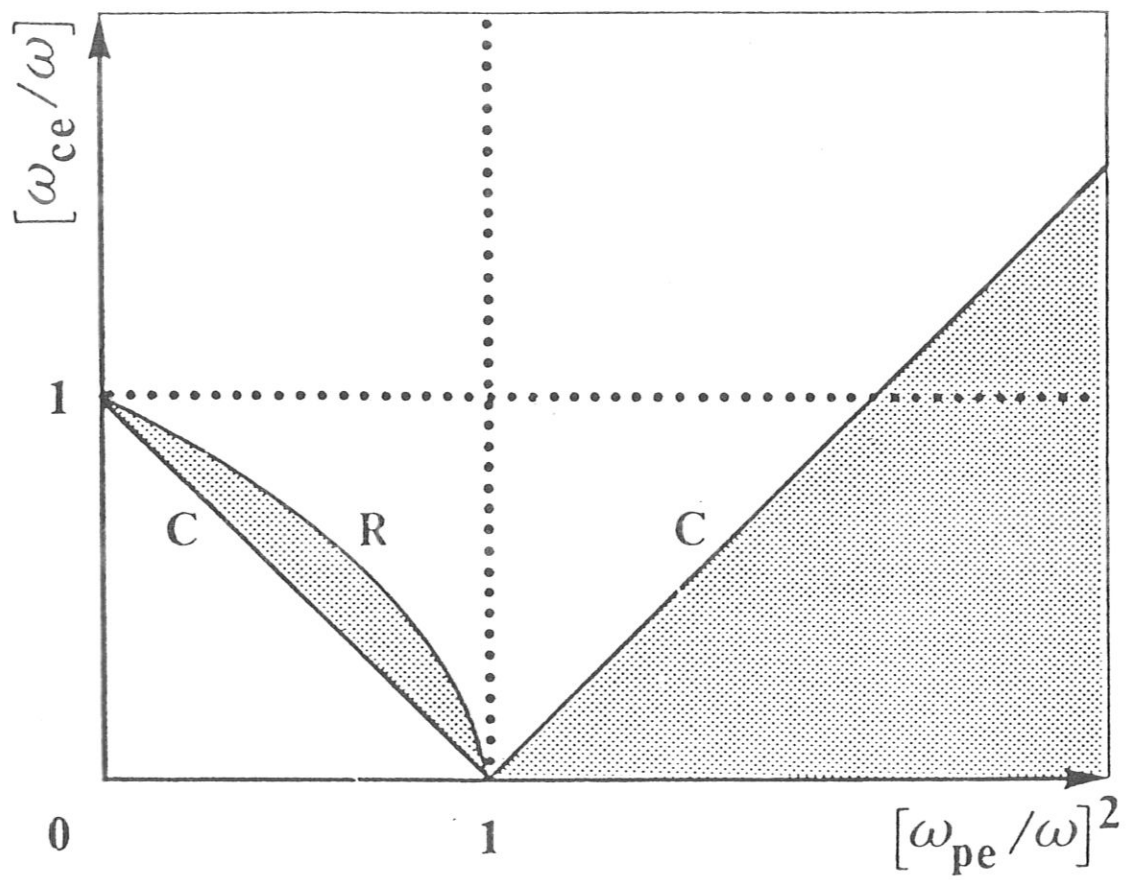


Fig. 7

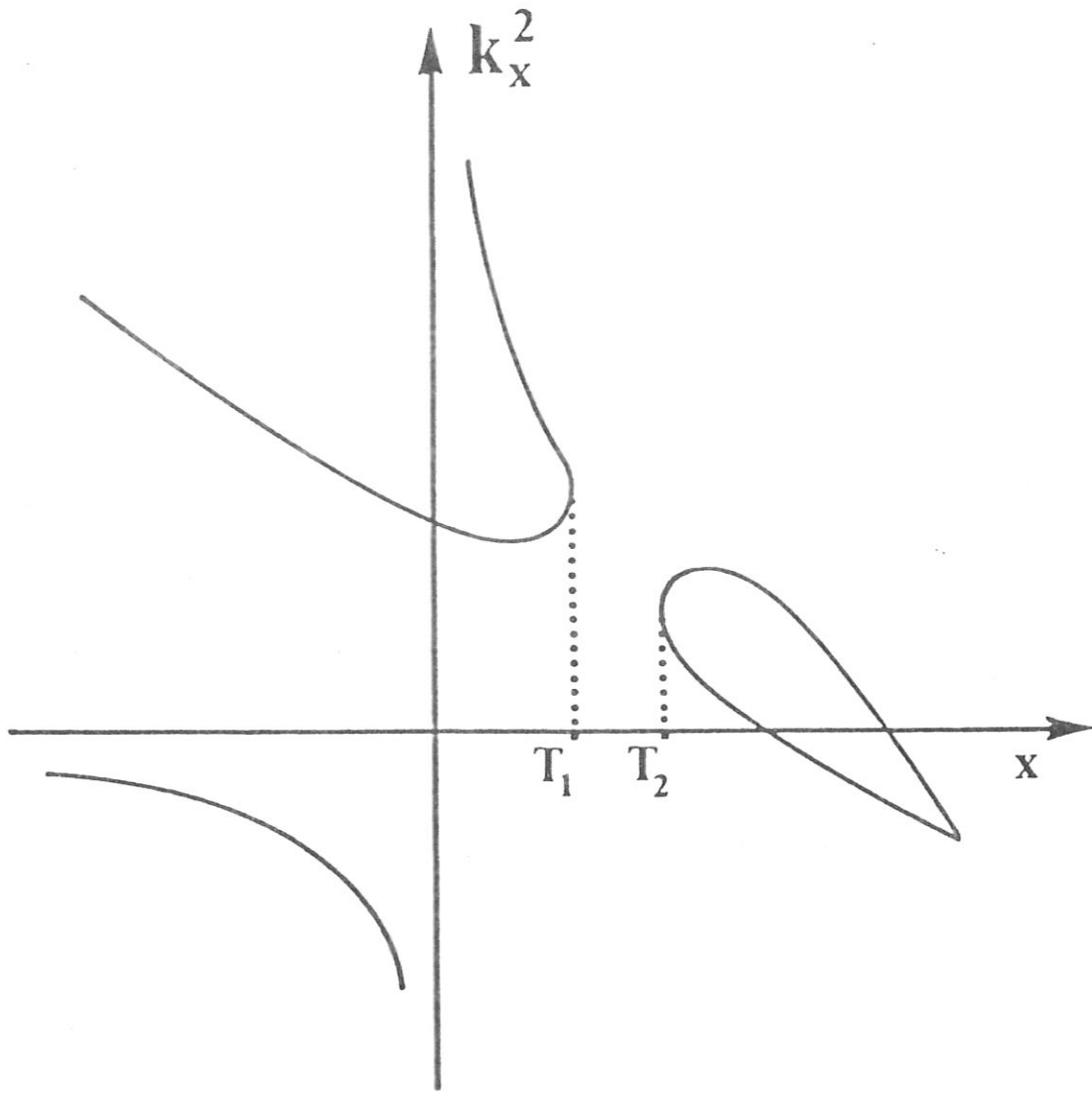


Fig. 8

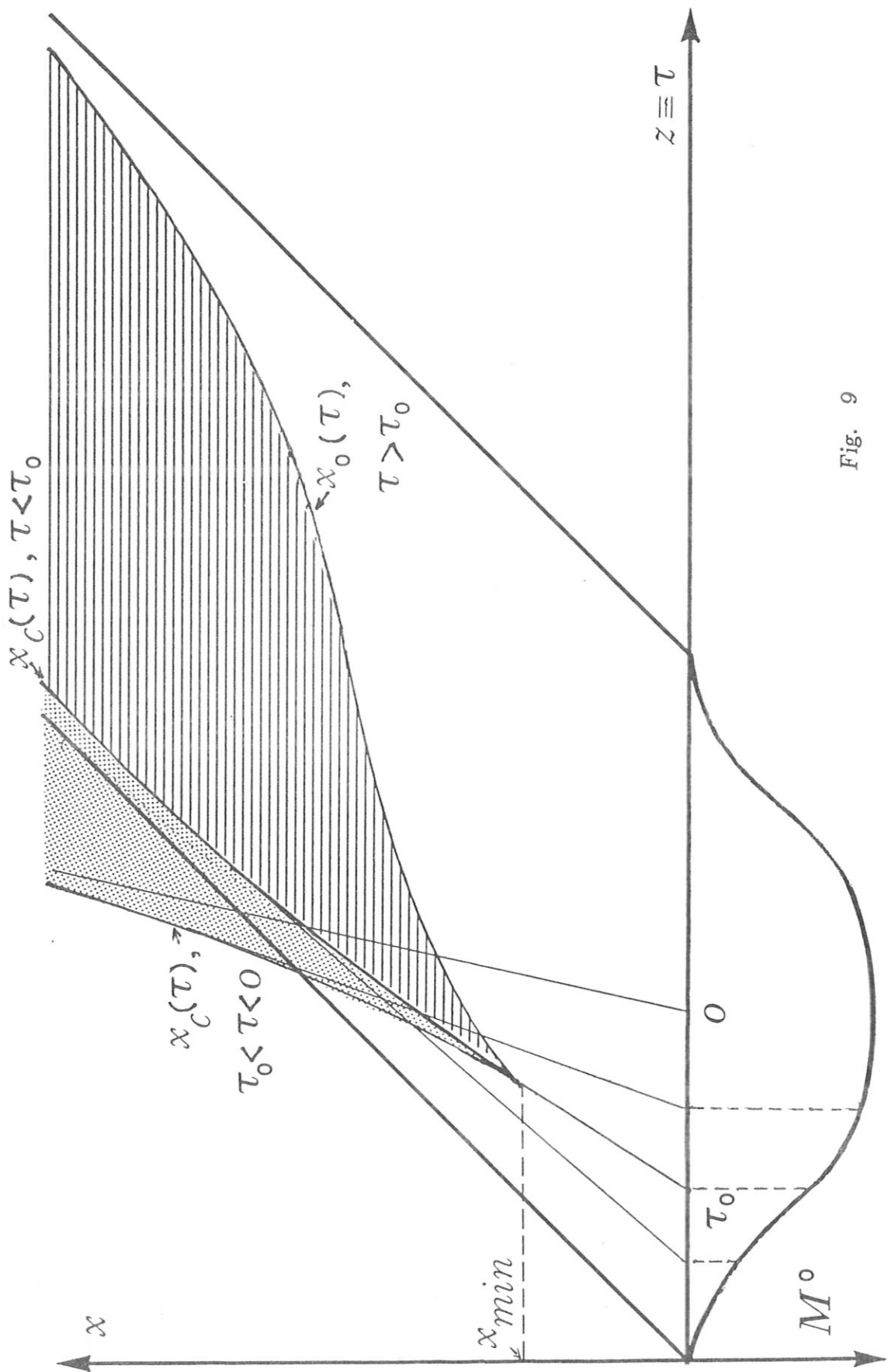


Fig. 9