MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Lyapunov Stability of Large Systems of van der Pol-like Oscillators and Connection with Statistics and Turbulence*

Henri Tasso

IPP 6/277 October 1988

*to be published in Proc. of "III. International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics" (Salice Terme, 26-30 Sept. 1988)

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

IPP 6/277 Henri Tasso

Lyapunov Stability of Large Systems of van der Pol-like Oscillators and Connection with Statistics and Turbulence

October 1988 (in English)

Abstract

For a system of van der Pol-like oscillators, Lyapunov functions valid in the greater part of phase space are given. They allow a finite region of attraction to be defined. Any attractor has to be within the rigorously estimated bounds. Under a special choice of the interaction matrices the attractive region can be squeezed to zero. In this case the asymptotic behaviour is given by a conservative system of nonlinear oscillators which acts as attractor.

Though this system does not possess in general a Hamiltonian formulation, Gibbs statistics is possible due to the proof of a Liouville theorem and the existence of a positive invariant or "shell" condition. The "canonical" distribution of the attractor is remarkably simple despite nonlinearities. Finally the connection of the van der Pol-like system and of the attractive region with turbulence problems in fluids and plasmas is discussed.

I. Introduction

The purpose of the van der Pol equation [1] was to study the nonlinear oscillations of a L-C circuit driven by a triode. The tension at the grid was taken as a solution of the equation [1]

$$\ddot{y} + (y^2-1)\,\dot{y} + y = 0 \,. \tag{1}$$

The term $-\dot{y}$ represents the amplification of the triode while $y^2\dot{y}$ is due to its nonlinear characteristic curve (see for example [2]).

Due to standard theorems [2] of Poincaré, Bendixson, Levinson and Smith the existence of an attracting limit cycle to eq. (1) is known. Practical calculations of the limit cycle are done by means of series expansions and numerical calculations. A typical phase plot is given in Fig. 1.

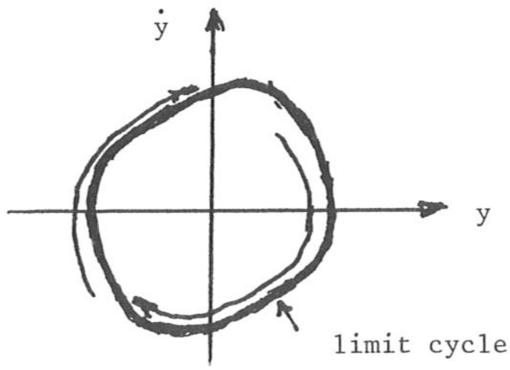

Fig. 1

Let us introduce here a modified van der Pol equation [3] for which the Lyapunov function and the limit cycle can be constructed easily. The modified equation is

$$\ddot{y} + (y^2+\dot{y}^2-1)\,\dot{y} + y = 0 \,. \tag{2}$$

Multiplying eq. (2) by $\dot{y}$ one obtains

$$\frac{1}{2}\frac{\partial}{\partial t}(y^2+\dot{y}^2) = -\dot{y}^2(y^2+\dot{y}^2-1) \; . \tag{3}$$

Due to Lyapunov stability theorems one obtains

Stability if $\dot{y}^2+y^2>1$

Instability if $\dot{y}^2+y^2<1$,

$y^2+\dot{y}^2=1$ being the equation of the limit cycle.

II. Stability of a System of van der Pol-like Oscillators

The existence theorems [2] for limit cycles are restricted to the case of a single oscillator. They cannot be extended to general systems of oscillators, in particular due to the possibility of more complex attractors like "strange" attractors [4,5]. Systems of oscillators of the kind given by eq. (2) turns out to be more tractable as shown by author's work [3] and as explained below. Consider the following system

$$\ddot{Y} + \left[(Y,AY)M+(\dot{Y},B\dot{Y})N-P\right]\dot{Y} + CY = 0 \tag{4}$$

where Y is a real vector of arbitrary length r. A,B,C,M,N,P are real r x r matrices and (...,...) is the scalar product. These matrices can be split in symmetric A_s, B_s... and antisymmetric parts A_a, B_a... .

Assume $C_a = 0$ and A_s, B_s, M_s, N_s, P_s and C be positive definite with largest eigenvalues $\alpha_1, \beta_1, \mu_1, \nu_1, \pi_1, \gamma_1$ and lowest eigenvalues

$\alpha_o, \beta_o, \mu_o, \nu_o, \pi_o, \gamma_o$ respectively. Take the scalar product of eq. (4) by $\dot{Y}$

$$\frac{1}{2}\frac{d}{dt}\left[(\dot{Y},\dot{Y})+(Y,CY)\right] = -\left[(Y,A_sY)(\dot{Y},M_s\dot{Y})+(\dot{Y},B_s\dot{Y})(\dot{Y},N_s\dot{Y})-(\dot{Y},P_s\dot{Y})\right] . \tag{5}$$

The bracket on the left-hand side of eq.(5) is positive definite and is a candidate for a Lyapunov function if the right-hand side can be made to have a definite sign. This is possible if one assumes

$$\gamma_o \geqq \frac{\alpha_1\mu_1}{\beta_1\nu_1} , \qquad \gamma_1 \leqq \frac{\alpha_o\mu_o}{\beta_o\nu_o} . \tag{6}$$

The proof (see [3]) is done by extracting two inequalities from eq. (5) which allow the use of Lyapunov's theorems if conditions (6) are satisfied. This result leads to the definition of an attractive region in the $(\dot{Y},\dot{Y})$, (Y,CY) plane (see Fig. 2).

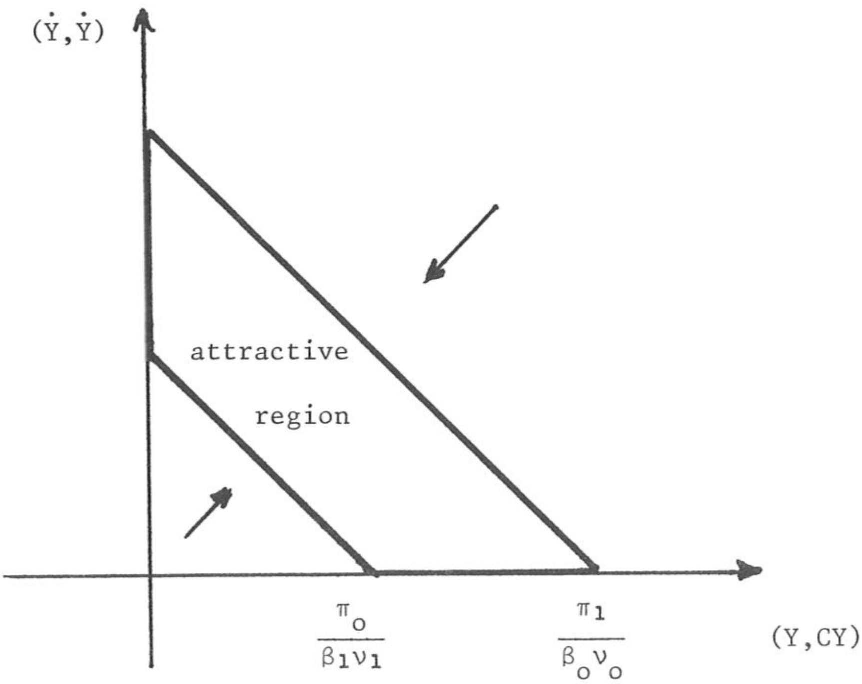

Fig. 2

Under conditions (6) any solution of system (4) will be "trapped" after some time in the attractive region defined by the bounds

$$(\dot{Y},\dot{Y}) + (Y,CY) \geq \frac{\pi_o}{\beta_1\nu_1} \tag{7}$$

and
$$(\dot{Y},\dot{Y}) + (Y,CY) \leq \frac{\pi_1}{\beta_o\nu_o} \quad . \tag{8}$$

The detailed asymptotic behaviour is very difficult to study, and one should expect, in general, a kind of high dimensional "strange" attractor.

If, however, the attractive region is made to shrink to zero in the $(\dot{Y},\dot{Y}),(Y,CY)$ plane, then the attractor of system (4) obeys itself a system of conservative nonlinear oscillators [6] together with a "shell" condition as will be shown below.

III. Special Cases of Attracting Systems

Under a special choice of the matrices A_s,B_s,C_s,M_s,N_s,P_s the attracting region of Fig. 2 can be made to shrink to zero. This choice is

$$\begin{aligned} A_s &= \alpha I,\ B_s = \beta I,\ C_s = \frac{\alpha\mu}{\beta\nu} I \quad , \\ M_s &= \mu I,\ N_s = \nu I,\ P_s = \pi\, I \quad , \end{aligned} \tag{9}$$

where I is the identity matrix and α,β,μ,ν and π are any real positive numbers. Relations (9) leave M_a,P_a and N_a undetermined so that we are led to distinguish between two cases

(A) $$M_a = P_a = N_a = 0.$$

In this case the attracting system is given by

$$\ddot{Y} + \frac{\alpha\mu}{\beta\nu} Y = 0 \tag{10}$$

and $$(\dot{Y},\dot{Y}) + \frac{\alpha\mu}{\beta\nu} (Y,Y) = \frac{\pi}{\beta\nu} \quad . \tag{11}$$

Eqs. (10) and (11) represent a system of r linear oscillators with a "shell" condition on their "amplitudes".

(B) M_a, P_a and N_a are any real antisymmetric matrices.

In this case the attracting system in reduced form is given by

$$\ddot{Y} + \left[(Y,Y)M_a, (\dot{Y},\dot{Y})N_a - P_a\right] \dot{Y} + Y = 0 \tag{12}$$

and the shell condition is

$$(\dot{Y},\dot{Y}) + (Y,Y) = \varepsilon = \frac{\pi}{\beta\nu} \quad . \tag{13}$$

Apart from the case of r = 2 oscillators which is completely integrable (see [7]), system (12) is not expected to be, in general, integrable (see [8]). It is shown in [6] that system (12) does not possess, in general, a Lagrangian formulation in terms of Y and that (13), the only known constant of motion, cannot play the role of a noncanonical Hamiltonian. In the next section it will be shown, however, that Y and $\dot{Y}$ constitute a convenient phase space to describe statistics of system (12) on the "ergodic" shell (13).

IV. Statistics of the Attracting System (12)

A conventional Gibbs statistics is not possible because of the lack of a Hamiltonian (see [6]). If we introduce

$$X = \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix} \quad ,$$

the components x_i of X obey

$$\sum_{i=1}^{2r} x_i^2 = \varepsilon \tag{14}$$

which is identical to the shell relation (13).

It can be shown [6] that system (12) represents an incompressible flow in the phase space X that is

$$\sum_{i=1}^{2r} \frac{\partial \dot{x}_i}{\partial x_i} = 2 \sum_{i,j=r+1}^{2r} x_i n_{ij} x_j = 0 \quad . \tag{15}$$

Liouville's theorem (15) and the shell condition (14) allow us to define a microcanonical distribution if an assumption of "physical" ergodicity is introduced. A canonical distribution could also be derived if the system were in contact with an "amplitude" bath.

This statistics leads to an equipartition in the amplitude expectations of the oscillators. It may be able to model some situations in which noise is observed but not turbulence in hydrodynamics where strongly decaying wave vector spectra are usually observed.

V. Connection with Turbulence and Outlook

System (12) reflects the asymptotic behaviour of system (4) under the choice (9). Choice (9) means that the oscillators are equally excited and damped. In that respect the statistics of system (12) (see section IV), in particular the equipartition of amplitudes is not surprising. If one likes to model turbulence spectra, one should give up choice (9). The oscillators should not be equally excited and not equally damped.

Unfortunately in such a case system (4) would not behave asymptotically as a system having a shell relation like (13) or (14) and a Liouville theorem would not be possible to prove, thus preventing the use of Gibbs distributions. Even the "stationary" statistical problem becomes very tough.

The existence of an attractive region (see section II) is, however, certainly useful especially if one is interested in upper bounds for the amplitude excursions of the oscillators. But a better estimate of the attractive region would be desirable.

The bounds given by (7) and (8) fall far apart according to the difference in eigenvalues like the lowest π_o and the largest π_1.

Since dissipative fluid systems are expected to have asymptotically a large but finite number of determining "modes" (see for example [9]), the existence of attractive regions for large ODE systems and the refinement of their bounds may become a powerful tool in fluid and plasma turbulence.

References

[1] van der Pol B. Phil. Mag. 43, 177 (1922)

[2] Hagedorn P. "Nichtlineare Schwingungen" Akademie Verlagsgesellschaft Wiesbaden (1978)

[3] Tasso H. Z. Naturforsch. 41a, 987 (1986)

[4] Smale S. "Differentiable Dynamical Systems" Bull. Amer. Math. Soc. 73, 747 (1967)

[5] Ruelle D., Takens F. Comm. Math. Phys. 20, 167, (1971)

[6] Tasso H. Z. Naturforsch. 42a, 1377 (1987)

[7] Tasso H. J. of Appl. Math. and Phys. 38, 143 (1987)

[8] Tasso H. Z. Naturforsch. 42a, 1237 (1987)

[9] Nicolaenko B., Scheurer B., Temam R. Physica 16D, 155, (1985)