MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Hysteresis in the Nonlinear Driven Drift and KdV Equations

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IPP 6/274 April 1988

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Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Abstract

The nonlinear driven/damped differential equation

$$\partial\phi/\partial t + a\ \partial^3\phi/\partial t\partial^2 y + b\ \partial^3\phi/\partial^3 y + c\ \partial\phi/\partial y + f\ \phi\partial\phi/\partial y = -\epsilon\sin(Ky - \Omega t) - \gamma\phi$$

with $a = 0$ (KdV equation) or $b = 0$ (drift equation) is numerically studied in a parameter region where the energy tends to a constant, E_s, for $t \to \infty$. It is found that $E_s(\epsilon)$ traces a hysteresis curve when the driving amplitude ϵ is cyclically varied.

1. Introduction

Recently, there has been interest in the properties of nonlinear systems which continuously depend on space and time and are perturbed by periodic external driving forces. For example, see /1 - 3/ for numerical studies of the driven-damped sine Gordon equation and /4/ for the driven-damped nonlinear Schrödinger equation. In /1 - 4/ the transitions chaotic $\leftrightarrow$ nonchaotic, in both space and time, are investigated as functions of the initial conditions and driving amplitudes.

Here, an analogous situation in nonlinear plasma drift wave theory is considered. We report, however, on a nonlinear phenomenon which occurs before chaotic bahaviour sets in: when the driving amplitude ϵ is cyclically increased and decreased, the plasma energy and other properties follow a *hysteresis* curve when the parameters of the driver are in some finite regions of parameter space.

The equation we consider is

$$\frac{\partial \phi}{\partial t} + a \frac{\partial^3 \phi}{\partial t \partial^2 y} + b \frac{\partial^3 \phi}{\partial y^3} + c \frac{\partial \phi}{\partial y} + f\phi \frac{\partial \phi}{\partial y} = - \epsilon \sin(Ky - \Omega t) - \gamma \phi , \qquad (1)$$

with $b = 0$ (case 1) or $a = 0$ (case 2). In case 1 eq. (1) without the driving and damping terms on the right-hand side is the one-dimensional drift equation (also called the regularized Korteweg-deVries (rKdV) equation /6/) as used in, for example, /5/, while in case 2 it is the KdV equation itself. Both equations have solutions in the form of solitary waves or soliton-like structures /6/, but it is only for the KdV equation that the solitary structures are true solitons, i.e. unchanged after collisions. Otherwise both equations are often considered equivalent, with the substitution $b = -ac$, from the long-wavelength linear approximation $\partial\phi/\partial t + c\partial\phi/\partial y = 0$, /6/.

Drift waves /7/ are driven by the gradient of the plasma density n. Their frequency is in the region $\omega \approx \kappa_n c_s << \Omega_i = eB/m_i c$ = ion gyrofrequency, where $\kappa_n = (\mathrm{d}n/\mathrm{d}x)/n$ is the inverse of the density gradient scale length, $c_s^2 = T_e/m_i$, T_e = electron temperature and m_i = ion mass. The usual assumptions for drift waves are that the electrons have a Boltzmann distribution

$$n_e(\mathbf{r},t) \;=\; n(x)\,\exp[\;e\Phi(\mathbf{r},t)/T_e(x)\;]\;, \tag{2}$$

while the ion fluid is governed by the continuity equation

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\mathbf{v}) \;=\; 0\;, \tag{3}$$

and both are related by the quasineutrality condition $n_i = n_e$. For propagation perpendicular to the magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, i.e. $\partial/\partial z = 0$, one has approximately $\mathbf{v}_\perp = \mathbf{v}_E + \mathbf{v}_P$, where $\mathbf{v}_E = -\,\hat{\mathbf{z}}\times\mathbf{E}/B$ and $\mathbf{v}_P = (B\Omega_i)^{-1}\partial\mathbf{E}/\partial t$ are the $\mathbf{E}\times\mathbf{B}$ and polarization drifts, respectively, with $\mathbf{E} = -\nabla\Phi$. In the the one-dimensional limit $k_x(k_y\rho_s)^2 << \kappa_T$ /5/ one obtains eq. (4) for the dimensionless potential $\phi = e\Phi/T_e$:

$$\left(1 \;-\; \rho_s^2\frac{\partial^2}{\partial y^2}\right)\frac{\partial\phi}{\partial t} \;-\; \kappa_n c_s\rho_s\,\frac{\partial\phi}{\partial y} \;+\; \kappa_T c_s\rho_s\,\phi\frac{\partial\phi}{\partial y} \;=\; 0\;, \tag{4}$$

where $\rho_s = c_s/\Omega_i$ and $\kappa_T = (\mathrm{d}T_e/\mathrm{d}x)/T_e$. This is the left-hand side of eq. (1) with $b = 0$. The nonlinearity is caused by the temperature gradient.

An external driving term periodic in space and time is introduced *adhoc* in eq. (4). Apart from the obvious case of an externally applied electromagnetic wave, it might represent a simple approximation to internal modes originating from effects not explicitly included in the equation. A damping term models the absorption of energy imparted by the external forces. Since KdV-type equations are so ubiquitous in physics, our model equation might also be relevant in other contexts.

2. Hysteresis

We solve eq. (1) with the Fourier mode ansatz $\phi(y,t) = \sum_{k=0}^{N-1} \phi_k(t) \exp(\ iky\ )$. The nonlinear term is evaluated in y space by using FFT with dealiasing. $N = 128$ is found to be sufficient for the present purposes. Time integration is done with a simple predictor-corrector scheme. The Fourier ansatz implies periodic boundary conditions, in conformity with y being a poloidal angular coordinate for toroidal plasmas.

We arbitrarily select $c = 1$, $f = -6$, $\gamma = 0.1$ and $K = 1$. In case 1 we take $a = -0.28711$, $b = 0$ with frequency $\Omega = 0.525$, while in case 2 we use $a = 0$, $b = 0.28711$ and $\Omega = 0.3$.

In the absence of external forces and damping the energy $E(t) = (2\pi)^{-1} \int_0^{2\pi} [\phi^2(y,t) - a\,(\partial\phi(y,t)/\partial y)^2\ ]\ dy/2$ is a constant of the motion. With the present parameters and $\epsilon \neq 0$, $E(t)$ also asymptotivally tends to a constant, E_s, for $t \to \infty$. We plot E_s vs. ϵ in Figs. 1 and 2 for cases 1 and 2, respectively.

The figures are obtained in the following way: To get started, we pick a small value of ϵ and use a profile $\phi(y,t=0)$ which, if undisturbed by driving and damping, would develop into a solitary wave of speed u, namely $\phi^{(0)}(y) = \phi_a + (\phi_b - \phi_a)\ \mathrm{sn}^2(c_s y, k)$. Here, sn is a Jacobian elliptic function and $c_s = \sqrt{f(\phi_c - \phi_a)/[12(ua-b)]}$, $u = c + f(\phi_a + \phi_b + \phi_c)/3$ and $k = \sqrt{(\phi_b - \phi_a)/(\phi_c - \phi_a)}$. We arbitrarily use $\phi_a = 0$, $\phi_b = 0.0625$, $\phi_c = 0.125$.

Equation (1) is solved until E reaches a constant value E_s as mentioned above. In this stage the solution $\phi(y,t)$ is also found to be a wave travelling with constant shape but with phase velocity Ω/K. This solution ϕ is used as a new initial condition $\phi(y,t=0)$ for a run with slightly increased driving amplitude ϵ. This procedure is repeated again and again and gives the lower branch in Figs. 1 and 2. At a critical $\epsilon = \epsilon_h$ the energy jumps to a higher level and continues on the upper branch in the figures.

Now, if the procedure is stopped somewhere in this upper branch and is reversed and ϵ

is decreased in small steps, it turns out that the system does not retrace the original path in the E_s - ϵ plane. Rather, E_s remains in the upper branch and jumps back to the lower branch at a critical value ϵ_l which is smaller than ϵ_h. From this point onwards the system continues down on the lower branch. This creates the hysteresis curve in Figs. 1 and 2.

Within wide margins the choice of the initial function $\phi^{(0)}(y)$ has no effect on the hysteresis if it is obtained by the method just described: different $\phi(y, t = 0)$ are attracted to the same hysteresis curve, and once a point on it is reached, $\phi(y, t)$ is independent of its previous history. (It is possible to derive hysteresis curves with another method, in which the initial conditions do influence ϵ_l and ϵ_h, see /8/.)

In further investigations /8/ we study the dependence of the hystereses on Ω. Both branches may become unstable to Hopf bifurcations and a selfsimilar structure in (ϵ, Ω) space develops.

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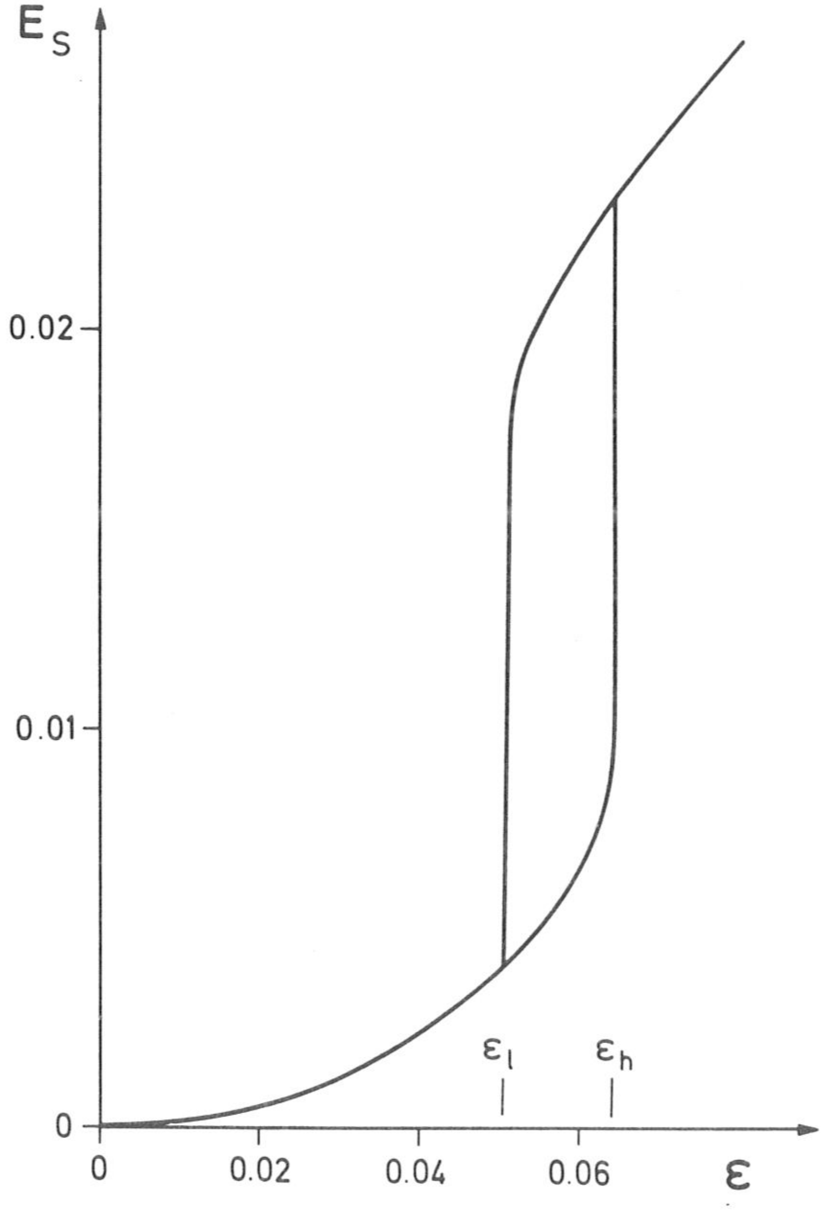
Figure captions

Fig. 1: Hysteresis curve for energy E_s as a function of the driving amplitude ϵ, for the drift wave equation.

Fig. 2: Same as Fig. 1, for the Korteveg-deVries equation.



E_S
0.02
0.01
0
ε_l
ε_h
0
0.02
0.04
0.06
ε

Figure 1

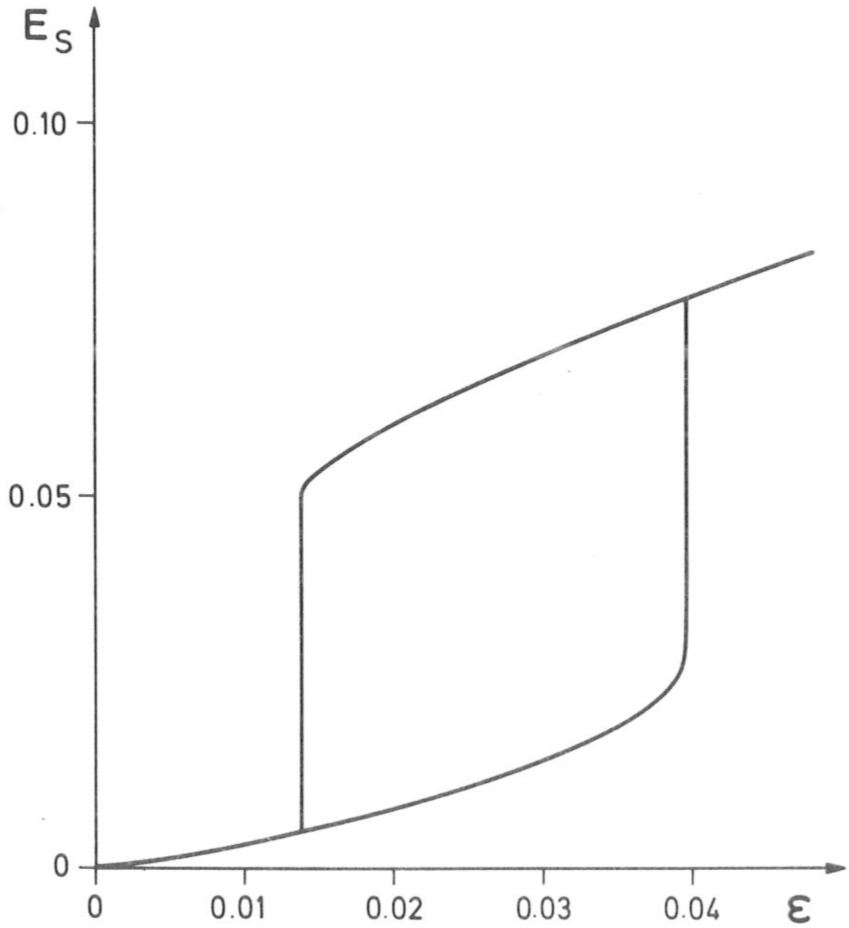


Figure 2