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Negative Energy Waves in the Framework of Vlasov-Maxwell Theory

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Abstract

On the basis of a variational formulation of the Vlasov-Maxwell theory it was recently shown that, for instance, all magnetically confined plasmas allow the existence of negative energy waves. Such waves can become nonlinearly and dissipatively unstable and might therefore be of importance in explaining anomalous transport. The proof of this result uses infinitely strongly localized perturbations. This is, however, not necessary: in this paper it is shown by discussing general, homogeneous, magnetized plasmas that the necessary localization is related to the average gyroradius r_g of the relevant particle species. For unstable plasmas the extent or wavelengths of negative energy waves can be of the order of r_g, whereas for linearly stable plasmas the extent can be a small fraction of r_g.

1. Introduction

Linear negative energy waves are of interest in the context of nonlinear and dissipative instabilities /1/, /2/. They might also have a bearing on, for example, problems of anomalous transport. This paper presents conditions for the existence of negative energy waves derived via a variational formulation of the Vlasov-Maxwell theory /3/, /4/, /5/.

In Ref. /5/ Noether's theorem is used to obtain a second-order energy expression in the perturbations which is the wave energy. Using localized perturbations, it is shown there for any equilibrium that negative energy waves exist if for at least one particle species ν

$$\vec{k} \cdot \vec{v} \, \vec{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \vec{v}} > 0 \tag{1}$$

holds for some $\vec{v}$ and $\vec{x}$ and for some directions $\vec{k}$. This generalizes a result recently obtained by Morrison for homogeneous isotropic plasmas /2/. Here $f_\nu^{(0)}(\vec{x}, \vec{v})$ is the unperturbed distribution function for species ν in a frame of reference in which the total energy of the equilibrium is smallest.

Since the Vlasov theory becomes invalid for length scales smaller than the Debye length, the question is to what degree it is necessary to localize the perturbations. For a homogeneous plasma with $\vec{B}^{(0)} = 0$ no localization is necessary. For general inhomogeneous systems with $\vec{B}^{(0)} \neq 0$ the localization needed should be similar to that for general homogeneous magnetized plasmas, which are therefore investigated in this paper.

2. Second-order wave energy for a homogeneous magnetized plasma

The constant unperturbed magnetic field $\vec{B}^{(0)}$ and the unperturbed vector potential $\vec{A}^{(0)}$ are taken as

$$\vec{B}^{(0)} = (0, 0, B^{(0)}) \quad , \quad \vec{A}^{(0)} = (0, B^{(0)} x , 0) . \tag{2}$$

In Ref. /5/ it is shown that for $\vec{B}^{(0)} = 0$ the minimum of the second-order wave energy $E^{(2)}$ is obtained for a vanishing perturbation $\vec{A}^{(s)}$ of the vector potential:

$$\vec{A}^{(s)} \equiv 0 , \tag{3}$$

which is a possible choice in the sense of an initial condition. In this paper eq.(3) is also chosen for $\vec{B}^{(0)} \neq 0$, which, however, might no longer correspond to the minimum of $E^{(2)}$ and therefore overestimate the necessary localization. Further we take

$$|F_\nu^{(s)}|^2 \sim \delta(v_x) \tag{4}$$

which is possible because of $f_\nu^{(0)} = f_\nu^{(0)}(v_x^2 + v_y^2, v_z)$.

With eqs. (2), (3) and (4) the second-order wave energy derived in /5/ becomes

$$E^{(2)} = \sum_\nu \frac{1}{4m_\nu} \int d^3x\, d^3v\, f_\nu^{(0)} \left\{ \left| \frac{\partial F_\nu^{(s)}}{\partial \vec{x}} - \omega_\nu \frac{\partial F_\nu^{(s)}}{\partial v_y} \vec{e}_x \right|^2 + \right.$$

$$+ \vec{v} \cdot \left[\left(\frac{\partial F_\nu^{(s)}}{\partial \vec{x}} - \omega_\nu \frac{\partial F_\nu^{(s)}}{\partial v_y} \vec{e}_x \right) \nabla \cdot \frac{\partial F_\nu^{(s)*}}{\partial \vec{v}} + conj.\ compl. \right] +$$

$$+ \frac{1}{16\pi} \int d^3x \left| \vec{E}^{(s)} \right|^2 , \tag{5}$$

$\omega_\nu = e_\nu B^{(0)}/(m_\nu c)$,
$\vec{e}_x$ is the unit vector in the x-direction ,
$\vec{E}^{(s)}$ is the complex amplitude of the electric field perturbation and
$F_\nu^{(s)} = F_\nu^{(s)}(\vec{x}, \vec{v}, t)$ is a generating function for the perturbation of the particle positions and velocities in eq.(5):

$$\vec{x}_\nu^{(s)} = \frac{1}{m_\nu} \frac{\partial F_\nu^{(s)}}{\partial \vec{v}} , \quad \vec{v}_\nu^{(s)} = -\frac{1}{m_\nu} \left(\frac{\partial F_\nu^{(s)}}{\partial \vec{x}} - \omega_\nu \frac{\partial F_\nu^{(s)}}{\partial v_y} \vec{e}_x \right) . \tag{6}$$

The electric field energy can be made equal to zero, being a bilinear expression of phase space integrals involving linearly the functions $F_\nu^{(s)}$, without influencing the particle contributions to $E^{(2)}$, see Ref. /5/. With

$$F_\nu^{(s)} \sim e^{i\, \vec{k}\cdot\vec{x}}$$

relation (5) becomes

$$E^{(2)} = V \sum_\nu \frac{1}{4m_\nu} \int d^3v \left[- \left| F_\nu^{(s)} \right|^2 \vec{k} \cdot \vec{v}\, \vec{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \vec{v}} + \right.$$

$$+ i\, \omega_\nu\, k_x\, f_\nu^{(0)}\, F_\nu^{(s)*} \frac{\partial F_\nu^{(s)}}{\partial v_y} +$$

$$+ i\, \omega_\nu\, v_x \left(\vec{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \vec{v}} F_\nu^{(s)} \frac{\partial F_\nu^{(s)*}}{\partial v_y} - \frac{\partial f_\nu^{(0)}}{\partial v_y} F_\nu^{(s)}\, \vec{k} \cdot \frac{\partial F_\nu^{(s)*}}{\partial \vec{v}} \right) +$$

$$\left. + \omega_\nu^2\, f_\nu^{(0)} \left| \frac{\partial F_\nu^{(s)}}{\partial v_y} \right|^2 \right] . \tag{7}$$

V is a large periodicity volume. $|\vec{k}| -> \infty$ again yields the general condition (1). For finite k we discuss separately parallel and perpendicular wave propagation.

3. Wave propagation parallel to $\vec{B}^{(0)}$

In this case

$$\frac{\partial F_\nu^{(s)}}{\partial v_y} = 0 \quad , \quad v_z \frac{\partial}{\partial v_z} \int f_\nu^{(0)}(\, v_y^2 \,,\, v_z)\, dv_y \;>\; 0 \tag{8}$$

allows $E^{(2)} < 0$ without any restriction on $|\vec{k}|$.
In general, one will have conditions between (1) and (8). This is qualitatively similar to perpendicular wave propagation, which is treated more explicitly in the next section.

4. Wave propagation perpendicular to $\vec{B}^{(0)}$

For this case eq. (7) becomes with $\vec{k} = (0, k, 0)$

$$E^{(2)} = V \sum_\nu \frac{1}{4m_\nu} \int d^3v \left(\omega_\nu^2\, f_\nu^{(0)} \left| \frac{\partial F_\nu^{(s)}}{\partial v_y} \right|^2 - \left| F_\nu^{(s)} \right|^2 k^2\, v_y\, \frac{\partial f_\nu^{(0)}}{\partial v_y} \right) . \tag{9}$$

From eq. (9) it follows that

$$k\, \frac{\partial F_\nu^{(s)}}{\partial v_y} \not\equiv 0 \tag{10}$$

is necessary for $E^{(2)} < 0$. Minimization of $E^{(2)}$ under the constraint

$$V \sum_\nu \frac{1}{4m_\nu} \int d^3v\, \omega_\nu^2\, f_\nu^{(0)} \left| \frac{\partial F_\nu^{(s)}}{\partial v_y} \right|^2 = 1 \tag{11}$$

with a Lagrange parameter $\frac{1}{\lambda} - 1$ yields for the minimum of $E^{(2)}$

$$E^{(2)} = 1 - \frac{1}{\lambda} . \tag{12}$$

This is negative for $0 < \lambda < 1$. With the dimensionless quantities

$$\hat{v}_y = \frac{v_y}{v_{th\ \nu}} , \quad \lambda\, k^2\, v_{th\ \nu}^2 \,/\, \omega_\nu^2 = \lambda\, k^2\, r_{g\nu}^2 = \Lambda , \tag{13}$$

the corresponding Hermitian eigenvalue problem is

$$\Lambda\, \hat{v}_y\, \frac{\partial f_\nu^{(0)}}{\partial \hat{v}_y}\, F_\nu^{(s)} + \frac{\partial}{\partial \hat{v}_y} \left(f_\nu^{(0)} \frac{\partial F_\nu^{(s)}}{\partial \hat{v}_y} \right) = 0 . \tag{14}$$

Negative $E^{(2)}$ are associated with

$$k^2 r_{g\nu}^2 = \frac{\Lambda}{\lambda} > \Lambda > 0 . \tag{15}$$

The smallest k and therefore the least localization corresponds to the smallest eigenvalue $\Lambda = \Lambda_{min}$:

$$k_{min} r_g = \sqrt{\Lambda_{min}} \tag{16}$$

$\Lambda > 0$ requires that $f_\nu^{(0)}$ have a minimum with respect to $\hat{v}_\perp = \sqrt{\hat{v}_x^2 + \hat{v}_y^2}$. The minimum wave vector is then qualitatively given by

$$k_{min}^2 \, r_{g\nu}^2 \sim \frac{1}{\Delta \hat{v}_y \, \hat{V}_y} \frac{\frac{1}{2}(f_\nu^{max} + f_\nu^{min})}{f_\nu^{max} - f_\nu^{min}} , \tag{17}$$

f_ν^{max} : the maximum of $f_\nu^{(0)}(v_y^2, v_z)$, f_ν^{min} : its relative minimum in $\Delta \hat{v}_y$,
$\Delta \hat{v}_y$: distance between the maximum and the minimum of $f_\nu^{(0)}$,
$\hat{V}_y$: velocity somewhere between the maximum and the minimum of $f_\nu^{(0)}$.
A numerical solution of the eigenvalue problem (14), (11) was obtained for

$$f_\nu^{(0)}(v_\perp^2, \, v_z) = \frac{(1 - \alpha/\beta)^{-3/2}}{(2\pi)^{3/2} \, v_{th}^3} \, e^{-\frac{1}{2}(1-\alpha/\beta)^{-1}\hat{v}_z^2} \left(e^{-\frac{1}{2}\hat{v}_\perp^2} - \alpha \, e^{-\frac{1}{2}\beta \hat{v}_\perp^2} \right) ,$$

$$0 \le \alpha < 1 \quad , \quad \beta > 1 \, . \tag{18}$$

$f_\nu^{(0)}$ has a minimum with respect to $v_\perp$ for $\alpha\beta > 1$. At the same time the system is linearly stable for (see Appendix)

$$\alpha = \beta^{-\epsilon} \quad , \quad \frac{1}{2} \le \epsilon < 1 \, . \tag{19}$$

Table 1 gives for a number of ϵ's and α's in the range

$$0.5 \le \alpha \le 0.9 \quad , \quad 0.1 \le \epsilon \le 0.9 \, . \tag{20}$$

the exact numerically obtained values $(k_{min} r_g)_{num}$ of $k_{min} r_g$ together with an approximation $(k_{min} r_g)_{app}$ of this quantity given by

$$(k_{min} r_g)_{app} = 1.13 \sqrt{\frac{f_{max} + f_{min}}{2 \, H_{max} \, (f_{max} - f_{\min})}} , \tag{21}$$

with

$$H_{max} = \frac{1}{2} \, \hat{v}_{y \; max}^2 = \frac{ln \; \alpha \; \beta}{\beta - 1} ,$$

$$f_{max} = f_\nu^{(0)}(v_{y \; max}^2, \, v_z) \quad , \quad f_{min} = f_\nu^{(0)}(0, \, v_z) \, .$$

Relation (21) corresponds to relation (17) for the case that the minimum of $f_\nu^{(0)}$ is at $\hat{v}_y = 0$, in which case $\hat{V}_y \approx \Delta \hat{v}_y$ and $\frac{1}{2}(\Delta \hat{v}_y)^2 = H_{max}$.

Table 1 contains for more detailed information also the values of the quantities $\alpha\beta$, H_{max} and f_{max}/f_{min} . The main result is that relation (21) or, equivalently, relation (17) provides a reasonable approximation for $k_{min}r_g$. This is in agreement with the character of the eigenfunctions to the lowest eigenvalue shown in Figs 1 together with the corresponding distribution functions. As expected, the eigenfunctions are of the form which was assumed for deriving relation (17). The only exception is the rather exotic case $\epsilon = 0.1$, $\alpha = 0.5$. Table 1 tells us, furthermore, that the localization $\frac{1}{k_{min}r_g}$ necessary for negative energy waves tends to be stronger for stable plasmas than for unstable ones.

5. Conclusions

Condition (1), which is fulfilled in any magnetically confined plasma, is only obtained with infinitely strongly localized perturbations. Negative energy waves should, however, be dangerous only if their wavelengths are not too small. By treating a magnetized, homogeneous plasma for illustration it is shown via a Hermitian eigenvalue problem that the necessary localization should be generally related to the average gyroradius $r_{g\nu}$ of the relevant particle species ν . For unstable plasmas the least localization is of the order of $r_{g\nu}$, whereas for stable plasmas it is a smaller fraction of $r_{g\nu}$. Since the Vlasov equation is valid only for length scales larger than the Debye length, all such negative energy waves should be physically meaningful if the gyroradius of the relevant species is larger than the Debye length.

Acknowledgement

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Appendix

Stability regime for waves propagating perpendicularly to $\vec{B}$

We discuss here waves with their electric field vectors parallel to $\vec{B}$. In this case the dispersion relation is

$$\frac{\omega^2}{c^2}\,\epsilon_{33}\;-\;k^2\;=\;0 \tag{A1}$$

with

$$\epsilon_{33}\;=\;1\;+\;\sum_{\nu}\frac{\omega_{p\nu}^2}{\omega^2}\left(-\,1\;+\;\int\limits_0^\infty v_\perp\,dv_\perp\,\frac{\partial f_\nu^{(p)}}{\partial v_\perp^2/2}\,\sum_{n=1}^{\infty}\frac{2n^2\omega_\nu^2}{\omega^2-n^2\omega_\nu^2}\,J_n(\frac{kv_\perp}{\omega_\nu})\right)\;; \tag{A2}$$

$\omega_{p\nu}$ is the plasma frequency of the species ν ,
$J_n(z)$ is the Bessel function of order n and
$f_\nu^{(p)}(v_\perp^2)$ is given by

$$\begin{aligned} f_\nu^{(p)}(v_\perp^2)\;&=\;2\pi\int\limits_{-\infty}^{+\infty}dv_z\;v_z^2\;f_\nu^{(0)}(v_\perp^2,\,v_z)\\ &=\;e^{-\frac{1}{2}\,\hat{v}_\perp^2}\;-\;\alpha\,e^{-\frac{1}{2}\,\beta\,\hat{v}_\perp^2}\;. \end{aligned} \tag{A3}$$

(see eq.(18)). With

$$x = Re(\omega)\qquad,\qquad y = Im(\omega) \tag{A4}$$

we can decompose the dispersion relation (A1), (A2) into its real and imaginary parts:

$$\begin{aligned} x^2\;-\;y^2\;-\;k^2c^2\;+\;\sum_{\nu}\omega_\nu^2\Bigg(-\,1\;+\;2\sum_{n=1}^{\infty}\frac{n^2\omega_\nu^2\;(x^2-y^2-n^2\omega_\nu^2)}{(x^2-y^2-n^2\omega_\nu^2)^2\;+\;4x^2y^2}\;\times\\ \times\int\limits_0^\infty v_\perp\;dv_\perp\;\frac{\partial f_\nu^{(p)}}{\partial v_\perp^2/2}\;J_n^2(\frac{kv_\perp}{\omega_\nu})\Bigg)\;=\;0.\;, \end{aligned} \tag{A5a}$$

$$\begin{aligned} x\;y\;\Bigg\{\;1\;-\;2\sum_{\nu}\omega_{p\nu}^2\sum_{n=1}^{\infty}\frac{n^2\omega_\nu^2}{(x^2-y^2-n^2\omega_\nu^2)^2\;+\;4x^2y^2}\;\times\\ \times\int\limits_0^\infty v_\perp\;dv_\perp\;\frac{\partial f_\nu^{(p)}}{\partial v_\perp^2/2}\;J_n^2(\frac{kv_\perp}{\omega_\nu})\;\Bigg\}\;=\;0\;. \end{aligned} \tag{A5b}$$

The second equation (A5b) can only be fulfilled with $xy \neq 0$ if for some ν and n

$$\int\limits_0^\infty v_\perp\;dv_\perp\;\frac{\partial f_\nu^{(p)}}{\partial v_\perp^2/2}\;J_n^2(\frac{kv_\perp}{\omega_\nu})\;>\;0\;. \tag{A6}$$

With $f_\nu^{(p)}$ from eq. (A3) one finds

$$\int_0^\infty v_\perp \, dv_\perp \, \frac{\partial f_\nu^{(p)}}{\partial v_\perp^2/2} \, J_n^2(\frac{kv_\perp}{\omega_\nu}) \;=\; -\, e^{-z} \, I_n(z) \;+\; \alpha \, e^{-z/\beta} \, I_n(z/\beta) \;, \qquad (A7)$$

$$z \;=\; k^2 \, v_{th}^2 \;/\; \omega_\nu^2 \;. \qquad (A8)$$

$I_n(z)$ is the modified Bessel function of order n. From the asymptotic behaviour of I_n one infers that α and β related by

$$\alpha \;=\; \beta^{-\epsilon} \quad , \quad \frac{1}{2} \leq \epsilon < 1 \qquad (A9)$$

do not allow positive values of the integral (A7) even for non-monotonic distribution functions, i.e. $\alpha\beta > 1$. The latter is always fulfilled with (A9) for $\beta > 1$.

With (A9) we can write (A7) in the form

$$\int_0^\infty v_\perp \, dv_\perp \, \frac{\partial f_\nu^{(p)}}{\partial v_\perp^2/2} \, J_n^2(\frac{kv_\perp}{\omega_\nu}) \;=\; -\, \frac{1}{z^\epsilon}\Big(z^\epsilon \, e^{-z} \, I_n(z) \;-\; (z/\beta)^\epsilon \, e^{-z/\beta} \, I_n(z/\beta)\Big) \;. \quad (A10)$$

If we can prove that

$$w_n \;=\; x^\epsilon \, e^{-x} \, I_n(x) \qquad (A11)$$

is a monotonically increasing function of x, then the integral (A10) is negative and the system is stable with respect to the considered perturbations. From the equation for $I_n(x)$,

$$x^2 \, \frac{d^2 I_n}{dx^2} \;+\; x \, \frac{dI_n}{dx} \;-\; (x^2 + n^2) \, I_n \;=\; 0 \;, \qquad (A12)$$

one finds for w_n the equation

$$x^2 w_n'' \;+\; 2x(x - \epsilon + \frac{1}{2})w_n' \;-\; (n^2 + \; 2x(\epsilon - \frac{1}{2}) \;-\epsilon^2)w_n \;=\; 0 \;. \qquad (A13)$$

This can also be written as

$$x^{1+2\epsilon}e^{-2x}\frac{d}{dx}(x^{1-2\epsilon}e^{2x}w_n'(x)) \;=\; (n^2 + \; 2x(\epsilon - \frac{1}{2}) \;-\epsilon^2)w_n \;, \qquad (A14)$$

from which it follows that

$$x^{1-2\epsilon}e^{2x}w_n'(x) \;=\; \int_0^x \frac{e^{2z}}{z^{1+2\epsilon}} \, (n^2 + \; 2z(\epsilon - \frac{1}{2}) \;-\epsilon^2)w_n(z) \; dz \;. \qquad (A15)$$

We need only consider $n \geq 1$ and $\epsilon < 1$. In this case the right-hand side of eq. (A15) converges and vanishes for $x = 0$, as required by the left-hand side. Furthermore, since $w_n(z) \geq 0$ and $1/2 \leq \epsilon < 1$, the integrand in (A15) is positive for $z > 0$ and therefore it follows that

$$w_n'(x) > 0 \qquad for \qquad x > 0 \;. \qquad (A16)$$

This proves that (A9) describes systems that are stable sith respect to the perturbations considered.

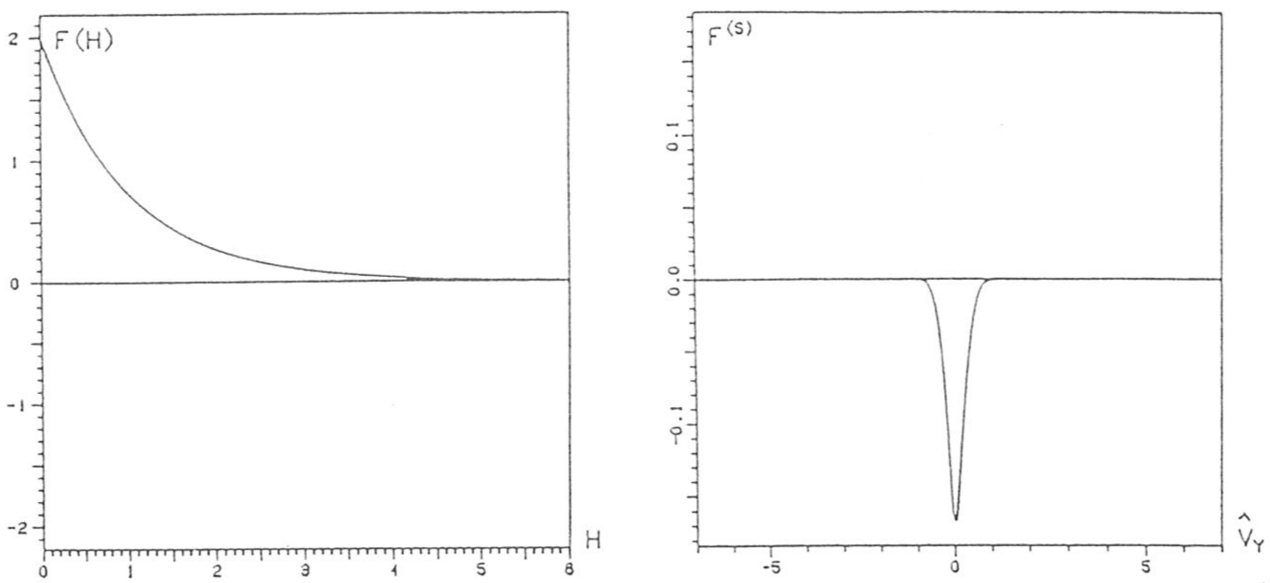
Table 1:
Parameters for the distribution functions (18):
$\alpha = \beta^{-\epsilon}$, see eq.(19);
$H_{max} = ln(\alpha\beta)/(\beta - 1)$: value of $H = \hat{v}_y^2/2$ at which $f_\nu(v_y^2, v_z)$ has its maximum;
$f_{max} = f_\nu(v_{y\ max}^2, v_z)$;
$f_{min} = f_\nu(0, v_z)$;
$(k_{min}r_g)_{app}$: approximation of $k_{min}r_g$ given by eq. (21)
$(k_{min}r_g)_{num}$: numerical value of $k_{min}r_g$

ϵ	α	β	$\alpha\beta$	H_{max}	f_{max}/f_{min}	$(k_{min}r_g)_{app}$	$(k_{min}r_g)_{num}$
0.1	0.5	1024.00	512.00	0.0061	1.99	17.82	11.43
	0.7	35.40	24.78	0.093	2.95	3.73	3.34
	0.9	2.87	2.58	0.51	3.92	1.46	1.49
0.3	0.5	10.08	5.04	0.18	1.51	4.21	4.01
	0.7	3.28	2.30	0.36	1.61	2.74	2.71
	0.9	1.42	1.28	0.58	1.65	2.11	2.12
0.5	0.5	4.00	2.00	0.23	1.19	5.64	5.58
	0.7	2.04	1.43	0.34	1.21	4.46	4.45
	0.9	1.23	1.11	0.45	1.21	3.85	3.85
0.7	0.5	2.69	1.35	0.18	1.05	11.71	11.84
	0.7	1.66	1.17	0.23	1.06	9.99	10.11
	0.9	1.16	1.05	0.28	1.06	9.02	9.13
0.9	0.5	2.16	1.08	0.066	1.005	61.34	64.53
	0.7	1.49	1.04	0.081	1.005	54.44	56.99
	0.9	1.12	1.01	0.094	1.005	50.33	52.54

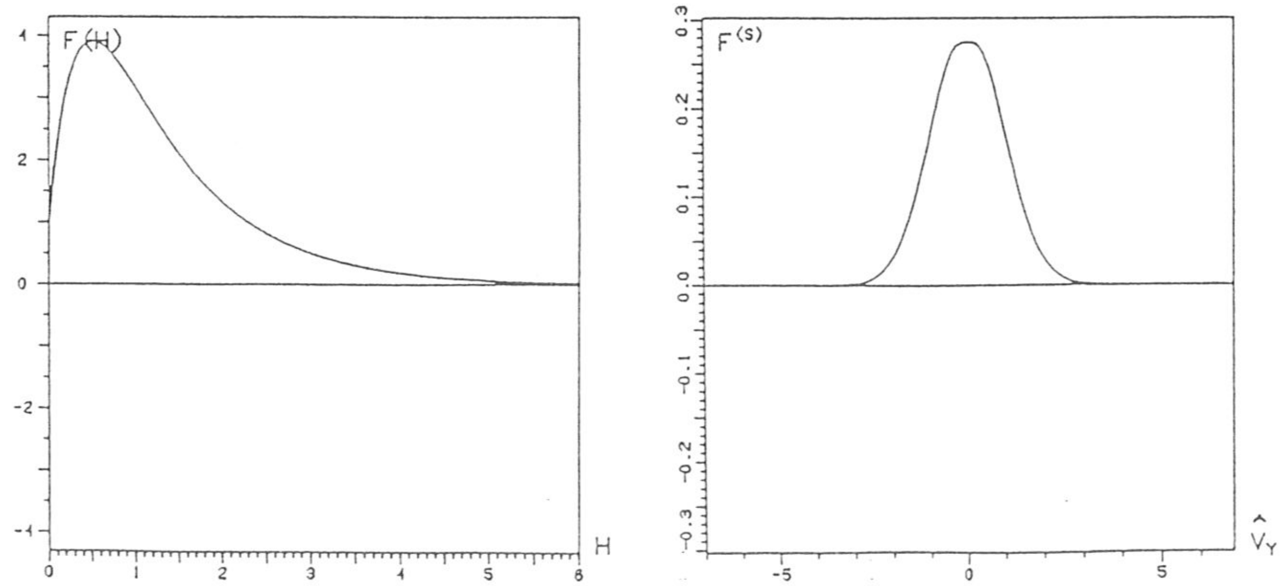
Figs. 1a-b

Distribution function $F = \frac{f^{(0)}(v_y^2,0)}{f^{(0)}(0,0)} = \frac{1}{1-\alpha}\left(e^{-H} - \alpha e^{-\beta H}\right)$ versus $H = \frac{1}{2}\hat{v}_y^2$

and eigenfunction $F^{(s)}$ of eq.(14) versus $\hat{v}_y = v_y/v_{th}$ for $\Delta = \Delta_{min}$



a) $\epsilon = 0.1$, $\alpha = 0.5$, $\beta = 1024.$

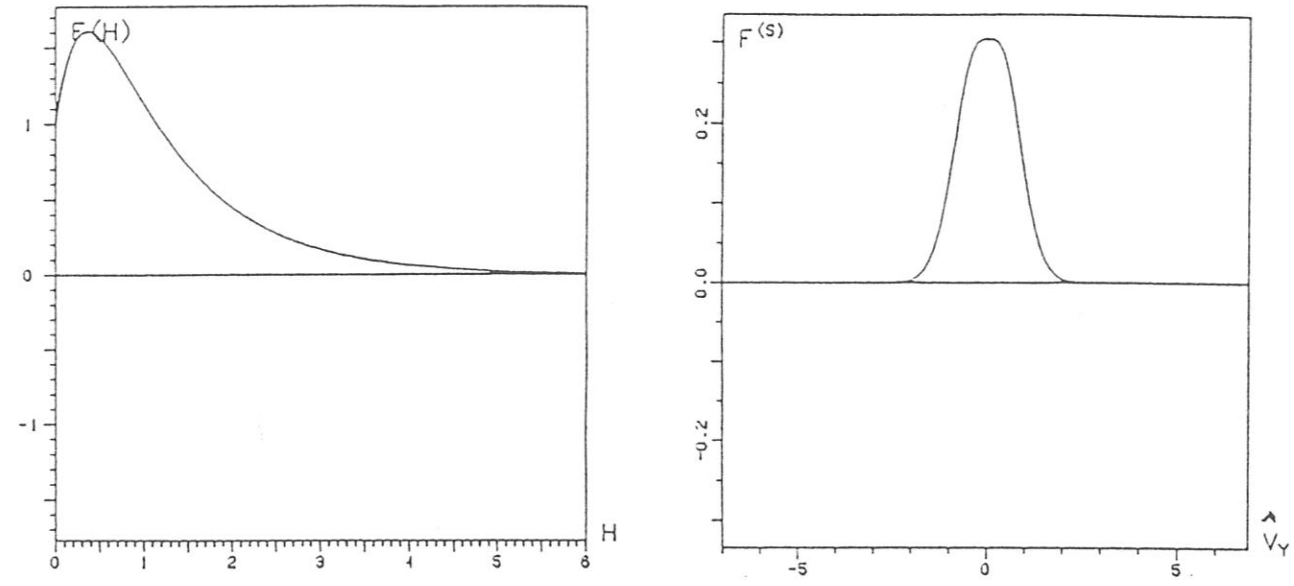


b) $\epsilon = 0.1$, $\alpha = 0.9$, $\beta = 2.87$

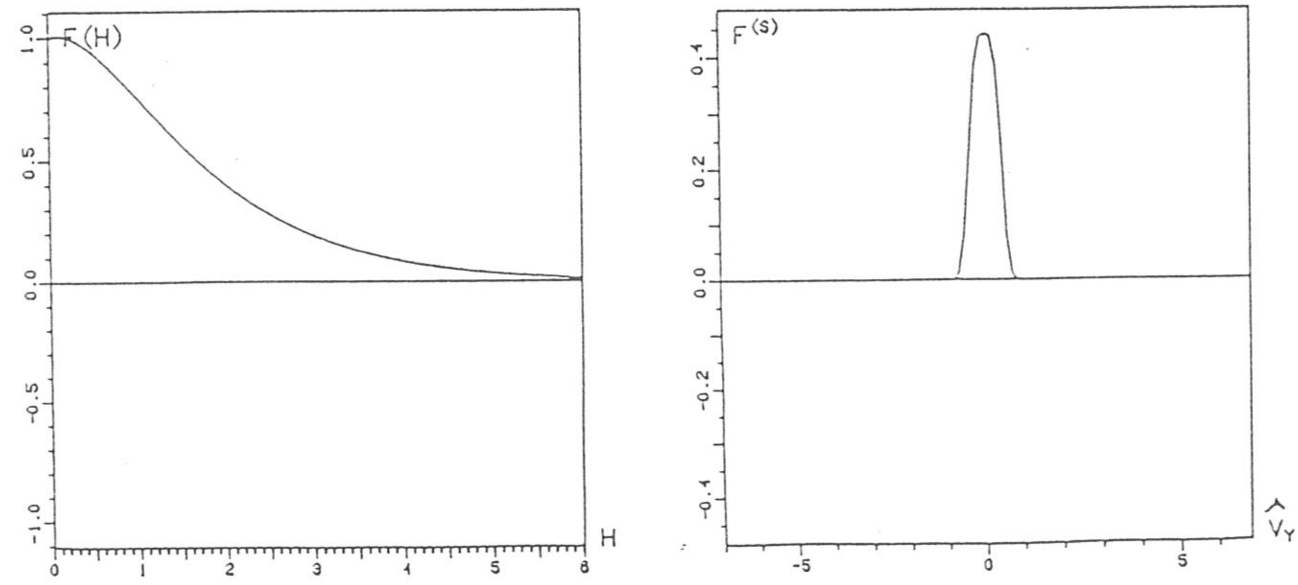
Figs. 1c-d

Distribution function $F = \frac{f^{(0)}(v_y^2,0)}{f^{(0)}(0,0)} = \frac{1}{1-\alpha}\left(e^{-H} - \alpha e^{-\beta H}\right)$ versus $H = \frac{1}{2}\hat{v}_y^2$

and eigenfunction $F^{(s)}$ of eq.(14) versus $\hat{v}_y = v_y/v_{th}$ for $\Delta = \Delta_{min}$



c) $\epsilon = 0.3$, $\alpha = 0.7$, $\beta = 3.28$



d) $\epsilon = 0.9$, $\alpha = 0.9$, $\beta = 1.12$