MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Exact Resonance Broadening Theory of Diffusion in Random Electric Fields

A. Salat

IPP 6/ 267 August 1987

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Abstract

Particle motion in random electric fields is considered on the assumption that orbit stochasticity causes the velocity increments at different times to be independent random events with Gaussian probabilities (Wiener process). The resulting resonance broadening term is substantially different from the one derived by Dupree /2/. The diffusion coefficient D is found to be a function of t/t_d, where t_d is the average diffusion time across the resonance region in velocity space. For $t/t_\mathrm{d} << 1$ the diffusion is quasilinear. Dupree's high-amplitude case (autocorrelation time >> trapping time) with $D \sim (< E^2 >)^{\frac{3}{4}}$ turns out to be inconsistent with a diffusion process: the particles are lost from the resonance region before diffusion is established.

1. Introduction

One of the standard topics in plasma physics is the motion of charged particles in random electric fields. In one dimension the equations of motion are

$$
\begin{aligned}
\dot{v}(t) &= \frac{q}{m}\, E(x(t),t) \ , \\
\dot{x}(t) &= v(t) \ .
\end{aligned}
\tag{1.1}
$$

For $E(x,t) = \sum\limits_k E_k e^{i(kx-\omega_k t)}$ where the phases of E_k are random, quasilinear theory /1/ predicts that the average particle motion in velocity space is described by a diffusion process, $< v^2(t) >= D_0 t/2$, with diffusion constant

$$
D_0(v) = \pi \, \frac{q^2}{m^2} \sum_k |E_k|^2 \, \delta(kv - \omega_k) \ . \tag{1.2}
$$

Equation (1.2) is derived on the assumption that in $E(x,t)$ the orbits $x(t)$ can be approximately replaced by the unperturbed orbits $x(0) + vt$. Obviously, the assumption breaks down for sufficiently strong fields. Dupree /2/ was the first to suggest a diffusive process for weak and strong fields, with diffusion constant D_{Du}, implicitly determined by

$$
D_{\mathrm{Du}}(v) = \frac{q^2}{m^2} \int\limits_0^\infty d\tau \sum_k |E_k|^2 \, e^{i(kv - \omega_k)t \; - \; \frac{1}{3}k^2 D_{\mathrm{Du}} \tau^3} \ . \tag{1.3}
$$

The last term in the exponent reflects the average effect of deviations from the unperturbed orbit and acts as a broadening of the wave-particle resonance at $kv = \omega_k$. The theory was later generalized /3/ to plasmas with magnetic fields. In this case there is also diffusion in configuration space, orthogonal to the magnetic field. This diffusion has a strong destabilizing effect on drift wave turbulence and transport /3/, particularly in sheared magnetic fields /4/.

Dupree's result, eq. (1.3), has been rederived by other authors /5 - 7/ by different methods. Benford and Thomson /6/ and Molvig et al. /7/, in particular, start by assuming a

Markov process with Gaussian probabilities, i.e. a Wiener process. In /7/ this assumption is justified by the "mixing" property of the particle orbits in fields with overlapping island structure /8/. Particles which are initially arbitrarily close together are separated exponentially fast and end up in totally different orbits.

Dupree's result, on the other hand, has also been criticized from several points of view /9 - 14/. Cook and Sanderson /9/ claim that Dupree's result is not valid in the regime where it differs markedly from the quasilinear value. In /10/ it is stated that trapped particles are misrepresented. The numerical experiments performed in /11 - 13/ yield results that disagree with eq. (1.3). In /14/, while the Markovian nature of the particle motion is supported by numerical experiments, it is pointed out in a special context that the derivation of eq. (1.3) in /7/ is erroneous and that the correct treatment leads to a breakdown of the diffusion scenario in sufficiently strong fields.

Resonance broadening theories and even the quasilinear diffusion coefficient in the domain where it used to be considered valid have met with another type of criticism in /15/, /16/, where it is claimed that selfconsistency between particles and field is not properly treated in the case of, for example, Langmuir or drift wave turbulence. Selfconsistency effects on diffusion are taken into account in, for example, renormalized theories and in DIA turbulence theory: see /17/, /18/. It is argued in /7/, however, that such theories are essentially expansions in the *Eulerian* field amplitudes and that their convergence is doubtful when the stochasticity of the *Lagrangian* fluctuations along the orbits is essential. Future selfconsistent theories will certainly have to incorporate both orbit stochasticity and the reaction of particle motion back onto the fields.

In the following, particle diffusion in given stochastic fields will again be considered. As in /6/, /7/, /14/, it is assumed that the increment Δv of particle velocity is a Markovian quantity and is distributed with Gaussian probability. The consequences for particle diffusion are discussed without further a priori approximations. As in /13/, /14/, a time-dependent diffusion coefficient D is obtained. It is derived here in a more general framework than in /14/, and the interpretation of the time dependence is made clearer than in /13/.

In Section 2 an equation for the diffusion coefficient D is derived from the Wiener process assumption for a simple model of Eulerian field correlations. In Section 3 D and in particular its deviation from the quasilinear value D_0 are numerically studied. The results and a resonance broadening term in a version obtained from first principles are interpreted in terms of particles leaking out of the finite resonance region. As in /9/, it is further concluded that Dupree's result, eq. (1.3) say, is invalid in the regime where D substantially deviates from D_0. In this high-amplitude regime particle motion is so intense that the particles leave the resonance zone before a diffusion process has time to be established.

2. Diffusion coefficient for a Wiener process

We consider the average motion of particles with charge q and mass m_p in an ensemble of waves

$$E(x,t) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} E_{m,n} \exp\left[i(k_m x - \omega_n t)\right] \tag{2.1}$$

with $k_m = 2\pi m/L$, $\omega_n = 2\pi n/T$ where L and T are some fundamental periodicity intervals.

With the definitions

$$\begin{aligned} x(t) &= x(t=0) + v\,t + \Delta x(t)\,, \\ v(t) &= v + \Delta v(t) \end{aligned} \tag{2.2}$$

the equations of motion for the deviations from straight-line orbits are

$$\begin{aligned} \Delta \dot{x}(t) &= \Delta v(t)\,, \\ \Delta \dot{v}(t) &= \frac{q}{m_p} E(x(t),t)\,. \end{aligned} \tag{2.3}$$

The diffusion coefficient D will be defined as

$$D(v) = \frac{1}{2}\frac{d}{dt} < [\Delta v(t)]^2 > . \tag{2.4}$$

The averaging operator $< >$ refers to the realizations of the chaotic orbits mentioned in the introduction. Since this stochasticity results from "infinitesimally" small variations of starting positions, mode amplitudes, etc., $< >$ commutes with all Eulerian quantities. One thus obtains

$$\begin{aligned} D &= \frac{1}{2}\left(\frac{q}{m_p}\right)^2 \frac{d}{dt} < \int_0^t dt'\, E(x(t'),t') \int_0^t dt''\, E(x(t''),t'') > \\ &= \left(\frac{q}{m_p}\right)^2 \int_0^t dt' < E(x(t'),t')E(x(t),t) > \\ &= \left(\frac{q}{m_p}\right)^2 \sum_{m,n}\sum_{m',n'} E_{m,n}E^*_{m',n'} \exp\left[i(k_m - k_{m'})x(0)\right] \exp\left[i(\omega_{n'} - k_{m'}v)t\right] \\ &\quad \cdot \int_0^t dt' \exp\left[i(\omega_n - k_m v)t'\right] < \exp\{i[k_m \Delta x(t') - k_{m'}\Delta x(t)]\} > . \end{aligned} \tag{2.5}$$

For a Wiener process with $<\Delta x>=0$ it holds /19/ that for all a, b

$$<\exp\{i[a\Delta x(t)-b\Delta x(t')]\}> = \exp\left\{-\frac{1}{2}<[a\Delta x(t)-b\Delta x(t')]^2>\right\}. \quad (2.6)$$

The probability density for finding a particle at time t with velocity increment Δv when at $t=0$ it had $\Delta v=0$ is given, in conformity with eq. (2.4), by

$$P(\Delta v,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left[-\frac{(\Delta v)^2}{4Dt}\right], \qquad t>0\,. \quad (2.7)$$

The joint probability density for finding the particle at time t' with $\Delta v'$ and at a later time t with Δv is the product

$$P(\Delta v,t;\Delta v',t') = P(\Delta v',t')\cdot P(\Delta v-\Delta v',t-t')\,. \quad (2.8)$$

From $\Delta\dot{x}=\Delta v$ and straightforward application of eqs. (2.7), (2.8) one obtains /14/, for $\tau=t-t'\geq 0$,

$$\begin{aligned}&<[k_m\Delta x(t')-k_{m'}\Delta x(t)]^2> = \\ &= \frac{2}{3}D\left[(k_{m'}-k_m)^2t'^3+3k_{m'}(k_{m'}-k_m)t'^2\tau+3k_{m'}^2t'\tau^2+k_{m'}^2\tau^3\right].\end{aligned} \quad (2.9)$$

For given fields eqs. (2.5), (2.6) and (2.9) implicitly determine D. If it proves to be (approximately) time-independent over an extended time intervall the assumed diffusion process is confirmed, while for strongly time dependent D particle motion consists of another type of process whose exact nature is then unknown in general.

For times t which are large compared with the autocorrelation time $t_0 = \max\{\Delta\omega^{-1},\,(\Delta k\,v)^{-1}\}$ of the waves the terms $m\neq m'$, $n\neq n'$ in eq. (2.5) phase mix away when averaged over the initial position ($\Delta\omega$, $\Delta k=$ width of the wave packet in ω_n, k_m), leaving

$$D(v) = \left(\frac{q}{m_p}\right)^2\sum_{m,n}|E_{m,n}|^2\int_0^t dt'\exp\left[i(k_mv-\omega_n)t'-\frac{1}{3}Dk_m^2t'^2(3t-2t')\right]. \quad (2.10)$$

Before being damped away, the omitted terms are rapidly oscillating on a typical time scale ω_0^{-1}. It is useful to normalize time, distance and velocity with ω_0^{-1}, k_0^{-1}, $(\omega_0/k_0)^{-1}$

respectively, where k_0 and ω_0 are the centre of the wave packet in k- and ω-space. Going over to a continuous spectrum and keeping the old notation for the normalized quantities, one obtains

$$\begin{aligned}\hat{D}(v) &= \sigma^2 \int_{-\infty}^{+\infty} dk \left(\frac{k}{k_0}\right)^2 \int_{-\infty}^{+\infty} d\omega \; s(k,\omega) \\ &\cdot \int_0^t dt' \exp\left[i\left(\frac{k}{k_0}v - \frac{\omega}{\omega_0}\right)t' - \frac{1}{3}\left(\frac{k}{k_0}\right)^2 \hat{D}t'^2(3t-2t')\right] ,\end{aligned} \tag{2.11}$$

where $\hat{D} = k_0^2 D/\omega_0^3$, $E = -\partial_x \Phi(x,t)$ and

$$\sigma^2 = \left(\frac{q}{m_p}\right)^2 \left(\frac{k_0}{\omega_0}\right)^4 < \Phi^2 >_0 , \tag{2.12}$$

$$s(k,\omega) = |\Phi(k,\omega)|^2 / \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \; d\omega \; |\Phi(k,\omega)|^2 . \tag{2.13}$$

σ is the normalized, dimensionless field amplitude which uses the space-time average $< \Phi^2 >_0$ of Φ^2. An alternative representation is

$$\hat{D} = \frac{-\sigma^2}{2\sqrt{\pi}} \int_0^t dt' \int_{-\infty}^{+\infty} dx \; \frac{1}{r} \exp\left[-\frac{(x-vt)^2}{4r^2}\right] \frac{\partial^2}{\partial x^2} g(x,t) , \tag{2.14}$$

where $r^2 = \frac{1}{3}\hat{D}t'^2(3t-2t')$ and

$$g(x,t) = \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \; s(k,\omega) \exp\left[i\left(\frac{k}{k_0}x - \frac{\omega}{\omega_0}t\right)\right] . \tag{2.15}$$

$g(x,t)$ equals the correlation function $< \Phi(x,t)\Phi(0,0) > / < \Phi(0,0)^2 >$ provided $< \Phi(x,t)\Phi(x',t') >$ depends on $|x-x'|$, $|t-t'|$ only, where the averaging is done over the Eulerian wave phases.

3. Numerical results and interpretation

The expressions (2.11), (2.15) for $\hat{D}$ can be evaluated further for simple wave spectra such as

$$s(k,\omega) = \frac{1}{4\pi k_0 \omega_0 \epsilon} \sum_{\pm} \exp\left[-\left(\frac{k \pm k_0}{\epsilon k_0}\right)^2\right] \exp\left[-\left(\frac{\omega \pm \omega_0}{\epsilon \omega_0}\right)^2\right], \quad (3.1)$$

where the sum extends over all four combinations of + and - in the exponents. The corresponding correlation function is

$$g(x,t) = \cos x \cdot \cos t \cdot \exp\left[-\left(\frac{\epsilon x}{2}\right)^2\right] \exp\left[-\left(\frac{\epsilon t}{2}\right)^2\right]. \quad (3.2)$$

Equations (3.1), (3.2) correspond to a symmetric wave packet with relative width $\Delta\omega/\omega_0 = \Delta k/k_0 \approx \epsilon$, where $\epsilon \lesssim 1$. ϵ^{-1} is the dimensionless autocorrelation or coherence time t_0 of the field. For comparison, Appendix A presents results for a monochromatic and unidirectional wave packet as considered in, for example, /12/, /13/.

As a result of the ω- and k-quadratures one obtains

$$\hat{D}(v) = \sigma^2 \int_0^t dt' \exp\left(-\frac{\epsilon^2 v^2 t'^2 + 4r^2}{4c}\right) \exp\left(-\frac{\epsilon^2 t'^2}{4}\right) \cdot \cos t' \cdot (A \sin \frac{vt'}{c} + B \cos \frac{vt'}{c}) \quad (3.3)$$

with $A = -\epsilon^2 vt'/c^{\frac{3}{2}}$, $B = (4 + 2\epsilon^2 c - \epsilon^4 v^2 t'^2)/(4c^{\frac{3}{2}})$ and

$$\begin{aligned} c &= 1 + \epsilon^2 r^2 , \\ r^2 &= \frac{1}{3} \hat{D} t'^2 (3t - 2t') . \end{aligned} \quad (3.4)$$

The quasilinear diffusion coefficient $\hat{D}_0$ is obtained from eq. (3.3) by setting $\hat{D} = r = 0$ in the integrand and extending the t'-integration to infinity. This yields

$$\begin{aligned} \hat{D}_0 = \sigma^2 \frac{\sqrt{\pi}}{2\epsilon} \frac{1}{(1+v^2)^{\frac{3}{2}}} \Bigg\{ & \left[\frac{(1+v)^2}{(1+v^2)} + \frac{\epsilon^2}{2}\right] \exp\left[-\frac{(1-v)^2}{\epsilon^2(1+v^2)}\right] \\ + & \left[\frac{(1-v)^2}{(1+v^2)} + \frac{\epsilon^2}{2}\right] \exp\left[-\frac{(1+v)^2}{\epsilon^2(1+v^2)}\right] \Bigg\} . \end{aligned} \quad (3.5)$$

$\hat{D}(v)$ was numerically determined from eq. (3.3) by a standard integration routine and a simple iteration process: starting with the quasilinear value $\hat{D}_0$ an improved $\hat{D}$ was evaluated and reinserted in r until convergence was achieved.

A typical result for D/D_0 as a function of time for different values of the field amplitude σ is shown in Fig. 1. Up to a few times the autocorrelation time $t_0 = \epsilon^{-1}$ (marked with a circle) the diffusion coefficient builds up from zero. In this regime terms have been left out which damp away at later times (see Section 2) so that details are not to be taken seriously. For later times D/D_0 strongly depends on the amplitudes: for "small" σ D equals its quasilinear value D_0 for a very extended interval of time. For "medium-sized" σ the quasilinear value persists for a shorter time interval, while for "large" σ no steady value is achieved at all and D declines without having reached D_0. In order to interpret these results properly, it is useful to consider additional time scales involved.

In a seemingly naive picture a particle diffuses in velocity space until at a critical "diffusion" time t_d it reaches the boundary $\pm v_d$ of the wave packet's interval of phase velocities $\Delta(\omega/k)$ and drops out of resonance. With $\Delta\omega/\omega_0 = \Delta k/k_0 \approx \epsilon$ and $v = O(1)$ one has $v_d \approx \epsilon$, and with $< \Delta v^2 >= 2\hat{D}t$ the order of magnitude of t_d is

$$t_d = \epsilon^2/\hat{D} \; . \tag{3.6}$$

For the quasilinear value $\hat{D}_0$ of $\hat{D}$, eq. (3.5), one has $t_d = \epsilon^3/\sigma^2$. For $t >> t_d$ $\hat{D}$ should then go to zero since more and more particles reach the boundary of the resonance region, where the probability of diffusing back inwards is much smaller than that of diffusing outwards from the inside. In Appendix B it is proved that $\hat{D}$ indeed goes to zero asymptotically. The estimate $\hat{D} < ct^{-\frac{1}{3}}$ is confirmed by numerical results (see Fig. 2).

The role of the parameter t_d is evident in the integrand of eq. (3.3). The ratio

$$\frac{4r^2}{\epsilon^2 v^2 t'^2} = \frac{4\hat{D}(3t-2t')}{3\epsilon^2 v^2} = O\left(\frac{t}{t_d}\right) \tag{3.7}$$

is small in relation to one for $t << t_d$. In this regime one has $\epsilon^2 r^2 = O(\epsilon^2 \hat{D} t'^2 t)$ $\lesssim O(\hat{D}t) << \epsilon^2 \lesssim 1$ so that $c \approx 1$. Hence, provided t is not too small either, the difference to the quasilinear case $r = 0$, $c = 1$ should be small. This is confirmed in Fig. 1, where the time $t = t_d$ is marked with a dash. (For $\sigma = 10^{-3}$ and $\sigma = 10$ t_d is to the right and to the left, respectively, of the t-interval shown.) At $t = t_d$ there is already a substantial decrease of D/D_0, while for $t_0 << t << t_d$ D is close to D_0.

Figures 3 and 4 show the role of t/t_d very clearly: D/D_0 is completely determined by t/t_d independently of whether σ is large or small, provided $t >> t_0$. In Fig. 3 D/D_0 is plotted at $t = t_d$ as a function of the wave amplitude σ. In the second abscissa the values of $t_d = \epsilon^3/\sigma^2$ corresponding to the σ above are indicated. σ is made to vary by three orders of magnitude and t_d by six orders, and yet D/D_0 does not change, up to the regime where t and t_d become comparable. D/D_0 depends somewhat on velocity and ranges from 0.67 to 0.92 for $v = 0.85$ - 1.15 and $\epsilon = 0.8$. In Figure 4 D/D_0 is plotted at $t = 0.1t_d$, i.e. at $t << t_d$. Indeed, D/D_0 is much closer to unity now, ranging from 0.89 to 1.04, and again these values are independent of σ.

In Figure 5 D/D_0 is shown explicitly as a function of t/t_d for a case with $t_d/t_0 \approx 10^3$. Initially, the deviation from the quasilinear value develops differently for different v, while at later stages the trend is the same for all v.

The fact that the deviation of D from the quasilinear value is not a function of σ as such but of the ratio t/t_d, and the above discussion of t_d suggest the interpretation that the "resonance broadening" term in the exponent of eq. (2.11) describes nothing but the loss of particles from the finite resonance region in velocity space in conformity with the original naive picture. The loss process begins as soon as the field is switched on since there is a finite probability at any time for a particle to jump to the border of the interaction region. For $t << t_d$ the probability of having "escaped" is still small. It increases with time until at $t = t_d$ the particle loss is substantial. Therefore, if the observation time t_{ob} is much smaller than t_d (with $t_{ob} >> t_0$), an apparently steady- state quasilinear diffusion is obtained, while for t_{ob} comparable to t_d the diffusion coefficient becomes a (decreasing) function of time, in conformity with Fig. 1. This effect has indeed been observed in numerical particle simulation /13/. Ishihara and Hirose /13/ also presented time-dependent D/D_0 based on analytic expressions similar to eqs. (2.11), (2.14), but the universal nature of the dependence on t/t_d was not recognized.

We finally compare the present theory with that of Dupree /2/ and others /5 - 7/. A major role is played there by the Kolmogorov (or trapping) time t_K, in which a particle diffuses over one wavelength k_0^{-1}. Its order of magnitude according to eq. (2.9) with, for example, $k_{m'} = 0$ is

$$t_K = \hat{D}^{-\frac{1}{3}} . \tag{3.8}$$

(In Dupree's high-amplitude case /2/ one has $\hat{D} \approx \sigma^{\frac{3}{2}}$, yielding $t_K \approx \sigma^{-\frac{1}{2}}$, which is the oscillation period of a particle trapped in a single wave of rms amplitude σ, hence the name.) From the definitions it follows that t_K and t_d are related by

$$\frac{t_d}{t_0} = \left(\frac{t_K}{t_0}\right)^3 . \tag{3.9}$$

In the resonance broadening theory /2/ D substantially differs from the quasilinear value in the high-amplitude case only, characterized by $t_K << t_0$ (equivalent to $\sigma >> \epsilon^2$). This then implies $t_d << t_0$. In this regime, however, according to the results above, D varies on a time scale comparable to t_0 (see Figs. 1 and 6). Additional variations on this time scale, omitted from the figures, originate from the terms $m \neq m'$, $n \neq n'$ of eq. (2.5). Figure 6 shows D/D_0 as a function of σ in the region $t_K \leq t_0$ and its rapid time variation. (For comparison, Dupree's result D_{Du}/D_0 is also shown. D and D_{Du} are close to each other at the instant $t \approx t_K$ only.) Consequently, in the high-amplitude case the assumption of (almost) constant D, made in eq. (2.7), is not satisfied and the particle motion is *not* a diffusion process. This is also confirmed in the particle simulation studies /11/.

Dupree's diffusion coefficient D_{Du} is obtained formally from eq. (2.5) if $< [\Delta x(t) - \Delta x(t')]^2 >$ is replaced by $< [\Delta x(t-t')]^2 >$ in the resonance broadening term (with $k_m = k_{m'}$) and the time integration is extended to infinity. This is done explicitly in

/5 - 7/ (and applied to obtain D_{Du} in Fig. 6). Equation (2.9), shows, however, that this approximation is not valid, i.e. is not consistent with the assumed diffusion process.

The present theory, in contrast, keeps the resonance broadening term in its original selfconsistent form. In conclusion, it essentially describes the fact, that owing to diffusion, particles stay in the resonance zone for finite times only. This causes a time-dependent deviation of the diffusion coefficient from its quasilinear value. Furthermore, if the wave amplitude is very large, particles get out of the interaction region so fast that a regular diffusion process is not established, and Dupree's theory, constructed specifically for the large-amplitude case, does not seem appropriate.

Appendix A

For the wave spectrum

$$s(k,\omega) = \frac{1}{2\sqrt{\pi}}\,\frac{1}{k_0\epsilon}\left\{\exp\left[-\left(\frac{k-k_0}{\epsilon k_0}\right)^2\right]\delta(\omega-\omega_0) + \exp\left[-\left(\frac{k+k_0}{\epsilon k_0}\right)^2\right]\delta(\omega+\omega_0)\right\} \tag{A.1}$$

the expressions corresponding to eqs. (3.2), (3.3) and (3.5) are

$$g(x,t) = \cos(x-t)\,\exp\left[-\left(\frac{\epsilon x}{2}\right)^2\right], \tag{A.2}$$

$$\hat{D}(v) = \sigma^2 \int_0^t dt'\,\exp\left(-\frac{\epsilon^2 v^2 t'^2 + 4r^2}{4c}\right)\cdot\left(A\sin\frac{v-c}{c}t' + B\cos\frac{v-c}{c}t'\right) \tag{A.3}$$

and

$$\hat{D}_0 = \sigma^2\,\frac{\sqrt{\pi}}{\epsilon}\,\frac{1}{|v|^3}\,\exp\left[-\left(\frac{v-1}{\epsilon v}\right)^2\right]. \tag{A.4}$$

Appendix B

If the absolute value in eq. (2.11) is taken and $3t - 2t'$ is replaced by t, the integrand is increased. Replacing the upper limit of integration by infinity, one obtains

$$
\begin{aligned}
\hat{D} &< \sigma^2 \int_{-\infty}^{+\infty} dk \left(\frac{k}{k_0}\right)^2 \int_{-\infty}^{+\infty} d\omega \, |s(k,\omega)| \int_0^{\infty} dt' \exp\left[-\frac{1}{3}\left(\frac{k}{k_0}\right)^2 \hat{D}t'^2 t\right] \\
&= \frac{\sigma^2}{\sqrt{\hat{D}t}} \frac{\sqrt{3\pi}}{2} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \left|\frac{k}{k_0} s(k,\omega)\right|
\end{aligned}
\tag{B.1}
$$

so that

$$
\hat{D} < \left(\frac{a^2\sigma^4}{t}\right)^{\frac{1}{3}} , \tag{B.2}
$$

where a is the factor coming with $\sigma^2/\sqrt{\hat{D}t}$ in eq. (B.1) and depends on ϵ only. For the wave packet (3.1), for example, a reduces to

$$
\begin{aligned}
a &= \frac{\sqrt{3}}{4} \int_{-\infty}^{+\infty} dx \, \exp(-x^2) \, (|1+\epsilon x| + |1-\epsilon x|) \\
&< \frac{\sqrt{3}}{2} \left[(\sqrt{\pi} + \epsilon \exp\left(\frac{-1}{\epsilon}\right)\right] .
\end{aligned}
\tag{B.3}
$$

<u>References</u>

/1/ A.A. Vedenov, E.P. Velikhov and R.Z. Sagdeev, Nucl. Fusion <u>1</u>, 82 (1961); W.E. Drummond and D. Pines, Nucl. Fusion Suppl. Part <u>3</u>, 1049 (1962).

/2/ T.H. Dupree, Phys. Fluids <u>9</u>, 1773 (1966)

/3/ T.H. Dupree, Phys. Fluids <u>10</u>, 1049 (1967);

T.H. Dupree and D.J. Tetreault, Phys. Fluids <u>21</u>, 425 (1978).

/4/ S.P. Hirshman and K. Molvig, Phys. Rev. Lett. <u>42</u>, 648 (1979);

K. Molvig, S.P. Hirshman and J.C. Whitson, Phys. Rev. Lett. <u>43</u>, 582 (1979).

/5/ J. Weinstock, Phys. Fluids <u>11</u>, 1977 (1968).

/6/ G. Benford and J.J. Thomson, Phys. Fluids <u>15</u>, 1496 (1972).

/7/ K. Molvig, J.P. Freidberg et al. in "Long-Time Prediction in Dynamics" pp.319-343; C.W. Horton, L.E. Reichl, V.G. Szebehely, Eds.; John Wiley and Sons, New York 1983.

/8/ G.M. Zaslavskii and B.V. Chirikov, Usp. Fiz. Nauk <u>105</u>, 3 (1971) [Sovj. Phys. Uspekhi <u>14</u>, 549 (1972)].

/9/ I. Cook and A.D. Sanderson, Plasma Phys. <u>16</u>, 977 (1974).

/10/ P. Rolland, J. Plasma Phys. <u>15</u>, 57 (1976).

/11/ K. Graham and J. Fejer, Phys. Fluids <u>19</u>, 1054 (1976).

/12/ F. Doveil and D. Grésillon, Phys. Fluids <u>25</u>, 1396 (1982).

/13/ O. Ishihara and A. Hirose, Phys. Fluids <u>28</u>, 2159 (1985).

/14/ A. Salat, Z. Naturforsch. A <u>38</u>, 1189 (1983).

/15/ S.M. Krivoruchko, V.A. Bashko and A.S. Bakai, Zh. Eksp. Teor. Fiz. <u>80</u>, 579 (1981) [Sovj. Phys. JETP <u>53</u>, 292 (1981)].

/16/ G. Laval and D. Pesme, Phys. Fluids 26, 52 (1983).

/17/ P.H. Diamond and M.N. Rosenbluth, Phys. Fluids 24, 1641 (1981).

/18/ J.A. Krommes and P. Similon, Phys. Fluids 23, 1553 (1980).

/19/ E. Parzen, "Modern Probability Theory and its Applications", John Wiley and Sons, New York 1976.

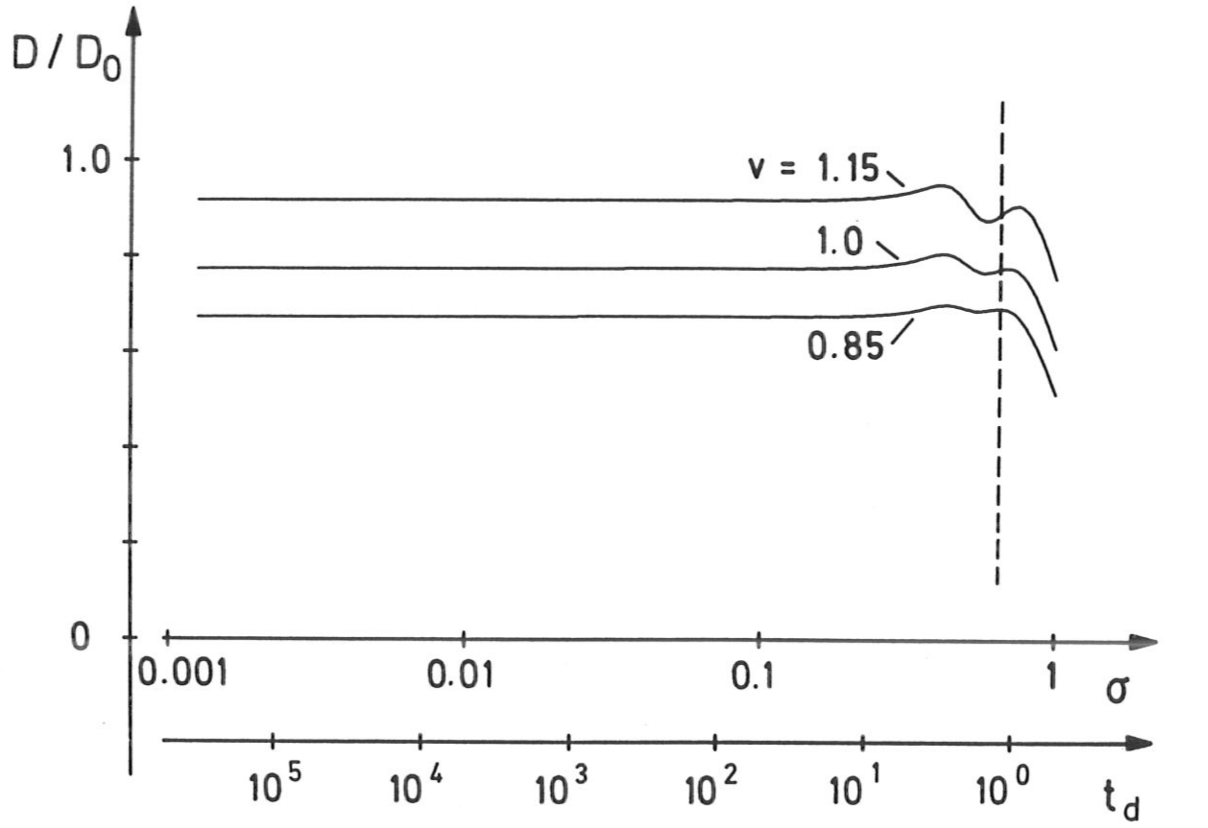

D / D0
1.0
v = 1.15
1.0
0.85
0
0.001
0.01
0.1
1
σ
10 5
10 4
10 3
10 2
10 1
10 0
t d

Fig. 3

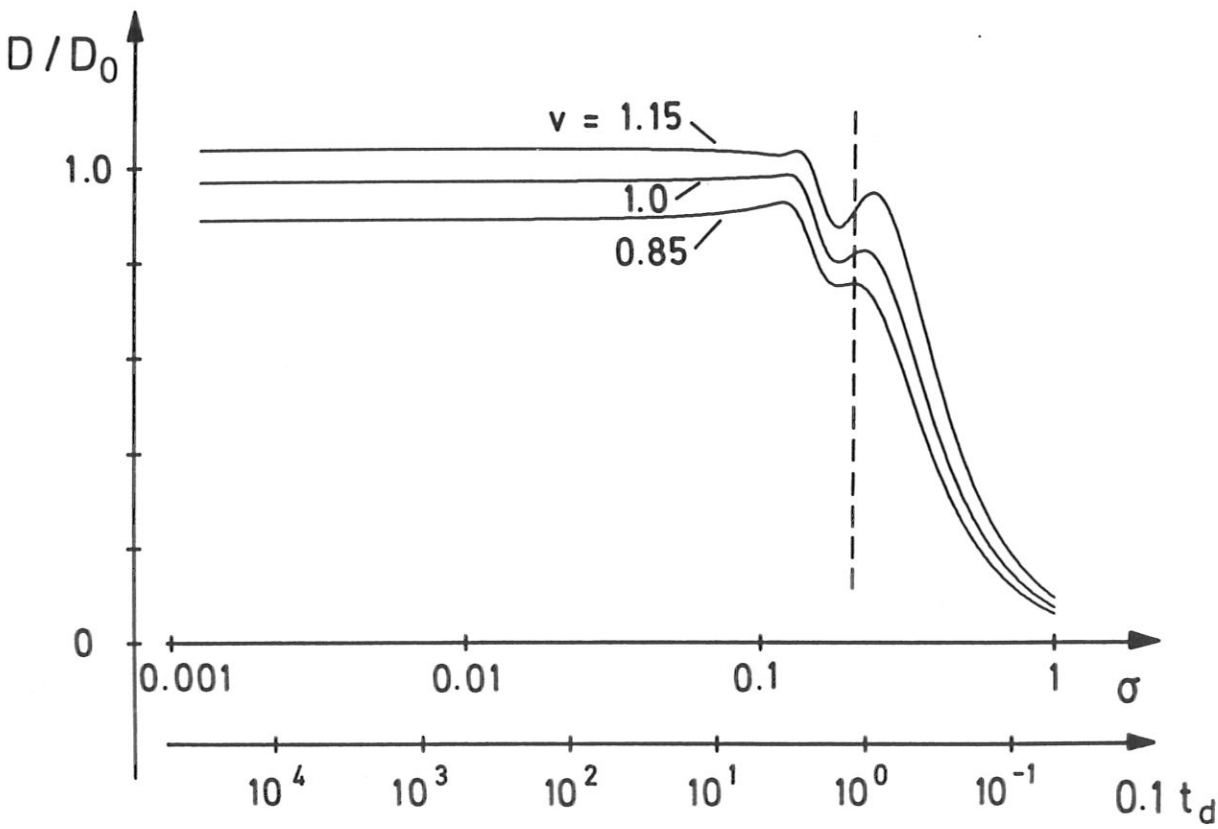
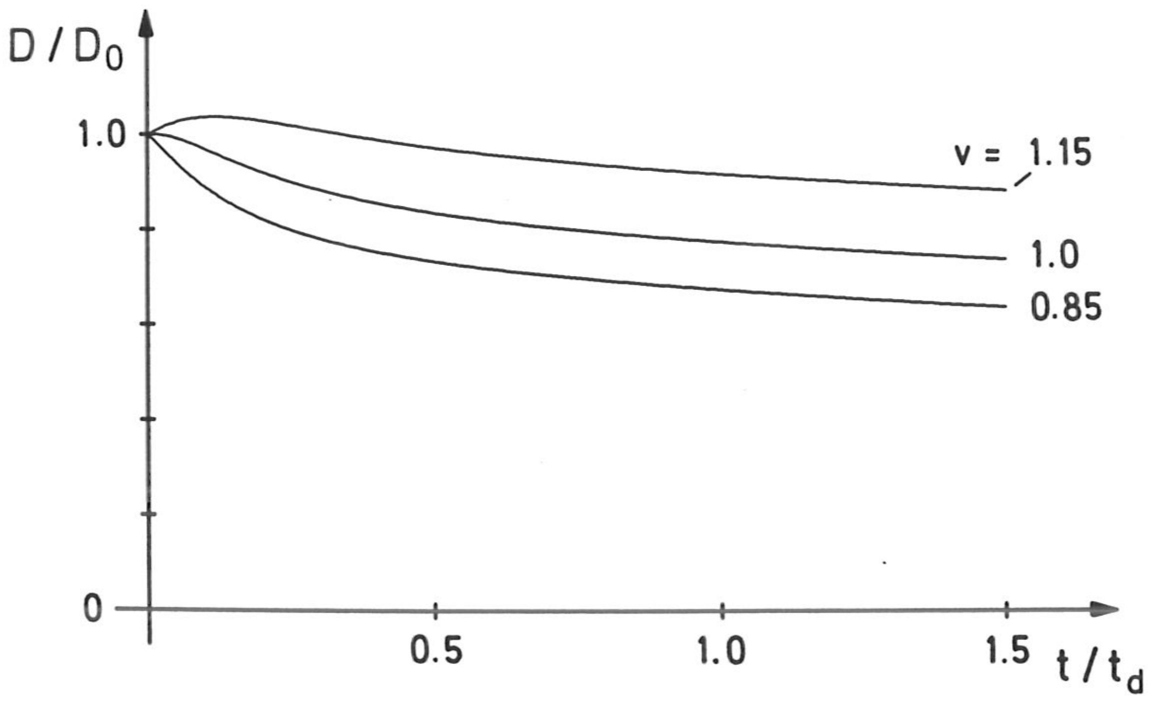
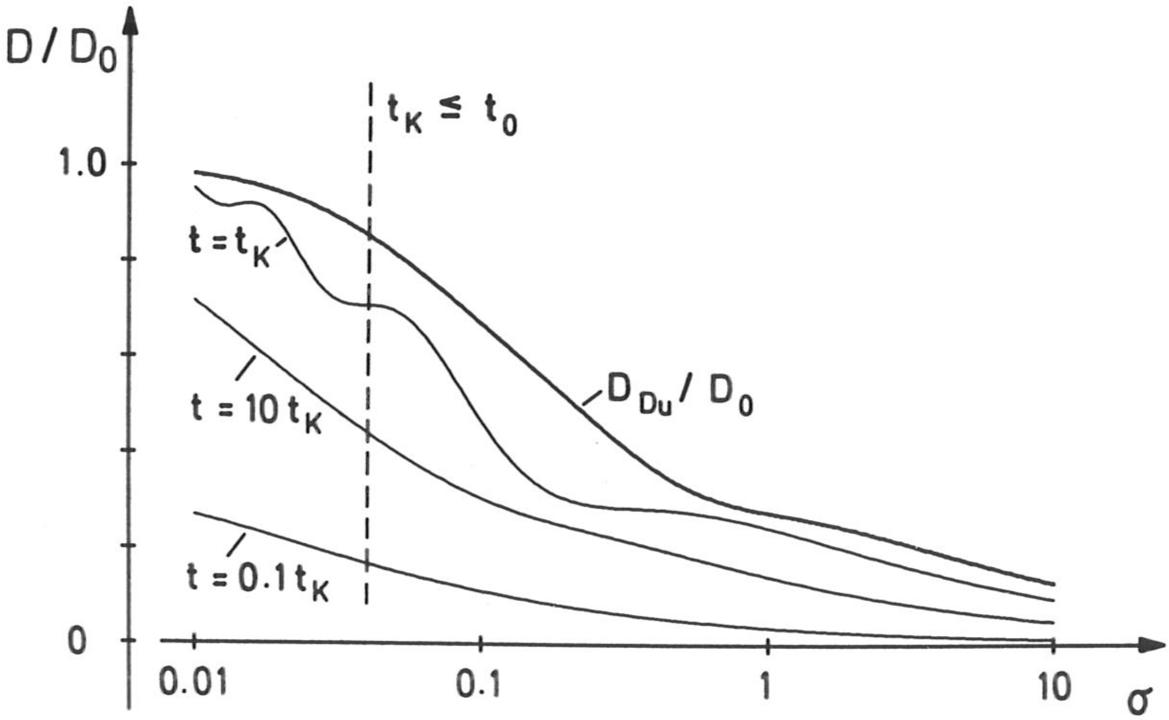

D / D0
v = 1.15
1.0
1.0
0.85
0
0.001
0.01
0.1
1
σ
10 4
10 3
10 2
10 1
10 0
10 -1
0.1 t d

Fig. 4



D / D0
1.0
0
v = 1.15
1.0
0.85
0.5
1.0
1.5
t / td

Fig. 5



D / D0
tK ≤ t0
1.0
t = tK
t = 10 tK
t = 0.1 tK
DDu / D0
0
0.01
0.1
1
10
σ

Fig. 6