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OF ION CYCLOTRON HEATING
OF TOKAMAK PLASMAS

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A MODEL FOR THE NUMERICAL SIMULATION OF ION CYCLOTRON HEATING OF TOKAMAK PLASMAS

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Abstract

We present a complete set of equations for the numerical simulation of ion cyclotron heating of tokamak plasmas. The model includes the full geometry of the tokamak equilibrium, full parallel dispersion, and perpendicular dispersion to second order in the Larmor radius. It is therefore capable of describing correctly ion cyclotron damping at the fundamental and first harmonic, as well as mode conversion to the ion Bernstein wave and/or the shear Alfvén wave, depending on the heating scenario. It includes also electron magnetic pumping and Landau damping, the latter to lowest order in m_e/m_i .

Relying on the knowledge gained from slab and ray tracing analysis, we also situate with respect to this standard model some of the further approximations which are commonly encountered in the literature. Finally, two procedures for the numerical solution of the standard model are proposed.

As a complement to a "ray tracing equation" i.e. a relation between the \mathbf{E} field and the oscillating current in the plasma. A potential variational approach to this problem has been developed by Chabrier and Sauter (1975) and other techniques have been used for the solution of the time-averaged Fokker-Planck equation by Kruer et al (1977). Here we greatly simplify our task by assuming that most ion species contribute harmonically resonances with a constant speed along the magnetic field lines. We will refer to this approximation as the "ballistic" model.

In section 2 we introduce the physical quantities and the differential operators needed

1. Introduction

H.F. waves in the ion cyclotron frequency domain are a favourite auxiliary heating method for large tokamak plasmas [1–4]. Considerable effort has therefore been devoted to the theory of their coupling, propagation, and absorption. The complex geometry of a tokamak was first simulated with a slab model including the appropriate variation of $|\vec{B}_0|$ and shear [5–6]. Considerable progress was made with the introduction of ray tracing [7–11], combined with a semi-analytic treatment of resonance layers [12]. A large amount of physics can be understood with these approaches; their limitations are, however, obvious. The next step should be the numerical solution of the full wave equation in two dimensions (the toroidal coordinate φ being ignorable, it can be eliminated by a development of the field in toroidal Fourier components). The first “global” codes have been recently developed [13–19].

Upon a closer look to the problem, however, one rapidly realizes that global codes for the simulation of ICRH of large tokamak plasmas are not far from saturating the capabilities of present-day computers. To keep things manageable, one is forced to make approximations, whose implications are not always easy to understand. Efforts to analytically unravel the behaviour of solutions of the wave equations in two dimensions have sometimes further confused rather than clarified the issues [20, 21]. The main goal of this report is to present a complete model (i.e. a self-contained set of equations) retaining the essential physics yet amenable to numerical solution with standard techniques. Relying on the knowledge gained from slab and ray tracing simulations, we also try to situate with respect to this “standard” model some of the further approximations which are commonly encountered in the literature. Finally, two procedures for the numerical solution of the standard model are proposed.

The first difficulty encountered in the realization of this programme is that of reducing the full set of Maxwell plus quasi-linear Vlasov-Fokker-Planck equations to an approximate but more manageable one, in which the information about the velocity space dynamics is compacted into a “constitutive relation”, i.e. a relation between the h.f. field and the oscillating current in the plasma. A powerful variational approach to this problem has been developed by Gambier and Samain [22]; related techniques have been used for the solution of the bounce-averaged Fokker-Planck equation by Kerbel et al. [23]. Here we greatly simplify our task by assuming that most ions cross cyclotron harmonic resonances with a constant speed along the magnetic field lines. We will refer to this approximation as the “ballistic” model.

In Section 2 we introduce the plasma equilibrium and the differential operators needed

later. The set of equations to be solved, based on the work of Swanson [24, 25], Colestock and Kashuba [26], and Brambilla and Ottaviani [27] is presented in Section 3. There we also briefly discuss the approximations included in the “ballistic” model, and we deal with the limit of vanishing electron inertia in a novel way, which allows to retain electron Landau damping, while eliminating from the equations the very small component of the h.f. electric field parallel to the static magnetic field. Finally, in Section 4, the equations are recast in the weak-variational form appropriate for numerical discretization: the boundary conditions and the power balance equation are also formulated.

In spite of the simplifications due to the ballistic approximation and the zero electron inertia limit, the equations of Sections 3–4 are very difficult to solve numerically due to the dispersive nature of the plasma response to h.f. waves. Dispersion has two consequences. On the one hand, finite Larmor radius (FLR) effects are responsible for the existence of short-wavelength, quasi-electrostatic modes, which are excited by linear mode conversion [25] in the vicinity of wave resonances. The need to resolve short-wavelength features sets heavy constraints on the mesh for the numerical discretization, whose severity increases with the plasma dimensions. On the other hand, free streaming of the particles along magnetic field lines makes the constitutive relation non-local in the vicinity of IC harmonics. Thus to describe kinetic absorption adequately, one has, in principle, to solve a set of integro-differential equations. To our knowledge, no code thus far has coped with both these problems.

In Section 5 we discuss the approximations which are most commonly made to alleviate these difficulties. We claim that some of these approximations, particularly the one consisting in taking the cold plasma limit of the constitutive relation, simplify the algebra rather than the numerical task; and that on occasions (including some of the most important ICH scenarios) they give quite misleading results, in spite of a flawless numerical solution.

Finally, in Section 6 we suggest two numerical schemes for the solution of the integro-differential equations of Sections 3–4. One is based on a mixed spectral-finite element discretization; the other uses finite elements in two dimensions, and iterates to cope with the non-local nature of the constitutive relation. Work towards the implementation of the spectral method is in progress [28]; the second method remains for the moment speculative. Our conclusion is that any numerical approach, to be successful, must be tailored as far as possible to the physics of the problem.

2. Description of the Plasma Equilibrium

It is convenient to describe the tokamak equilibrium in the parametric form

$$X = X(\psi, \vartheta) \quad Z = Z(\psi, \vartheta) \quad (1)$$

X, Z are horizontal and vertical Cartesian coordinates in the poloidal cross-section; ψ labels magnetic surfaces. and ϑ is some poloidal angle. The third coordinate will be the toroidal angle φ .

The covariant metric of coordinates $(\psi, \vartheta, \varphi)$ is

$$g_{ij} = \begin{vmatrix} N_\psi^2 & G & O \\ G & N_\vartheta^2 & O \\ O & O & R^2 \end{vmatrix} \quad (2)$$

$$g = (\det g_{ij})^{1/2} = RJ \quad (3)$$

where $R = R_0 + X$ is the distance from the vertical axis, and

$$\begin{aligned} N_\psi^2 &= \left(\frac{\partial X}{\partial \psi} \right)^2 + \left(\frac{\partial Z}{\partial \psi} \right)^2 = \frac{1}{N_\vartheta^2} (J^2 + G^2) \\ N_\vartheta^2 &= \left(\frac{\partial X}{\partial \vartheta} \right)^2 + \left(\frac{\partial Z}{\partial \vartheta} \right)^2 \\ G &= \frac{\partial X}{\partial \psi} \frac{\partial X}{\partial \vartheta} + \frac{\partial Z}{\partial \psi} \frac{\partial Z}{\partial \vartheta} \\ J &= \frac{\partial X}{\partial \psi} \frac{\partial Z}{\partial \vartheta} - \frac{\partial X}{\partial \vartheta} \frac{\partial Z}{\partial \psi} \end{aligned} \quad (4)$$

In these coordinates, the static magnetic field can be written

$$\vec{B} = B_0 R_0 \{ f(\psi) \vec{\nabla} \varphi \times \vec{\nabla} \psi + g(\psi) \vec{\nabla} \varphi \} \quad (5)$$

so that its toroidal and poloidal components are, respectively,

$$\begin{aligned} B_\varphi &= B_0 R_0 \frac{g(\psi)}{R} \\ B_r &= \frac{N_\vartheta}{RJ} B_0 R_0 f(\psi) \end{aligned} \quad (6)$$

In the general case, $G \neq 0$, so that $(\psi, \vartheta, \varphi)$ are not orthogonal coordinates. At each point we can, however, introduce an orthogonal triad of unit vectors

$$\begin{aligned} \vec{e}_\psi &= \frac{J}{N_\vartheta} \vec{\nabla} \psi \\ \vec{e}_r &= \frac{G}{N_\vartheta} \vec{\nabla} \psi + N_\vartheta \vec{\nabla} \vartheta \\ \vec{e}_\varphi &= R \vec{\nabla} \varphi \end{aligned} \quad (7)$$

($\vec{e}_r = \vec{e}_\varphi \times \vec{e}_\psi$ points in the direction of the poloidal magnetic field.)

For subsequent reference, we list here the most important differential operators in the $(\psi, \vartheta, \varphi)$ coordinates:

$$\vec{\nabla}\phi = \frac{1}{J}(N_r \frac{\partial\phi}{\partial\psi} - \frac{G}{N_r} \frac{\partial\phi}{\partial\vartheta})\vec{e}_\psi + \frac{1}{N_r} \frac{\partial\phi}{\partial\vartheta}\vec{e}_r + \frac{1}{R} \frac{d\phi}{d\varphi}\vec{e}_\varphi \quad (8)$$

$$\text{div}\vec{A} = \frac{1}{RJ} \left\{ \frac{\partial}{\partial\psi}(RN_r A_\psi) + \frac{\partial}{\partial\vartheta} \left[R \left(\frac{J}{N_r} A_r - GA_\psi \right) \right] \right\} + \frac{1}{R} \frac{\partial A_\varphi}{\partial\varphi} \quad (9)$$

$$\begin{aligned} \text{rot}\vec{A} = & \left[\frac{1}{N_r R} \frac{\partial}{\partial\vartheta}(RA_\varphi) - \frac{1}{R} \frac{\partial A_r}{\partial\varphi} \right] \vec{e}_\psi \\ & + \frac{1}{R} \left\{ \frac{\partial A_\psi}{\partial\varphi} - \frac{1}{J} \left[N_r \frac{\partial}{\partial\psi}(RA_\varphi) - \frac{G}{N_r} \frac{\partial}{\partial\vartheta}(RA_\varphi) \right] \right\} \vec{e}_r \\ & + \frac{1}{J} \left\{ \frac{\partial}{\partial\psi}(N_r A_r) - \frac{\partial}{\partial\vartheta} \left[\frac{1}{N_r} (JA_\psi + GA_r) \right] \right\} \vec{e}_\varphi \end{aligned} \quad (10)$$

A simple description of the h.f. response of the plasma, on the other hand, is only possible in a reference frame which singles out explicitly the direction of the static magnetic field. A triad of units vectors satisfying this condition can be obtained from the previous one by a simple rotation:

$$\begin{aligned} \vec{e}_\zeta &= \vec{e}_\psi \\ \vec{e}_\eta &= \cos\Theta \vec{e}_r - \sin\Theta \vec{e}_\varphi \\ \vec{e}_\zeta &= (\vec{B}/B) = \sin\Theta \vec{e}_r + \cos\Theta \vec{e}_\varphi \end{aligned} \quad (11)$$

where

$$\tan\Theta(\psi, \vartheta) = B_r/B_\varphi \quad (12)$$

We will refer to the (physical) components of a vector along the triad (11) as its "Stix" components. In addition, it is often convenient to introduce the rotating unit vectors

$$\vec{e}_\pm = \frac{1}{\sqrt{2}}(\vec{e}_\zeta \mp i\vec{e}_\eta) \quad (13)$$

They allow the decomposition of $\vec{E}_\perp = \vec{E} - (\vec{E} \cdot \vec{B})/\vec{B}$ in left (+) and right (-) circularly polarized components.

Although (ψ, η, ζ) coordinates are defined only in a local sense, it is of course straightforward to express the differential operators (8) to (10) in the Stix reference frame:

$$\vec{e}_\psi \cdot \vec{\nabla} \Phi \equiv \partial_\psi \Phi = \frac{1}{J} \left(N_\tau \frac{\partial \Phi}{\partial \psi} - \frac{G}{N_\tau} \frac{\partial \Phi}{\partial \Theta} \right)$$

$$\vec{e}_\eta \cdot \vec{\nabla} \Phi \equiv \partial_\eta \Phi = \frac{\cos \Theta}{N_\tau} \frac{\partial \Phi}{\partial \vartheta} - \frac{\sin \Theta}{R} \frac{\partial \Phi}{\partial \varphi} \quad (14)$$

$$\vec{e}_\zeta \cdot \vec{\nabla} \Phi \equiv \partial_\zeta \Phi = \frac{\sin \Theta}{N_\tau} \frac{\partial \Phi}{\partial \vartheta} + \frac{\cos \Theta}{R} \frac{\partial \Phi}{\partial \varphi}$$

$$\text{div} \vec{A} = (\partial_\psi + \nu_\psi) A_\psi + (\partial_\eta + \nu_\eta) A_\eta + (\partial_\zeta + \nu_\zeta) A_\zeta$$

$$\nu_\psi = \frac{1}{RJ} \left[\frac{\partial}{\partial \psi} (RN_\tau) - \frac{\partial}{\partial \vartheta} \left(\frac{RG}{N_\tau} \right) \right]$$

$$\nu_\eta = \frac{1}{RJ} \frac{\partial}{\partial \vartheta} \left(\frac{RJ}{N_\tau} \cos \Theta \right) \quad (15)$$

$$\nu_\zeta = \frac{1}{RJ} \frac{\partial}{\partial \vartheta} \left(\frac{RJ}{N_\tau} \sin \Theta \right)$$

$$\vec{e}_\psi \cdot \text{rot} \vec{A} = (\partial_\eta + \gamma_{\psi\eta}) A_\eta - (\partial_\zeta + \gamma_{\psi\zeta}) A_\zeta$$

$$\gamma_{\psi\eta} = \frac{1}{N_\tau R} \frac{\partial}{\partial \vartheta} (R \sin \Theta)$$

$$\gamma_{\psi\zeta} = \frac{1}{N_\tau R} \frac{\partial}{\partial \vartheta} (R \cos \Theta)$$

$$\vec{e}_\eta \cdot \text{rot} \vec{A} = (\partial_\zeta + \gamma_{\eta\zeta}) A_\zeta + \gamma_{\eta\eta} A_\eta - (\partial_\psi + \gamma_{\eta\psi}) A_\psi$$

$$\gamma_{\eta\psi} = \frac{\sin \Theta}{J} \frac{\partial}{\partial \vartheta} \left(\frac{J}{N_\tau} \right)$$

$$\gamma_{\eta\eta} = \cos \Theta \sin \Theta \left[\frac{1}{R} \partial_\psi R - \frac{1}{N_\tau} \partial_\psi N_\tau \right] + \partial_\psi \Theta \quad (16)$$

$$\gamma_{\eta\zeta} = \cos^2 \Theta \frac{1}{R} \partial_\psi R + \sin^2 \Theta \frac{1}{N_\tau} \partial_\psi N_\tau$$

$$\vec{e}_\zeta \cdot \text{rot} \vec{A} = -(\partial_\eta + \gamma_{\zeta\eta}) A_\eta + (\partial_\psi + \gamma_{\zeta\psi}) A_\psi + \gamma_{\zeta\zeta} A_\zeta$$

$$\gamma_{\zeta\psi} = \frac{\cos \Theta}{J} \frac{\partial}{\partial \vartheta} \left(\frac{J}{N_\tau} \right)$$

$$\gamma_{\zeta\eta} = \sin^2 \Theta \frac{1}{R} \partial_\psi R + \cos^2 \Theta \frac{1}{N_\tau} \partial_\psi N_\tau$$

$$\gamma_{\zeta\zeta} = \sin \Theta \cos \Theta \left\{ -\frac{1}{R} \partial_\psi R + \frac{1}{N_\tau} \partial_\psi N_\tau \right\} - \partial_\psi \Theta$$

We note the identities which will be useful later:

$$\begin{aligned}
\gamma_{\eta\zeta} + \gamma_{\zeta\eta} &= \nu_\psi \\
\gamma_{\zeta\psi} + \gamma_{\psi\zeta} &= \nu_\eta \\
\gamma_{\eta\psi} + \gamma_{\psi\eta} &= \nu_\zeta \\
\gamma_{\eta\eta} + \gamma_{\zeta\zeta} &= 0
\end{aligned} \tag{17}$$

3. The Constitutive Relation

For a wave with time dependence $\exp(-i\omega t)$, Maxwell equations can be written

$$\text{rot rot } \vec{E} = \frac{\omega^2}{c^2} (\vec{E} + \frac{4\pi i}{\omega} \vec{J}) \tag{18}$$

where \vec{J} , the current density within the plasma, is a linear functional of \vec{E} . In a slab model, where a dependence of the form $\exp(ik_\zeta \zeta)$, with $k_\zeta = k_\parallel$ a constant, can be assumed for the field, $\vec{J}(\vec{E})$ has been obtained from the linearized Vlasov equation correct to second order in the ion Larmor radius by Swanson [24] and by Colestock and Kashuba [26]. While the assumption of a constant k_\parallel does not exclude shear ([24, 12]), it is, nevertheless, incompatible with the full two-dimensional geometry of a tokamak plasma. In this case $J(\vec{E})$ is non-local even in the small Larmor radius limit.

The constitutive relation for this situation has been obtained by Brambilla and Ottaviani [27], under the additional assumption that ions cross cyclotron resonances with constant parallel velocity, and that their parallel distribution function is Maxwellian. Distinguishing the zeroth and second order contributions in the Larmor radius expansion,

$$\vec{J} = \vec{J}^{(0)} + \vec{J}_e^{(2)} + \sum_{ions} \vec{J}_i^{(2)} \tag{19}$$

the results of Ref. [27] can be written

$$\begin{aligned}
\vec{E} + \frac{4\pi i}{\omega} \vec{J}^{(0)} &= \hat{L} E_+ \vec{e}_+ + \hat{R} E_- \vec{e}_- + \hat{P} E_\zeta \vec{e}_\zeta \\
\frac{4\pi i}{\omega} \vec{J}_e^{(2)} &= \vec{\nabla}_\perp \times (\hat{\lambda}_{oe} (\vec{\nabla}_\perp \times \vec{E}_\perp)) \\
\frac{4\pi i}{\omega} \vec{J}_i^{(2)} &= \frac{1}{2} (\vec{\nabla}_\perp - i \vec{e}_\zeta \times \vec{\nabla}_\perp) \left[\hat{\lambda}_{2i} ((\vec{\nabla}_\perp - i \vec{\nabla}_\perp \times \vec{e}_\zeta) \vec{E}_\perp) \right]
\end{aligned} \tag{20}$$

where $\vec{E}_\perp = E_\psi \vec{e}_\psi + E_\eta \vec{e}_\eta = E_+ \vec{e}_+ + E_- \vec{e}_-$. The perpendicular differential operators are easily identified by inspection of eqs. (14) – (17); they are most easily expressed in rotating coordinates through

$$\begin{aligned}\partial_\pm &= \frac{1}{\sqrt{2}} \left[(\partial_\psi + \gamma_\eta) \pm i(\partial_\eta + \gamma_\psi) \right] \\ \partial^\pm &= \frac{1}{\sqrt{2}} \left[(\partial_\psi + \gamma_\eta) \pm i(\partial_\eta + \gamma_\psi) \right]\end{aligned}\quad (21)$$

Then

$$\begin{aligned}\vec{\nabla}_\perp \times \vec{E}_\perp &= -i(\partial_- E_+ - \partial_+ E_-) \vec{e}_\zeta \\ \vec{\nabla}_\perp \times \vec{A}_\zeta &= -1 \left[(\partial^+ A_\zeta) \vec{e}_+ - (\partial^- A_\zeta) \vec{e}_- \right] \\ \frac{4\pi i}{\omega} \vec{J}_i^{(2)} &= 2\partial^- (\lambda_2 \partial_+ E_+) \vec{e}_+\end{aligned}\quad (22)$$

Finally, in eq. (20) we have introduced the following integral operators:

$$\begin{aligned}\hat{L}E_+(\psi, \vartheta, \varphi) &= \left(1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} \left(1 - \frac{\Omega_{ce}}{\omega} \right) \right) E_+(\psi, \vartheta, \varphi) \\ &+ \sum_{ions} \frac{\omega_{pi}^2}{\omega^2} \int dv_\parallel \frac{e^{-v_\parallel^2/v_{thi}^2}}{\sqrt{\pi} v_{thi}} \left[i\omega \int_0^\infty d\tau e^{i(\omega - \Omega_i)\tau} E_+(\psi, \vartheta'(\tau), \varphi'(\tau)) \right] \\ \hat{R}E_-(\psi, \vartheta, \varphi) &= \left[1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} \left(1 + \frac{\Omega_{ce}}{\omega} \right) - \sum_i \frac{\omega_{pi}^2}{\omega^2} \frac{\omega}{\omega + \Omega_{ci}} \right] E_-(\psi, \vartheta, \varphi) \\ \hat{P}E_\zeta(\psi, \vartheta, \varphi) &= - \frac{\omega_{pe}^2}{\omega^2} \int dv_\parallel \frac{e^{-v_\parallel^2/v_{the}^2}}{\sqrt{\pi} v_{the}} \left[i\omega \int_0^\infty d\tau e^{i\omega\tau} E_\zeta(\psi, \vartheta'(\tau), \varphi'(\tau)) \right] \\ \hat{\lambda}_{n,\alpha} \vec{E}_\perp(\psi, \vartheta, \varphi) &= \frac{1}{2} \frac{\omega_{p\alpha}^2}{\Omega_{c\alpha}^2} \frac{v_{th\alpha}^2}{c^2} \int dv_\parallel \frac{e^{-v_\parallel^2/v_{th\alpha}^2}}{\sqrt{\pi} v_{th\alpha}} \\ &\cdot \left[i\omega \int_0^\infty d\tau e^{i(\omega - n\Omega_\alpha)\tau} \vec{E}_\perp(\psi, \vartheta'(\tau), \varphi'(\tau)) \right]\end{aligned}\quad (23)$$

$$\vartheta'(\tau) = \vartheta - \frac{v_\parallel}{N_r} \sin\Theta \cdot \tau \quad \varphi'(\tau) = \varphi - \frac{v_\parallel}{R} \cos\Theta \cdot \tau$$

\hat{L} and \hat{R} contain the polarization and the $\vec{E} \times B$ drift of the electrons, and the zero Larmor radius response of the ions: in particular, the antihermitean part of \hat{L} describes ion cyclotron damping (ICD) at $\omega = \Omega_{ci}$. \hat{P} is dominated by the very large parallel inertial response of the electrons: its antihermitean part describes electron Landau

damping (ELD). Among the FLR terms, the antihermitean part of $\hat{\lambda}_{oe}$ describes electron magnetic pumping (EMP), and the one of $\hat{\lambda}_{2i}$ first harmonic ICD. We recall that $\hat{\lambda}_{2i}$ is resonant at $\omega = 2\Omega_{ci}$, where all zero Larmor radius terms remain finite: this is responsible for a relatively large evanescence gap between the resonance layer and the confluence with the lowest ion Bernstein wave [29, 30]. Terms proportional to $\hat{\lambda}_{oi}$ and $\hat{\lambda}_{1i}$, which remains always small compared with homologous zeroth-order terms, have been omitted from eq. (20).

While the restriction to Maxwellian distribution functions can be trivially removed, the ballistic assumption $v_{\parallel} = \text{cte.}$ during transit through a cyclotron resonance deserves some comment. The typical duration of a resonance is [22]

$$T_{Res} \cong \sqrt{\frac{\pi R}{\Omega_o v_{\parallel} \sin \theta}} \quad (24)$$

during which the change in v_{\parallel} can be estimated to be

$$\Delta v_{\parallel} \cong \frac{v_{\perp}^2}{2} \sqrt{\frac{\pi \sin \theta}{\Omega v_{\parallel} R}} \quad (25)$$

This is certainly small for passing particles, but not for trapped particles on orbits with reflection points close to the resonance plane. Moreover, solution of the Fokker-Planck equation in arbitrarily specified field distributions [31–32] show a tendency to the accumulation of such particles, for which the ballistic approximation is obviously untenable. Nevertheless, in a tokamak, their number at any given time is bound to be small because of their rapid vertical diffusion. Thus, while h.f. -driven trapping might be important in producing escaping fast particles, it is unlikely to affect appreciably wave propagation.

At IC frequencies, electrons screen very efficiently any electric field component parallel to the static magnetic field: this is reflected in the fact that the coefficient of \hat{P} is m_i/m_e times larger than those of \hat{R} and \hat{L} . The difficulty of dealing with the resulting bad scaling is usually avoided by taking the limit $\omega^2/\omega_{pe}^2 \rightarrow 0$, which of course implies $E_{\zeta} \rightarrow 0$.

The limit of zero electron inertia, however, eliminates electron Landau damping altogether from eq. (18). This is well justified for the fast wave, for which EMP in any case exceeds ELD by at least one order of magnitude. On the other hand, ELD is in some important cases the only non-collisional absorption mechanism (but then a very efficient one, cfr. Sect. 5) for short wavelength modes, which are almost irrotational,

hence insensitive to EMP. A correct evaluation of electron heating therefore requires keeping ELD in the model.

To make this possible while still avoiding scaling problems, we use the ζ component of eq. (18) to estimate E_ζ iteratively:

$$\hat{P}E_\zeta = \frac{c^2}{\omega^2}(\text{rotrot}\vec{E}_\perp)_\zeta \quad (26)$$

Strictly speaking, this equation is not explicit, since \hat{P} is an integral operator. In any conceivable ICH scenario, however, short wavelength modes can only be excited when the antihermitean part of \hat{P} is very small, and are absorbed well before this condition is violated. We can therefore “linearize” around the cold limit, and invert eq.(27) as

$$E_\zeta \cong \frac{c^2}{\omega^2} \frac{\hat{P}^\dagger}{P_0^2}(\text{rotrot}\vec{E}_\perp)_\zeta \quad (P_0 = -\omega_{pe}^2/\omega^2) \quad (27)$$

where \hat{P}^\dagger denotes the hermitean conjugate of \hat{P} . This expression can then be used in the two remaining components of eq. (18) to eliminate E_ζ altogether while retaining ELD to leading order in m_e/m_i .

4. Weak Variational Formulation

In view of its numerical solution, it is convenient to recast eq. (18) in the so-called weak-variational, or Galerkin, form [33]:

$$\int dV \vec{F}^*(\vec{r}) \left\{ \text{rotrot}\vec{E} - \frac{\omega^2}{c^2} \left(\vec{E} + \frac{4\pi i}{\omega} \vec{J} \right) \right\} = 0 \quad (28)$$

It will be required that this equation be satisfied for all vectors \vec{F} belonging to a suitable space of test functions (to be specified later). As it is well known, this leads to convenient discretization procedures. In addition, by part integration, eq. (26) can be put into a form in which only first-order operators are present (except for the ELD term), boundary conditions can be explicitly stated, and which is closely related to the energy balance equation.

The vacuum term in eq. (28) can be transformed using the identity

$$\vec{F}^* \text{rotrot}\vec{E} = \text{rot}\vec{F}^* \cdot \text{rot}\vec{E} - \text{div}(\vec{F}^* \times \text{rot}\vec{E}) \quad (29)$$

The FLR terms in the plasma current density, and those arising from the substitution (27), are also easily integrated by parts, using the definitions (21)–(22), together with the identity (17). The final form is

$$\begin{aligned}
& \int_V dV \left\{ \text{rot} \vec{F}_\perp^* \cdot \text{rot} \vec{E}_\perp + \frac{\omega^2}{c^2} \left[-\vec{F}_+^* (\hat{L} E_+) - F_-^* (\hat{R} E_-) \right. \right. \\
& \quad \left. \left. + (\partial_\zeta \text{div}_\perp \vec{F}_\perp)^* \left(\frac{\hat{P}^\dagger}{P_0^2} (\partial_\zeta \text{div}_\perp \vec{E}_\perp) \right) \right. \right. \\
& \quad \left. \left. + 2(\partial_+ F_+)^* (\hat{\lambda}_{2i} (\partial_+ E_+)) + (\vec{\nabla}_\perp \times \vec{F}_\perp)^* (\hat{\lambda}_{oe} (\vec{\nabla}_\perp \times \vec{E}_\perp)) \right] \right\} \\
& = - \int_S d\vec{S} \cdot (\vec{F}^* \times \text{rot} \vec{E})
\end{aligned} \tag{30}$$

Here the volume integral extends to the whole vacuum vessel; the surface integral on its wall and on the antenna surface. It is convenient (and realistic) to assume that the plasma density decreases continuously towards the walls in such a way that F.L.R. contributions to the surface integral are negligible.

It is further convenient to impose that \vec{F} satisfies the same boundary conditions as \vec{E} at the wall, and that it is continuous at the antenna. Then if the wall is a perfect conductor, its contribution to the surface integral vanishes, while the antenna contribution can be rewritten

$$- \int d\vec{S} (\vec{F}^* \times \text{rot} \vec{E}) = \frac{4\pi i \omega}{c^2} \int dS (\vec{F}^* \cdot \vec{J}_a^\#) \tag{31}$$

where $\vec{J}_a^\#$ is the surface current density distribution in the antenna.

Finally, the power balance equation is immediately obtained from eq. (30) by identifying \vec{F} with \vec{E} , and taking the imaginary part:

$$\begin{aligned}
& -\frac{1}{2} \text{Re} \int (\vec{E}^* \cdot \vec{J}_a^\#) dS \\
& = \frac{\omega}{8\pi} \int dV \left\{ E_+^* (\hat{L}^a E_+) + (\partial_\zeta \text{div}_\perp \vec{E}_\perp)^* \left(\frac{\hat{P}^a}{P_0^2} (\partial_\zeta \text{div}_\perp \vec{E}_\perp) \right) \right. \\
& \quad \left. - 2(\partial_+ E_+)^* \left[\hat{\lambda}_{2i}^a \cdot (\partial_+ E_+) \right] - (\vec{\nabla}_\perp^a \times \vec{E}_\perp)_\zeta \left[\hat{\lambda}_{oe}^a (\vec{\nabla}_\perp \times \vec{E}_\perp)_\zeta \right] \right\}
\end{aligned} \tag{32}$$

(superscript a denotes the antihermitean part of the operators). This equations provide a useful global check on the accuracy of the numerical solution. Even more important, the integrand on the rhs is the local power absorption: the meaning of each term is immediately clear from the discussion following eq. (23).

In principle, eq. (32) could also be used to evaluate the radiation resistance R_a of the antenna, since

$$\frac{1}{2} R_a I_a^2 = -\frac{1}{2} \text{Re} \int (\vec{E}^* \vec{J}_a^\#) dS \tag{33}$$

In practice, however, this is hardly feasible. Indeed, to keep the problem a two-dimensional one, one is obliged to solve for one toroidal mode ($\exp in_\varphi \varphi$) at a time. While eq. (32) holds for each partial wave separately, eq. (33) only holds for the total field, i.e. implies a summation over n_φ . The difficulty of the task can be judged by recalling that the spectrum of a typical antenna in a medium-size tokamak extends over several tens of n_φ modes.

5. Discussion

For the reasons stated in the introduction, the equations of sec. 3, or their variational equivalent of sec. 4, still present formidable numerical problems. Thus all codes realized thus far have made one or more of the following further approximations.

1. Finite Larmor radius terms are omitted;
2. the integral operators $\hat{L}, \hat{\lambda} \dots$ are replaced by local ones;
3. the poloidal component of the static magnetic field is neglected.

In this section we will discuss these approximations, drawing heavily on knowledge gained from the dispersion relation [12, 30], from one-dimensional models [5, 6, 12, 25], and from the first published (and unpublished) results from two-dimensional codes [13–19].

To facilitate the discussion, it is convenient to recall first the qualitative features of the electric field distribution expected for the most important ICH scenarios (for more details, cfr. [10, 12] and references therein). They are most easily understood from the dispersion relation obtainable from eqs. (18)–(23) assuming a uniform plasma and plane waves:

$$\begin{aligned} \frac{\lambda_{2i}^{(o)}}{2} n_\perp^4 + [(n_\parallel^2 - S^{(o)}) + \lambda_{2i}^{(o)} (n_\parallel^2 - R^{(o)})] n_\perp^2 \\ + (n_\parallel^2 - R^{(o)}) (n_\parallel^2 - L^{(o)}) = 0 \end{aligned} \quad (34)$$

Here $R^{(o)}, L^{(o)}, S^{(o)} = \frac{1}{2}(R^{(o)} + L^{(o)})$ are the familiar elements of the plasma dielectric tensor in the zero Larmor radius approximation, immediately obtainable from eq. (23) under the assumption of a constant $k_\parallel = \omega n_\parallel / c$:

$$L^{(o)} = 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} \left(1 - \frac{\Omega_{ce}}{\omega}\right) - \sum_i \frac{\omega_{pi}^2}{\omega^2} x_{oi} Z(x_{1i}) \quad (35)$$

where $x_{ni} = (\omega - n\Omega_c)/k_{\parallel}v_{thi}$, and Z is the plasma dispersion function [35]. $-\lambda_{2i}^{(o)}n_{\perp}^2$ is the FLR correction to $L^{(o)}$ in the same limit [30]; the very small terms describing EMP and ELD have been omitted.

Equation (34) has two roots: one describing the fast Alfvén or magnetosonic wave,

$$n_{\perp}^2 \cong -\frac{(n_{\parallel}^2 - R^{(o)})(n_{\parallel}^2 - L^{(o)})}{(n_{\parallel}^2 - S^{(o)})} \quad (36)$$

and one describing a slow wave, which in the vicinity of a first IC harmonic resonance ($\omega = 2\Omega_{ci}$, $\lambda_{2i}^{(o)}$ resonant) is an ion Bernstein wave

$$n_{\perp}^2 \cong -2\frac{n_{\parallel}^2 - S^{(o)}}{\lambda_{2i}^{(o)}} \quad (37)$$

while near an isolated ion-ion resonance ($S^{(o)} \cong 0$, $\lambda_2^{(o)}$ negligible) it is the shear Alfvén wave (also called ion cyclotron wave for $\omega = 0(\Omega_{ci})$):

$$n_{\parallel}^2 \cong S^{(o)} \quad (38)$$

To apply eqs. (35)–(38) in the geometry of a tokamak, n_{\perp} and n_{\parallel} have to be expressed in terms of the horizontal and vertical components k_X, k_Z , of the wave vector, and of the toroidal wave number n_{φ} :

$$\begin{aligned} n_{\perp}^2 &= \frac{c^2}{\omega^2} \left\{ k_X^2 + k_Z^2 - 2\frac{n_{\varphi}}{R} \cos\Theta \sin\Theta \right. \\ &\quad \left. + \left(\frac{n_{\varphi}^2}{R^2} - k_r^2 \right) \sin^2\Theta \right\} \\ n_{\parallel} &= \frac{c}{\omega} \left(\frac{n_{\varphi}}{R} \cos\Theta + k_r \sin\Theta \right) \\ k_r &= \frac{1}{N_r} \left(\frac{\partial X}{\partial \vartheta} k_X + \frac{\partial Z}{\partial \vartheta} k_Z \right) \end{aligned} \quad (39)$$

In a region without strong density gradients, k_Z and the metric coefficients can be regarded as slowly varying quantities in a WKB sense, and eqs. (35)–(38) solved for k_X [5, 6, 12]. This leads to the following qualitative predictions:

(a) First harmonic heating of a single species plasma.

The main feature of the dispersion relation in this case is the confluence between the fast wave and the ion Bernstein wave which lies somewhat to the high magnetic field side (HMFS) of the $\omega = 2\Omega_{ci}$ resonance, and is separated from it by an evanescence gap [29]. The Bernstein wave propagates away from the resonance, and is mainly electrostatic; hence it is unaffected by either ICD or EMP. On the other hand, its perpendicular index

increases rapidly; refraction is likely to make n_{\parallel} increase as well, until conditions for efficient ELD will be met, and the wave will be absorbed by electrons.

It should be recalled that for modes with sufficiently large n_{φ} Doppler broadening of the IC resonance tends to suppress the evanescence layer, and weakens coupling to the Bernstein wave. Moreover, Gambier and Samain [22] have shown that linear mode conversion is also inhibited by too large density gradients. Hence short wavelength features and electron heating are expected to be confined to the central region of the plasma, to the HMFS of the $\omega = 2\Omega_{ci}$ resonance; they should be much more pronounced for low n_{φ} than for high n_{φ} modes.

(b) H^+ minority in D^+ .

When the fundamental resonance of the minority species coincides with the first harmonic of the majority, the topology of the dispersion curves is the same as in the previous case. The evanescence gap is adjacent to the ion-ion resonance $S = 0$, and moves to higher magnetic field as the minority concentration increases. Therefore the critical value of n_{φ} for the suppression of mode conversion (separating the “mode conversion” regime from the “minority” regime) also increases; it remains, nevertheless, well within the spectral range of typical antennas at low to moderate H^+ concentrations.

(c) He_3^{++} in H^+ or D^+ .

In these scenarios the ion-ion hybrid resonance is isolated, and FLR terms are very small. Mode conversion then occurs to the shear Alfvén wave (38), which propagates towards the LMFS. If the Z/A ratio of the minority is larger than that of the bulk ions (H_3^{++} in D^+), it propagates by the same token towards the fundamental resonance of the minority, where it suffers a parallel resonance of the Stix magnetic beach type; in the opposite situation (H_3^{++} in H^+) the cyclotron resonance is on the HMFS of $S = 0$, and inaccessible to this wave. Because of its very short wavelength, however, the shear Alfvén wave is even more sensitive to damping by the electrons than the ion Bernstein wave of situations a) and b).

We are now ready to understand the consequences of the approximations listed above; it will actually be sufficient to consider in details the most drastic one, consisting in taking the cold limit of the equations of eq. (30) (with $P_0 \rightarrow \infty$). The gain in simplicity is obvious: integral operators become algebraic, and the only differential operator in eq. (30) is the curl, whose discretization is relatively straightforward.

In this limit kinetic absorption is absent altogether from the model. To avoid singularity near ion-ion resonances, some collisional damping is assumed. The rationale for this procedure is that it gives the “correct” result in a plane-layered medium, in the sense

that the jump of Poynting flux found at the resonance in the model with collisions, and interpreted as dissipation, is just equal to the sum of what is locally absorbed plus what is coupled to short-wavelength waves in the more complete model. Since these waves are also finally absorbed not far from where they are excited, the two approaches essentially agree. Gambier and Samain [22] have given plausible arguments that this remains true also in the two-dimensional case.

This procedure has, however, two shortcomings. The most obvious is that, while absorption due to mode conversion might indeed be correctly estimated, direct absorption of the fast wave at $\omega = \Omega_{ci}$ (minority) and/or $\omega = 2\Omega_{ci}$ (bulk ions) is completely neglected. Thus the method can hardly be applied to the "minority" regime, and not at all to first harmonic heating. Of course, this defect could be at least qualitatively cured by reintroducing ion cyclotron damping approximatively, for example by evaluating the antihermitean part of $L^{(o)}$ as if k_{\parallel} were known and constant, say $k_{\parallel} = n_{\varphi}/R_{Res}$. At least for modes with large toroidal number this should be an acceptable approximation.

There is, however, a more subtle problem that vitiates the cold plasma approximation irretrievably in scenarios (a) and (b). As it is clear from what happens in case (c), neglecting FLR corrections to the hf plasma currents, and taking the zero electron inertia limit $m_e/m_i \rightarrow 0$, by no means eliminates short-wavelength modes from the model, as it would do in a plane-layered geometry. Near an ion-ion resonance, the cold plasma dispersion relation predicts, instead of the ion Bernstein wave, the excitation of the shear Alfvén wave (38), which, moreover, propagates towards the "wrong" side, i.e. the LMFS, of the resonance. The presence of this non-physical mode has plagued attempts to describe analytically mode conversion in tokamak plasmas [20–21]. Numerically, it poses very difficult resolution and convergence problems, since its wavelength is considerably shorter than that of the ion Bernstein wave.

The group velocity of the shear Alfvén wave is almost parallel to the static magnetic field. In a model without kinetic damping therefore a kind of internal resonant cavity modes can be excited when the appropriate quantization rules are satisfied along magnetic field lines between the points where they cut the ion-ion hybrid resonance: the mode conversion layer then acts as virtual antenna [33]. Short wavelength electric field patterns, localized on a discrete set of magnetic surfaces and with very large amplitude (limited only by collisional damping), have indeed been observed in numerical solutions of the cold plasma equations [16].

We would like to make two final comments on this subject. The first is that the cold plasma approximation might be adequate for heating scenario (c), provided that kinetic damping (ICD and ELD) is at least qualitatively taken into account. The second is

that the problems connected with the shear Alfvén wave disappear when the poloidal component of the static magnetic field is neglected (moreover, then, the assumption $k_{||} = n_{\varphi}/R$ becomes exact within the model). For this reason we are convinced that an unpublished code by Colestock [17], in which both B_{pol} and FLR terms are omitted but kinetic damping is included, is at the moment the best available, in spite of the rather drastic simplifications on which it is based.

The conclusions of the foregoing discussion for the general case should by now be clear. FLR effects have to be retained so that the model will describe the correct plasma modes in all situations. Parallel dispersion becomes then necessary, not only for an accurate evaluation of power absorption profiles, but also to make sure that short-wavelength modes suffer the correct amount of damping, and do not plague the numerical solution with unnecessary resolution and convergence problems. Because of the electrostatic nature of short wavelength modes, the last remark applies also to ELD, which is completely negligible for the externally excited fast Alfvén wave.

6. Numerical Considerations

A completely conventional application of finite element (FEL) techniques to eq. (30) is made impossible in practice by the nature of the integral operators \hat{L} , $\hat{\lambda}$, \hat{P} . In principle, one would begin with the expansion of the electric field into an appropriate set of interpolating functions, say [34]:

$$\vec{E}(\psi, \vartheta, \varphi) = \sum_{p,i,j} E_{ij}^p h_{ij}(\psi, \vartheta) \vec{e}_p(\psi, \vartheta) e^{in_{\varphi}\varphi} \quad (40)$$

Here p extends over the two possible polarizations ($\vec{e}_p \equiv (\vec{e}_{\psi}, \vec{e}_{\eta})$, or $\vec{e}_p \equiv (\vec{e}_{+}, \vec{e}_{-})$), while indexes i, j denumber the points $\psi = \psi_i, \vartheta = \vartheta_j$ of a mesh in the poloidal plane. Identifying in turn \vec{F} with each of the possible test functions,

$$\vec{F}(\psi, \vartheta, \varphi) = h_{i_o, j_o}(\psi, \vartheta) \vec{e}_{p_o} e^{in_{\varphi}\varphi} \text{ (all } p_o, i_o, j_o) \quad (41)$$

and performing the required integrations on the test functions and their derivatives, one would obtain a set of linear algebraic equations in the nodal values E_{ij}^p , whose inversion finally gives the numerical solution to the problem.

The construction of the stiffness matrix of the system involves the evaluation of integrals of the form

$$\int_{-\infty}^{+\infty} du \frac{e^{-u^2}}{\sqrt{\pi}} \int_0^{\infty} d\tau h_{ij}(\psi, \vartheta - \frac{v_{th\alpha} \sin \Theta}{N_r} u \tau) \cdot \exp i \left[(\omega - \nu \Omega_{\alpha}(\psi, \vartheta) - n_{\varphi} \frac{v_{th\alpha} \cos \Theta}{R} u) \tau \right] (\nu = 0, 1, 2) \quad (42)$$

at a sufficient number of points to make possible the further integrations over ψ and ϑ required by eq. (30). The difficulty of devising an efficient algorithm for this purpose is obvious from the oscillatory behaviour of the integrand, and from the necessity of following Landau prescription, $Im\omega \rightarrow 0+$, implicit in the definition of the operators $\hat{L}, \hat{\lambda}, \hat{P}$ [27]. Stationary phase approximations, being non-uniform in u , fail in the immediate vicinity of cyclotron resonances, where a correct evaluation is most imperative.

There is, moreover, a less obvious, but in our opinion even more decisive argument against this direct approach. The interpolating functions $h_{ij}(\psi, \vartheta)$ are localized within one element around $\psi = \psi_i, \vartheta = \vartheta_j$, whose support must be small compared to the shortest wavelengths expected. In most cases, moreover, they also have discontinuous derivatives of a low order (typically the first) at the extremes of this interval. Hence their Fourier spectrum in ϑ necessarily contains much higher components than the actual electric field distribution, the more so the better the resolution one wishes to achieve. In particular, heavily damped components will give a large contribution when $\hat{L}, \hat{\lambda}$ or \hat{P} are applied to the functions h_{ij} separately. In other words, a large amount of internal cancellation must occur for the correct evaluation of sums like

$$\hat{L}^{(a)}(\vec{E}) = \sum_{i,j} E_{ij}^+ \hat{L}^{(a)}(h_{ij}(\psi, \vartheta) \vec{e}_+) \quad (43)$$

Failure to achieve proper cancellation would result in large and uncontrollable errors in the estimates of damping.

Having ruled out the straightforward approach to FEL discretization, we propose now two methods which hopefully avoid problems of this kind. We will limit ourselves to the description of the main ideas, and to plausibility arguments about feasibility and convergence. At least for the first of the two schemes, whose numerical implementation is in progress [28], details will be given in a forthcoming report.

a) The spectral approach Clearly, much would be gained by using a set of basis functions $h_{ij}(\psi, \vartheta)$, having the same harmonic content as the solution. Ideally suited for this purpose are the Fourier modes themselves

$$h_{im}(\psi, \vartheta) = h_i(\psi) e^{im\vartheta} \quad -M \leq m \leq M \quad (44)$$

They have the further invaluable advantage that double integrals of the type (42) can then be expressed analytically in terms of the plasma dispersion function Z [35], for whose numerical evaluation well-tested algorithms are available.

In more conventional terms, eq. (44) means that the electric field is Fourier-expanded in ϑ ,

$$\vec{E}(\psi, \vartheta, \varphi) = \sum_m \vec{E}^{(m)}(\psi) e^{i(m\vartheta + n\varphi)} \quad (45)$$

so that the single partial differential equation (18) is replaced by a large system of coupled ordinary differential equations for the Fourier amplitudes $\vec{E}^{(m)}(\psi)$. This system is then discretized with standard one-dimensional FEL techniques. Details of this procedure will be given elsewhere [28].

Let us, nevertheless, consider somewhat more explicitly the contribution of the zeroth order current to the variational integral (3) as an example. It becomes

$$\begin{aligned} & \int \int \int R J \vec{F}^* \cdot \left(\vec{E} + \frac{4\pi i}{\omega} \vec{J}^{(0)} \right) d\psi d\vartheta d\varphi \\ &= \sum_{m'} \int d\psi \vec{F}^{(m)*}(\psi) \cdot \varepsilon(m; m'; n; \psi) \cdot \vec{E}^{(m')}(\psi) \end{aligned} \quad (46)$$

where

$$\begin{aligned} \varepsilon_{\psi\psi} &= \varepsilon_{\eta\eta} = \int_0^{2\pi} R J e^{i(m'-m)\vartheta} S^{m',n}(\psi, \vartheta) d\vartheta \\ \varepsilon_{\psi\eta} &= -\varepsilon_{\eta\psi} = -i \int_0^{2\pi} R J e^{i(m'-m)\vartheta} D^{m',n}(\psi, \vartheta) d\vartheta \\ S &= 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} - \frac{1}{2} \sum_i \frac{\omega_{pi}^2}{\omega^2} \left[\frac{\omega}{\omega + \Omega_{ci}} - x_{oi}^{m',n} Z(x_{1i}^{m',n}) \right] \\ D &= \frac{\Omega_{ce}}{\omega} - \frac{1}{2} \sum_i \frac{\omega_{pi}^2}{\omega^2} \left[\frac{\omega}{\omega + \Omega_{ci}} + x_{oi}^{m',n} Z(x_{1i}^{m',n}) \right] \end{aligned} \quad (47)$$

with

$$\begin{aligned} x_{\nu i}^{m,n}(\psi, \vartheta) &= \frac{\omega - \nu \Omega_{ci}(\psi, \vartheta)}{k_{\parallel}^{m,n} \cdot v_{thi}} \\ k_{\parallel}^{m,n}(\psi, \vartheta) &= \frac{m}{N_r} \sin \Theta + \frac{n}{R} \cos \Theta \end{aligned} \quad (48)$$

This shows that the spectral Ansatz (45) replaces the integral operator \hat{L} (and similarly $\hat{\lambda}, \hat{P}$) with a matricial algebraic operator, whose elements are expressed in known functions, and can be given a direct physical meaning.

Of course, the matrix $\varepsilon(m, m')$ has a very large band width in the pair of indexes m, m' . More generally, the stiffness matrix obtained with the spectral approach is block tridiagonal only in the pair of indexes referring to the ψ (radial) discretization; by contrast, a conventional FEL discretization in two dimensions would lead to a matrix

block tridiagonal in both pairs of indexes (we disregard for simplicity coupling among separate elements induced by the integral operators). From this point of view, eq. (44) has to be understood just as a choice of the base functions space which (compared to localized interpolating functions) gives up sparseness of the stiffness matrix in one pair of indexes, in favour of simplicity in the evaluation of its elements. Indeed their computation, although tedious, can be made reasonably fast by the systematic use of the Fast Fourier Transform and of the convolution theorem. In this way the effort required appears comparable to that demanded by the conventional FEL discretization of a purely differential system of similar complexity, with only little additional penalty from the integrodifferential nature of the equations to be solved.

Admittedly, Fourier modes are far from being even approximate eigenmodes of the problem. Strong coupling among a large number of components makes therefore the spectral expansion (45) appear somewhat artificial. Nevertheless, we may note that its convergence is ensured on physical grounds, a natural cut-off for $|m|$ being provided by the shortest wavelengths allowed by the dispersion relation (in this context, having included in the model the proper amount of damping is once again essential). This criterion can be exploited in practice to obtain a useful *a-priori* estimate of how large M should be taken in any particular case.

This line of arguments also shows that although the required number of Fourier modes might appear uncomfortably large, this is not a problem of the spectral method as such, but reflects the general difficulty of having to resolve short wavelength features. In a two-dimensional discretization, it would show up in a just as uncomfortably large number of mesh points.

b) An iterative approach

Finally, we explore the possibility of nevertheless adopting a two-dimensional FEL discretization, then coping with the integrodifferential operators iteratively. The principle is simple: the h.f. current is split into two parts, $\vec{J}(\vec{E}) = -\sigma \cdot \vec{E} + \delta \vec{J}$ where $\sigma \cdot \vec{E}$ is some local approximation to the constitutive relation, and eq. (18) is then solved by successive approximations according to the scheme

$$\text{rot rot } \vec{E}^{(n+1)} - \frac{\omega^2}{c^2} \left(\vec{E}^{(n+1)} + \frac{4\pi i}{\omega} \sigma \cdot \vec{E}^{(n+1)} \right) = \frac{4\pi\omega}{c^2} \delta \vec{J}(\vec{E}^{(n)}) \quad (50)$$

The success of such a procedure will depend on the appropriate choice of $\sigma \cdot \vec{E}$, and of the first guess for \vec{E} itself.

The simplest possibility is to evaluate $\sigma \cdot \vec{E}$ by neglecting the poloidal magnetic field in the definitions (23): then σ reduces to the local conductivity tensor evaluated with $k_{\parallel} = k_{\parallel}^{0,n\varphi} = n_{\varphi}/R$. With this choice, $\delta\vec{J}(\vec{E})$ can be evaluated with an acceptable efficiency by making the FFT of $\vec{E}^{(n)}$, evaluating $\vec{J}(\vec{E}^{(n)})$ as in the spectral method, subtracting the $k_{\parallel}^{0,n\varphi}$ term, and applying the inverse FFT, followed by some interpolation if the mesh for FEL and for FFT do not coincide. On the other hand, it is not allowed to simplify the l.h.s. of eq. (50) further by relegating FLR terms in $\delta\vec{J}$: this would alter the nature of the solutions of the homogeneous equation corresponding to eq. (50) (for example by the appearance of the shear Alfvén wave) in such a way that convergence could hardly be expected.

Similar considerations apply to the choice of the first guess for \vec{E} : probably the safest thing is to evaluate it from the homogeneous eq. (5), neglecting B_{pol} also in the differential part of the operators.

The scheme just sketched is likely to work for modes with a sufficiently large toroidal wavenumber n_{φ} , but might be inadequate for low n_{φ} , particularly for $n_{\varphi} = 0$. Improvements are not difficult to be found; it would, however, be futile to discuss them further before putting them to test.

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