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Entropy Principle for Tokamak Profiles

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Abstract

It is shown that experimental temperature and density profiles are consistent with the assumption that they relax towards profiles related by

$$\frac{T(\mathbf{x})}{T_0} = \left(\frac{n(\mathbf{x})}{n_0}\right)^{\gamma-1} exp\left(\alpha\left(1 - \frac{n_0}{n(\mathbf{x})}\right)\right) \tag{1}$$

$\gamma = 5/3$ = adiabatic constant, $\alpha \geq -(\gamma-1)n_{min}/n_0$ independent of $\mathbf{x}$;
in most cases one has $n_{min} = 0$.

Cases of incomplete relaxation are exceptions, e.g. pellet injection in which the temperature profiles are too flat compared with the corresponding density profiles .

Relation (1) follows from the entropy principle proposed here. According to it tokamak plasmas should relax towards states described by relations $T = T(n(\mathbf{x}))$, in which the total entropy of the plasma does not change when the plasma performs arbitrary internal motions slow enough so as not to alter the relation between T and n.

$\alpha = 1$ corresponds to the profiles obtained by BISKAMP [2] and KADOMTSEV [3] from an energy principle assuming the electrical conductivity being given by SPITZER's law $\sigma = const \cdot T(\mathbf{x})^{3/2}$.

To get agreement with experimental profiles, however, α values ranging from 0 to 3 are required.

Tokamak equilibria are usually describable in terms of two arbitrary functions of the poloidal flux ψ. For resistive plasmas with SPITZER's formula valid one can choose the temperature and the density as these functions. Equation (1) reduces this freedom to the free choice of one function, say $T(\psi)$, and of a special value of the parameter α. This is a feature that can be related to what is called "profile consistency" [1].

Examples are presented for cylindrical plasmas with circular cross-sections.

1. Introduction

Tokamak profiles usually show some universal features, for which COPPI [1] has introduced the notion "profile consistency". Because of this universality it is conceivable that these features should be describable by a certain general principle. BISKAMP [2] and KADOMTSEV [3] have proposed a variational principle which consists of an energy principle with the constraint of fixed total toroidal current. Interesting results of this variational principle are only obtained, however, if the constraint is not treated in a straightforward way, which would lead to surface current distributions, but in a more restricted way for which no physical interpretation exists at present.

In this paper we propose a variational principle which we call "entropy principle" and which aims at yielding relations between the density profile $n(\mathbf{x})$ and the temperature profile $T(\mathbf{x})$. It can be shown by comparison with experimental findings that tokamak plasmas indeed have the tendency to relax to such states in which these relations hold. The entropy principle results in the following equation between $T(\mathbf{x})$ and $n(\mathbf{x})$:

$$\frac{T(\mathbf{x})}{T_0} = \left(\frac{n(\mathbf{x})}{n_0}\right)^{\gamma-1} exp\left(\alpha \cdot \left(1 - \frac{n_0}{n(\mathbf{x})}\right)\right), \qquad (1)$$

α : parameter independent of the position vector $\mathbf{x}$, $\quad \alpha \geq -(\gamma-1)n_{min}/n_0$;

γ : adiabatic coefficient, $\qquad\qquad\qquad\qquad\qquad \gamma = \frac{5}{3}$;

$T_0 = T(\mathbf{x_0})$;

$n_0 = n(\mathbf{x_0})$;

$\mathbf{x_0}$: arbitrary reference point, e.g. the position of the magnetic axis,

$\qquad\qquad$ then $p \leq p_0$ and $n \leq n_0$ for monotonous profiles;

n_{min} : minimal value of $n(\mathbf{x})$;

In Sect.2 the entropy principle is formulated and relation (1) is derived. It is also shown that eq.(1) can be approximated by the polytropic relation

$$\frac{T(\mathbf{x})}{T_0} = \left(\frac{n(\mathbf{x})}{n_0}\right)^{\eta} ; \quad \eta = \frac{2}{3} + \frac{4}{3} |\alpha|^{0.9} sign(\alpha) . \qquad (2)$$

The approximation is poor for $\frac{n}{n_0} \ll 1$, i.e. near the plasma boundary. However, in most cases the experimental error bars are larger then the error made by using the

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The approximation is poor for $\frac{n}{n_0} \ll 1$, i.e. near the plasma boundary. However, in most cases the experimental error bars are larger then the error made by using the polytropic relation (2) instead of the exact relation (1). Relation (1) or (2) means that one temperature profile allows different density profiles belonging to different values of the parameter α. This is at variance with BISKAMP's /2/ and KADOMTSEV's /3/ theory, which yields a one-to-one relation between the density and temperature profiles for $j \sim T^{3/2}$. Their relation corresponds approximately to the special value $\alpha = 1$ in our theory. Agreement with experimental results requires, however, a wide range of α values including $\alpha = 1$. Relation (1) is checked in Sect.3 against experimental temperature and density profiles of a variety of tokamak discharges. In Sect.4 we present "theoretical" profiles $n(\mathbf{x})$ and $T(\mathbf{x})$ for cylindrical $\beta_p = 1$ plasmas which follow from relation (1) when it is combined with SPITZER's formula for the electrical conductivity.

2. The Entropy Principle

The entropy principle proposed in this paper is aimed at yielding relations between the density and the pressure profiles valid for states towards which, it is assumed, tokamak plasmas tend to relax. These states are characterized by such functions

$$p = p(n) \tag{3}$$

for which the entropy

$$S = \frac{1}{\gamma - 1} \int\limits_{plasma} d^3x \left(n(\mathbf{x}) \, \ln \left(p(n(\mathbf{x})) \, n(\mathbf{x})^{-\gamma} \right) + (\gamma - 1) \, s_0 \, n(\mathbf{x}) \right) \tag{4}$$

$s_0 = $ entropy constant,

no longer changes when the plasma performs arbitrary internal motions which are slow enough so as not to alter the relation between p and n. These motions can be described by an arbitrary displacement vector $\xi = \xi(\mathbf{x})$ vanishing at the plasma boundary. The variation of S is then given by

$$\delta S = \frac{1}{\gamma - 1} \int\limits_{plasma} d^3x \, \delta n(\mathbf{x}) \, \frac{\partial}{\partial n} \left(n \, \ln(p \, n^{-\gamma}) + (\gamma - 1) \, s_0 \, n \right) \tag{5}$$

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with

$$\delta n(\mathbf{x}) = -\nabla \cdot \left(\xi(\mathbf{x}) \, n(\mathbf{x})\right) \tag{6}$$

Eq.(6) describes conservation of the number of particles.

Partial integration in eq.(5) yields

$$\delta S = \frac{1}{\gamma - 1} \int_{plasma} d^3 x \, n \, \xi \cdot \nabla \cdot \left(\frac{\partial}{\partial n}\left(n \, ln(p \, n^{-\gamma}) + (\gamma - 1) \, s_0 \, n\right)\right) \tag{7}$$

This has to be zero for arbitrary $\xi(\mathbf{x})$. We therefore obtain

$$\frac{\partial}{\partial n}\left(n \, ln(p \, n^{-\gamma}) + (\gamma - 1) \, s_0 \, n\right) = \lambda \tag{8}$$

with

$$\nabla \lambda = 0.$$

Integration over n yields

$$n \, ln \, p \, n^{-\gamma} + \left((\gamma - 1) \, s_0 - \lambda\right) n = -\alpha \, n_0 \tag{9}$$

with

$$\nabla \alpha = 0, \qquad\qquad n_0 = n(\mathbf{x_0}).$$

$\mathbf{x_0}$ is chosen in the following as the position of the magnetic axis, thus $n(\mathbf{x}) \leq n_0$ for monotonous profiles. Writing down relation (9) for the position $\mathbf{x_0}$, we can express λ in terms of s_0, α, n_0 and $p_0 = p(\mathbf{x_0})$ to obtain

$$\frac{p}{p_0} = \left(\frac{n}{n_0}\right)^\gamma exp(\alpha \cdot (1 - \frac{n_0}{n})) \tag{10}$$

Inserting

$$p = n \, T$$

into eq.(10) yields relation (1), given in the introduction. The temperature relation (1) tells us that

$$\alpha \geq -(\gamma - 1) n_{min}/n_0$$

must hold in order to guarantee that $\frac{\partial T(\mathbf{x})}{\partial n(\mathbf{x})} \geq 0$ in the density regime of interest.

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It should be pointed out that $\alpha = 0$ means an isentropic plasma. In this case there exists an example which might throw some light on the physical background of our principle: an isentropic gas with a non-uniform temperature distribution in the direction of a gravitational force is just marginally stable; in addition, it is well known that gravitation and the curvature of magnetic field lines have similar effects as regards stability. As far as tokamak plasmas are concerned it is important, however, that α can be any non-negative number as will emerge in the next section, where relation (1) will be checked against experimental results.

Finally, we show that the polytropic relation (2) approximates relation (1) reasonably well, except for $\frac{n}{n_0} \ll 1$, as can be seen from Fig.1.

Fig.1

$\frac{T}{T_0}$ versus $\frac{n}{n_0}$

Solid: according to the exact relation (1)

Dashed: according to the polytropic relation (2)
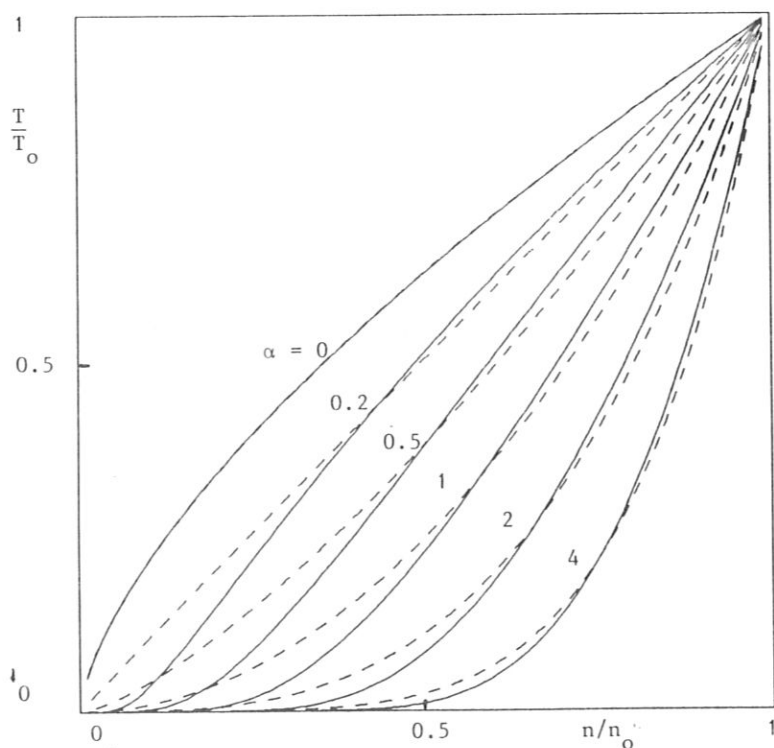
3. Comparison with Experimental Profiles

For a comparison with experimental profiles one needs for each position a pair of density and temperature values. Sometimes there is uncertainty as to the real position; this, however, does not matter if it is guaranteed that the values of n and T belong to the same position. We distinguish such data by a discrete lable k : $n(k)$; $T(k)$. We use directly measured points $(+)$ as well as points $(\diamond)$ on curves given in the literature in order to fit the experimental results. The values of α to be used in eqs.(1) and (2) are obtained from

$$\alpha = \frac{\sum\limits_{k=1}^{kmax} g_k \left(1 - \frac{n_0}{n}\right)_k \left(ln\frac{T}{T_0} - \frac{2}{3} ln\frac{n}{n_0}\right)_k}{\sum\limits_{k=1}^{kmax} g_k \left(1 - \frac{n_0}{n}\right)_k^2} \tag{11}$$

with

$kmax$: number of points considered.

This means a least square fit of $ln(\frac{T}{T_0})$ with the weight function g_k; for the latter we have taken

$$g_k = \left(\frac{n}{n_0}\right)_k^4 \left(1 - \frac{n}{n_0}\right)_k . \tag{12}$$

Eq.(12) gives small weight to the plasma centre $\frac{n}{n_0} \approx 1$ and the plasma boundary region $\frac{n}{n_0} \ll 1$. In these regions uncertainties are usually biggest and also the applicability of relation (1) might be doubtful.

In the following figures we reproduce $n(r)$ and $T(r)$ diagrams from **ALCATOR A** and **ASDEX** and compare relations (1) and (2) in $T - n$ diagrams with these experimental data; a transport code simulation of an **ASDEX** discharge done by **BECKER** /4/ is also considered.
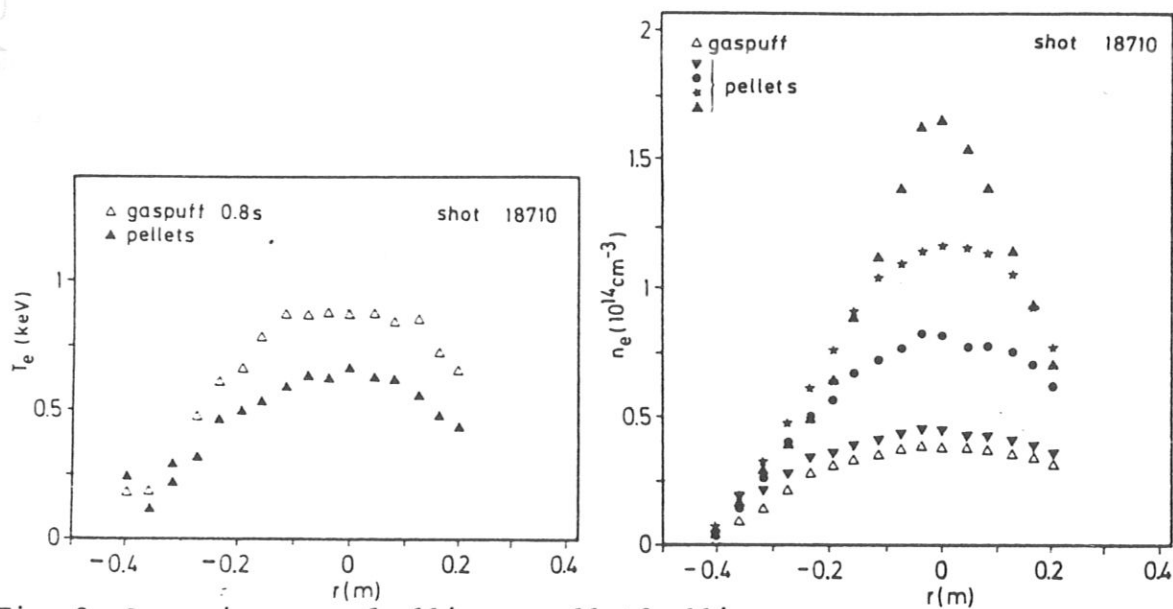
Fig.2a

Reproduction of Ref/6/, Fig.2

Fig. 2: Comparison: gasfuelling - pelletfuelling
left: $T_e(r)$ with gasfuelling (Δ) and after a series of 20 pellets ($\blacktriangle$)
right: $n_e(r)$ during gasfuelling (Δ) and pellet fuelling (dark symbols) showing the time evolution of $n_e(r)$; (t = 1,3-1,5-1,7-2,1 s).
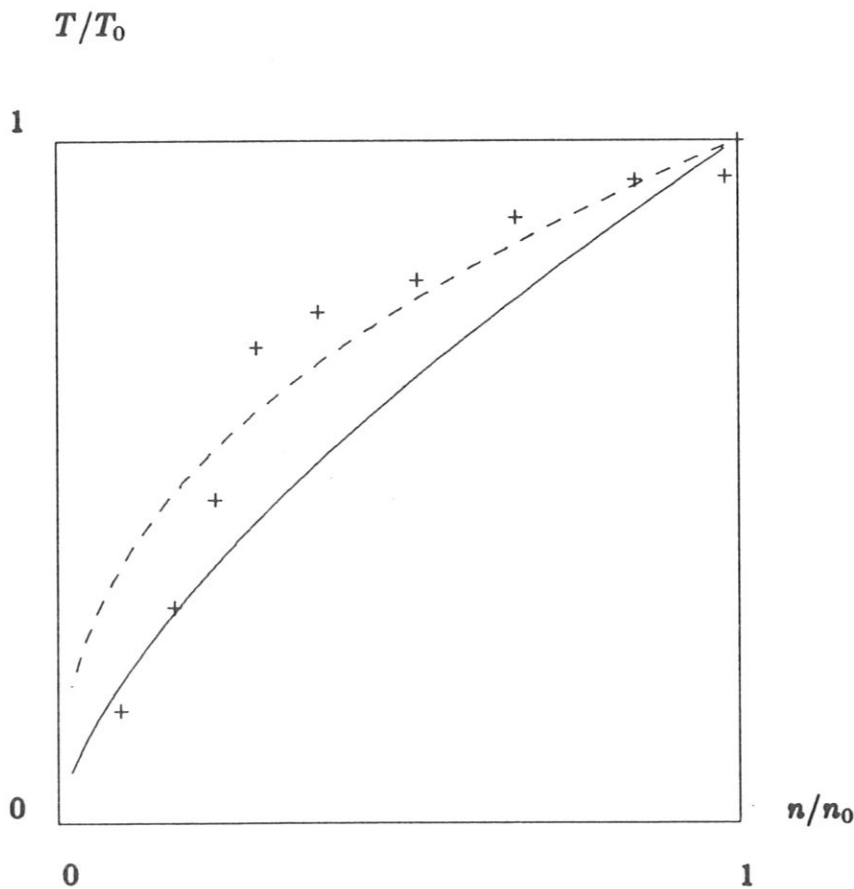
Fig.2b $\quad \frac{T}{T_0}$ versus $\frac{n}{n_0}$ for Ref/6/, Fig.2, pellets

$+$: experimental points,

$\diamond$: points on curves fitting experimental data,

solid line : relation (1) with $\alpha = 0$,

dashed line : relation (2a) with η from a fit analogous to eq.(11) for α.

$\eta = 0.47$

The entropy relation cannot be applied to this case of pellet refuelling reproduced here, because the plasma has not yet relaxed to "quasi- equilibrium" described by relation (1) or (3)+(4). The density profile $n(r)$ is too slender compared with the temperature profile $T(r)$, which causes $\eta < 2/3$, this not being possible in the quasi-equilibrium theory.

T/T_0

n/n_0

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Fig.2c $\quad \frac{T}{T_0}$ versus $\frac{n}{n_0}$ for Ref/6/, Fig.2, gas puff

+ : experimental points,

◇ : points on curves fitting experimental data,

solid line : relation (1) with α according to eq.(11),

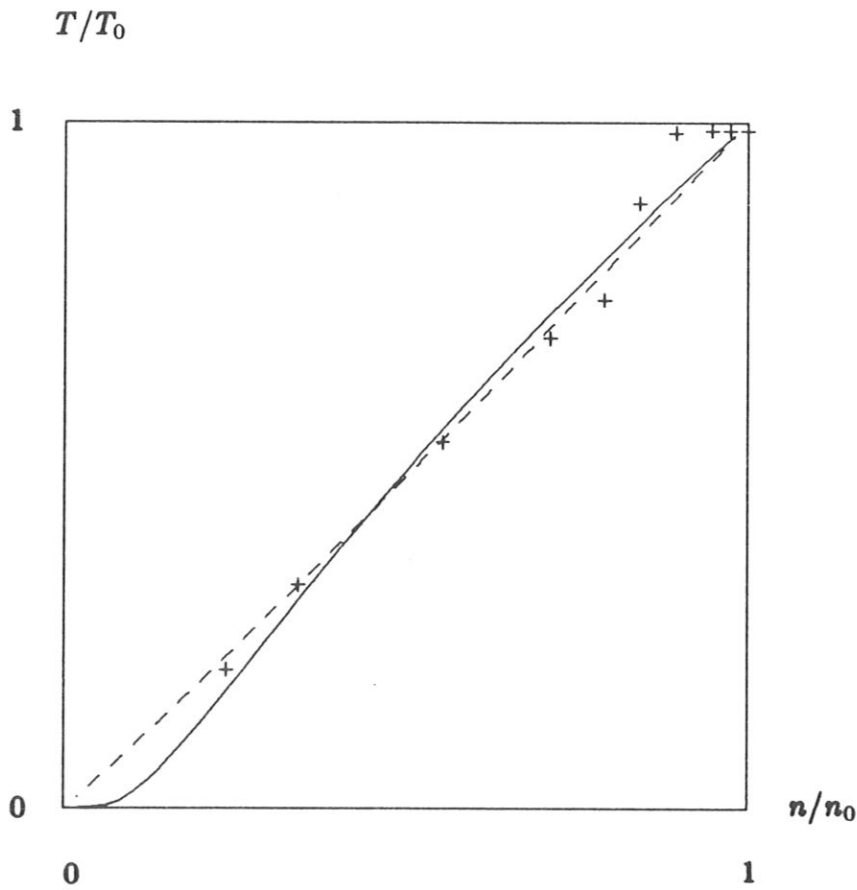dashed line : relation (2) with α according to eq.(11).

$\alpha = 0.27$

$\eta = 1.08$

T/T_0

Results of a transport simulation code from Ref(4)

$t = 1.3 s$ evolution. H-Phase

#18310, Transp. simulation, Run GDB570

ASG253/14 MARCH 86   JALUTION / 2).JCT33

- 1- *** TIME STEP 280 ***   TIME =   1333.537

ZONE	RADIUS CM	TE KEV	REG	TI KEV	REG	NE PART/CU CM
0	-1.06	2.431E+00	L	4.315E+00	L	5.837E+13
1	1.00	2.430E+00	NC	4.315E+00	NC	5.837E+13
2	3.00	2.392E+00	TE	4.307E+00	NC	5.749E+13
3	5.00	2.345E+00	TE	4.273E+00	NC	5.594E+13
4	7.00	2.303E+00	TE	4.191E+00	NC	5.385E+13
5	9.00	2.247E+00	TE	4.052E+00	NC	5.152E+13
6	11.00	2.185E+00	TC	3.161E+13	NC	4.901E+13
7	13.00	2.110E+00	TE	3.639E+00	NC	4.642E+13
8	15.00	1.987E+00	TE	3.419E+00	NC	4.405E+13
9	16.50	1.897E+00	CC	3.252E+10	NC	4.211E+13
10	18.00	1.803E+00	TE	3.033E+10	NC	4.015E+13
11	19.00	1.746E+00	TE	2.955E+10	NC	3.882E+13
12	20.00	1.675E+00	TC	2.845E+10	NC	3.749E+13
13	21.00	1.667E+00	TE	2.718E+10	NC	3.617E+13
14	22.00	1.539E+00	TE	2.586E+10	NC	3.484E+13
15	23.00	1.469E+00	TE	2.455E+00	NC	3.353E+13
16	24.00	1.390E+00	TE	2.322E+10	NC	3.215E+13
17	25.00	1.323E+00	TE	2.193E+00	NC	3.079E+13
18	26.00	1.253E+00	TE	2.053E+00	NC	2.945E+13
19	27.00	1.179E+00	TE	1.913E+00	NC	2.810E+13
20	28.00	1.104E+00	TE	1.773E+00	NC	2.675E+13
21	29.00	1.029E+00	TE	1.631E+00	NC	2.538E+13
22	30.00	9.544E-01	TE	1.482E+00	NC	2.400E+13
23	31.00	8.802E-01	TE	1.347E+00	NC	2.260E+13
24	32.00	8.064E-01	TE	1.209E+00	NC	2.118E+13
25	33.00	7.343E-01	TE	1.062E+00	NC	1.975E+13
26	34.00	6.625E-01	TE	9.135E-01	NC	1.830E+13
27	35.00	5.925E-01	TE	7.557E-01	NC	1.684E+13
28	36.00	5.238E-01	TE	5.341E-01	NC	1.537E+13
29	37.00	4.593E-01	TE	4.520E-01	NC	1.387E+13
30	37.80	4.084E-01	TE	3.493E-01	NC	1.265E+13
31	38.40	3.722E-01	TE	2.835E-01	NC	1.171E+13
32	38.90	3.431E-01	TE	2.382E-01	NC	1.092E+13
33	39.30	3.067E-01	NC	2.090E-01	NC	1.006E+13
34	39.60	2.699E-01	NC	1.942E-01	NC	9.108E+12
35	39.80	2.332E-01	NC	1.893E-01	NC	8.210E+12
36	40.00	1.751E-01	NC	1.895E-01	NC	6.817E+12
37	40.20	6.103E-02	NC	1.519E-01	NC	4.875E+12
38	40.40	3.486E-02	NC	1.192E-01	NC	3.463E+12
39	40.60	2.390E-02	NC	1.006E-01	NC	2.618E+12

r_{sep}

Separatrix

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Fig.3b $\quad \frac{T}{T_0}$ versus $\frac{n}{n_0}$ for some of the data given in Fig.3a.

$\diamond$: data used,

solid line : relation (1) with α according to eq.(11),

dashed line : relation (2) with α according to eq.(11).

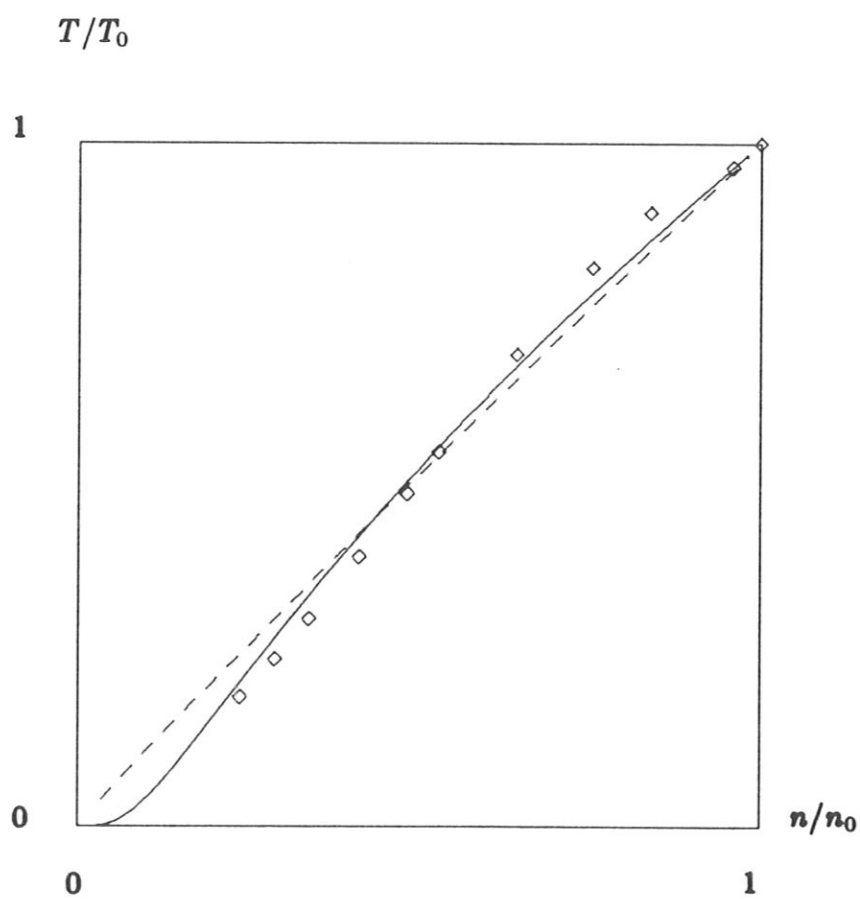$\alpha = 0.21$

$\eta = 0.99$

T/T_0

Reproduction of THOMSON scattering profiles for the L- and H- regimes from Ref/5/
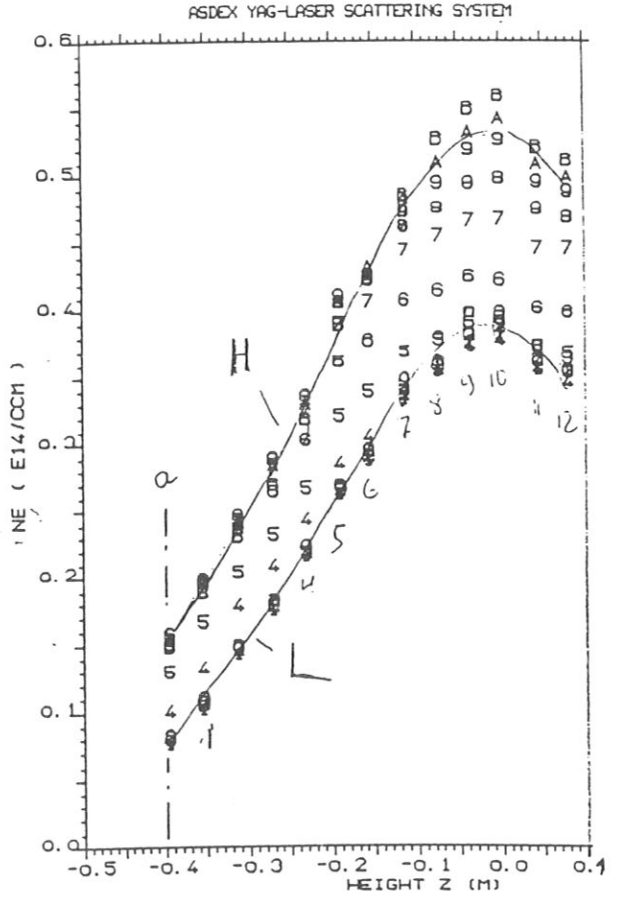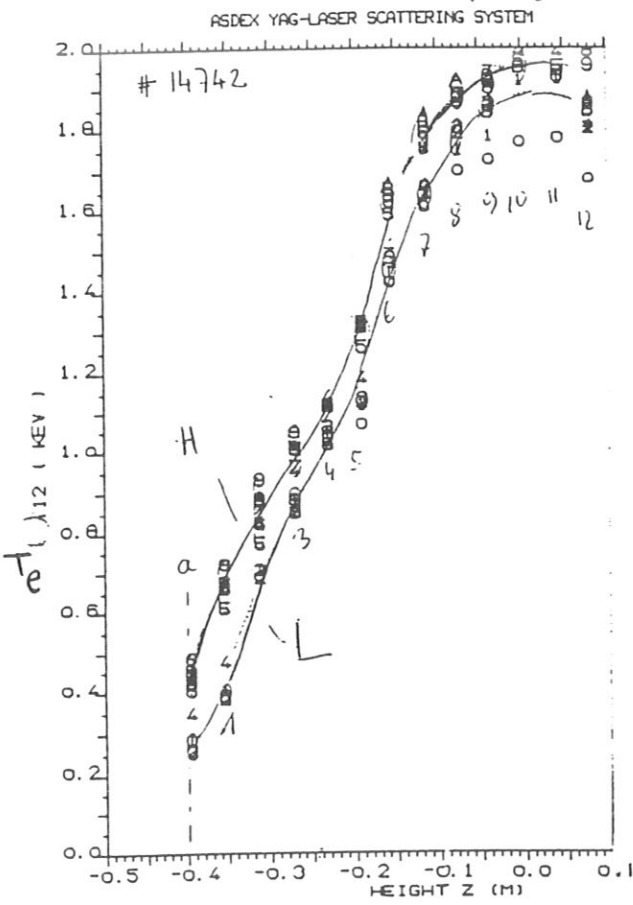

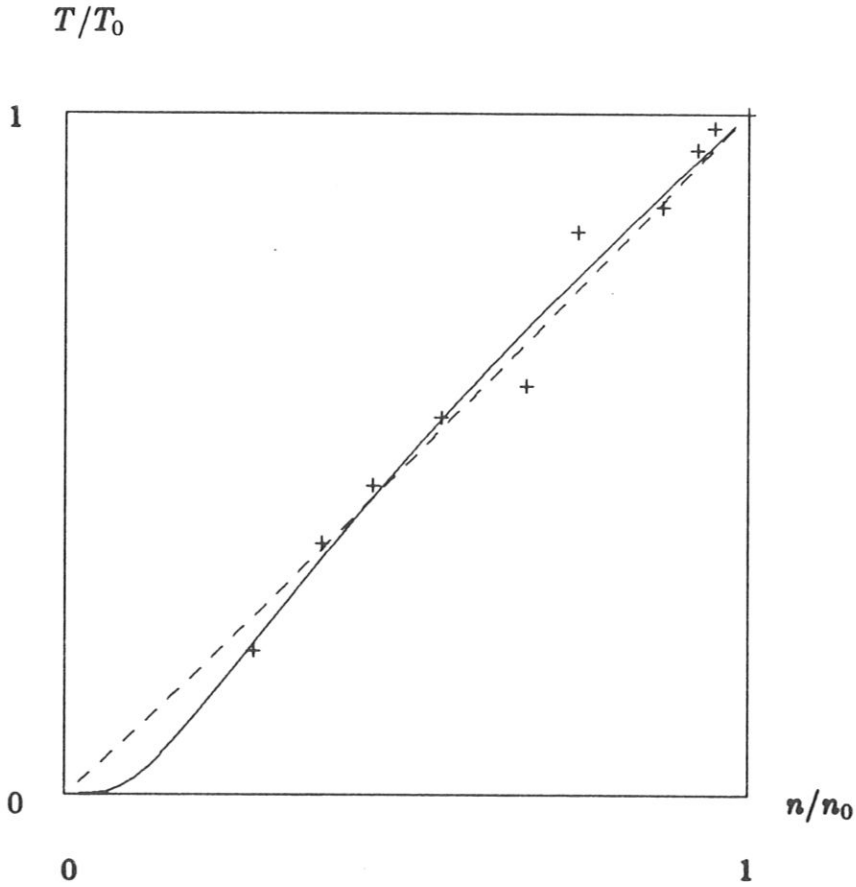Thomson scattering profiles

Fig.4b $\quad \frac{T}{T_0}$ versus $\frac{n}{n_0}$ for Ref/5/, L-regime

+ : experimental points,

$\diamond$: points on curves fitting experimental data,

solid line : relation (1) with α according to eq.(11),

dashed line : relation (2) with α according to eq.(11).

$\alpha = 0.24$

$\eta = 1.04$

Fig.4c $\quad \frac{T}{T_0}$ versus $\frac{n}{n_0}$ for Ref/5/, H-regime

+ : experimental points,

◇ : points on curves fitting experimental data,

solid line : relation (1) with α according to eq.(11),

dashed line : relation (2) with α according to eq.(11).
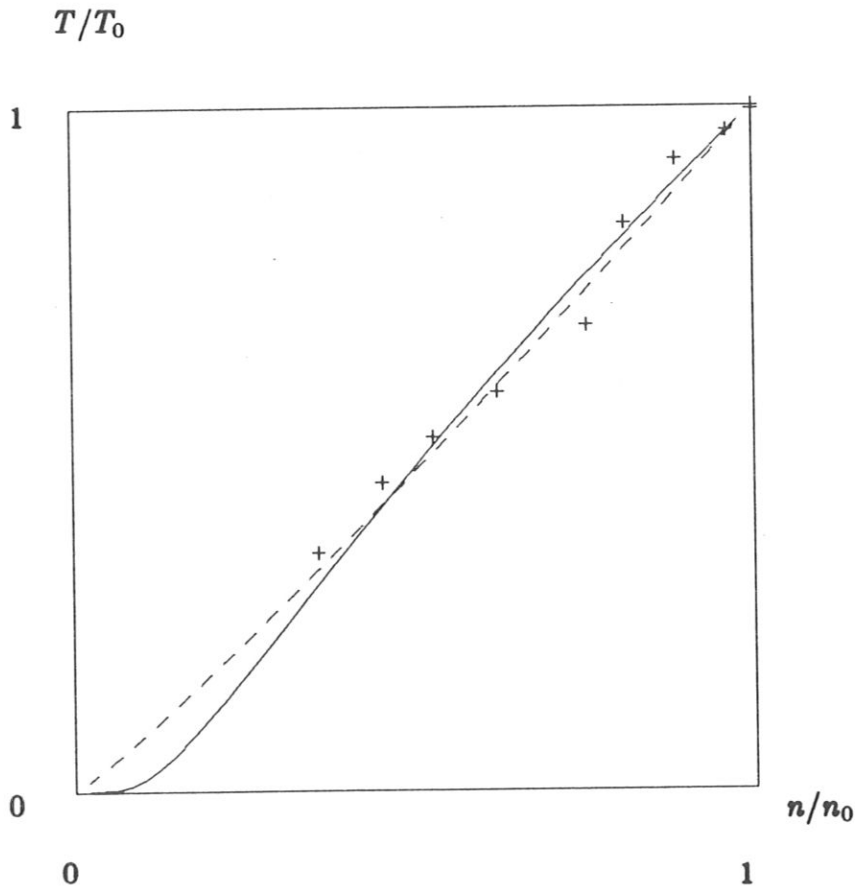
$\alpha = 0.27$

$\eta = 1.07$

T/T_0

Fig.5a
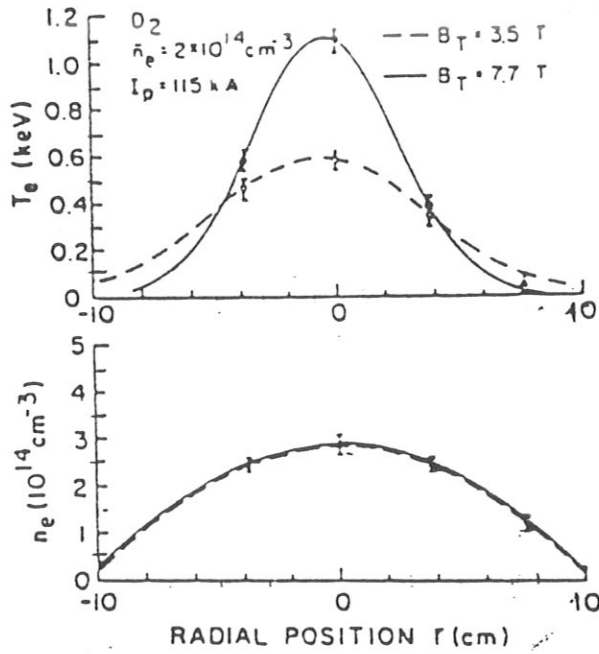
Reproduction of Ref/1/, FIGURE 2

FIGURE 2. Evidence of decoupling of the particle-density profile from the electron-temperature profiles at relatively low densities in the Alcator A device.[5] Here the linear average density, in deuterium plasmas, is $\bar{n}_e = 2 \times 10^{14}$ cm^{-3} and the plasma current $I \simeq$ 115 kA. The ratio q_s/q_0 has been varied by increasing B_T from 35 to 77 kG.
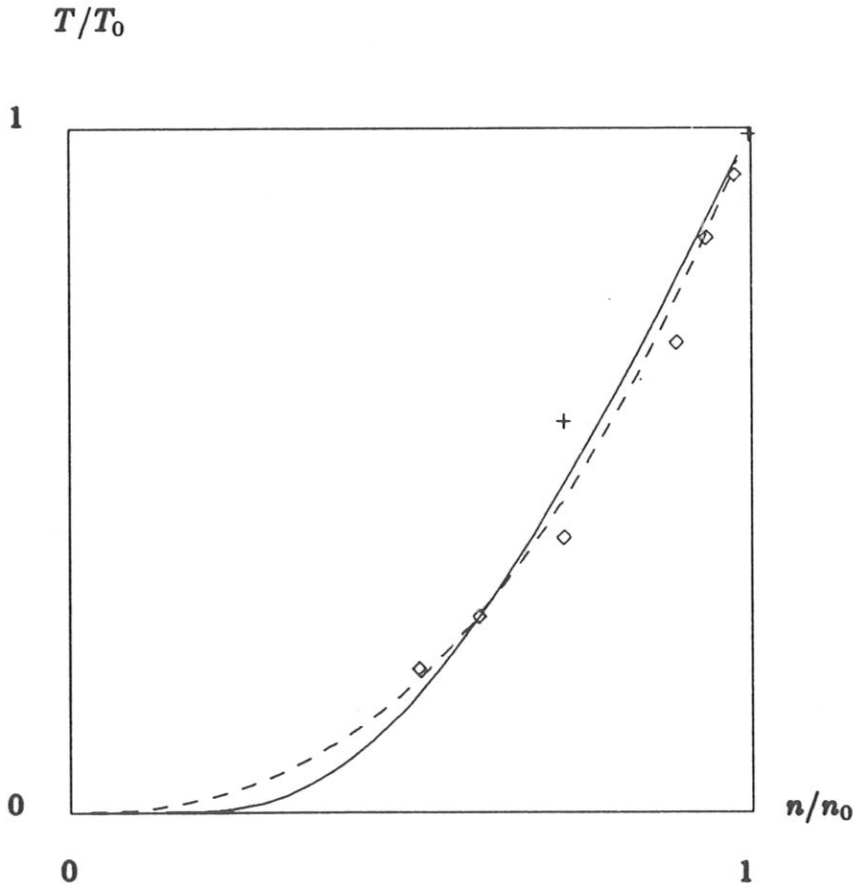
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Fig.5b $\frac{T}{T_0}$ versus $\frac{n}{n_0}$ for Ref/1/, FIGURE 2, dashed, $r \geq 0$

+ : experimental points,

◇ : points on curves fitting experimental data,

solid line : relation (1) with α according to eq.(11),

dashed line : relation (2) with α according to eq.(11).

$\alpha = 1.30$

$\eta = 2.36$

T/T_0

Fig.5c $\quad \frac{T}{T_0}$ versus $\frac{n}{n_0}$ for Ref/1/, FIGURE 2, solid, $r \leq 0$

$+$: experimental points,

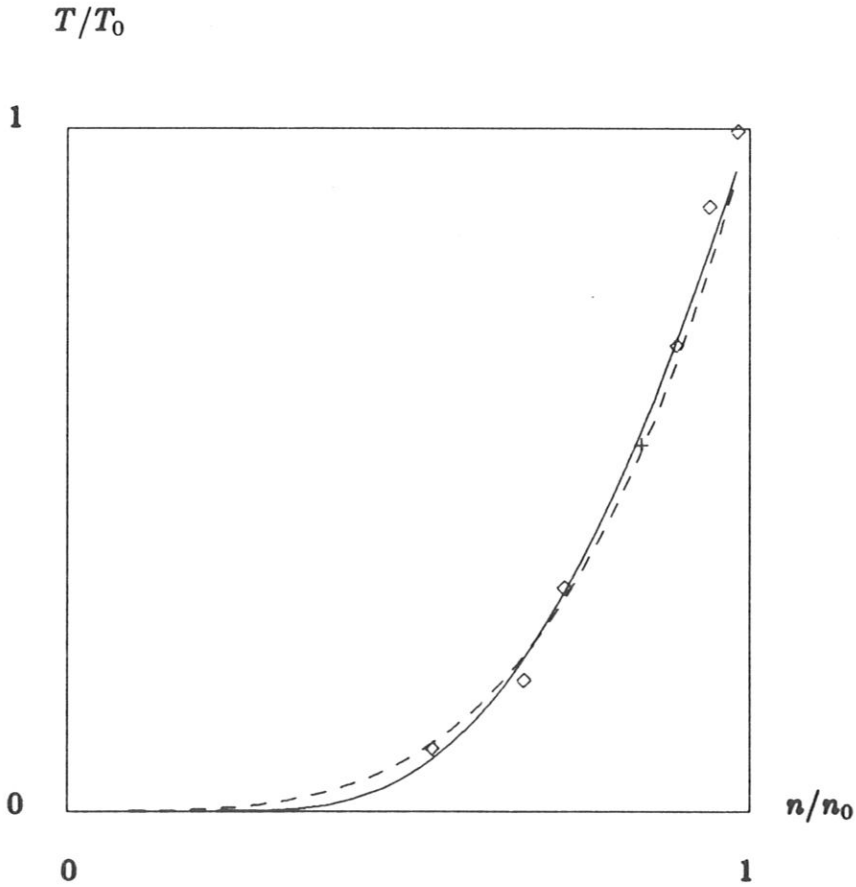$\diamond$: points on curves fitting experimental data,

solid line : relation (1) with α according to eq.(11),

dashed line : relation (2) with α according to eq.(11).

$\alpha \;=\; 2.46$

$\eta \;=\; 3.66$

T/T_0

4. "Theoretical" Profiles for Cylindrical Plasmas

In this section we present theoretical profiles which are obtained on the assumption that the toroidal field B_t is large compared with the poloidal field B_p, and that therefore OHM's law for the toroidal direction is simply

$$j_t \sim T^{3/2} . \tag{13}$$

Details are given in the Appendix.

$\beta_p = 1$ means in its simplest form $B_t = const.$ This situation is described by eqs.(A13a-d). It turns out that in this case only $\alpha = 0$ allows $n = 0$ at a plasma boundary placed on a finite radius a. For $\alpha > 0$ the plasma boundary can be defined by a certain arbitrarily chosen value of a.

Figs.6 show $n(\tilde{r})$ and $T(\tilde{r})$ with $\tilde{r} \sim r$ defined in eqs.(A8) and (A12) for the parameters listed in Table 1. This table also contains the values of the safety factor ratio q_a/q_0 according to eq.(A11). For the plasma boundary we have taken $\tilde{a} = 10$, except for $\alpha = 0$, with $\tilde{a}/a = \tilde{r}/r$.
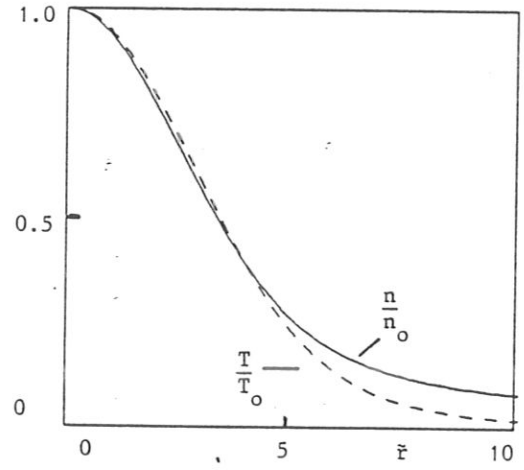
Table 1

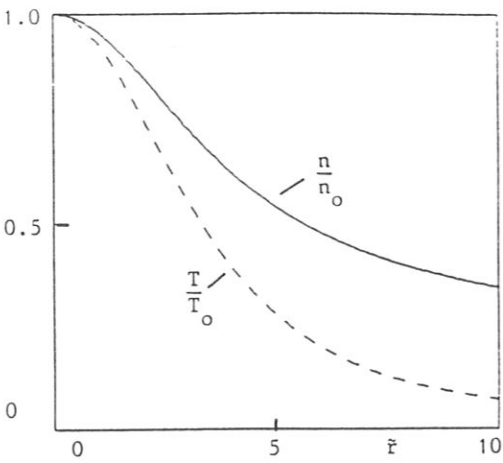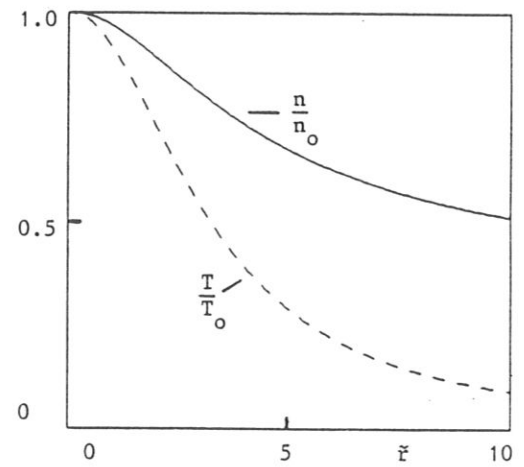| Fig	α	q_a/q_0
6a	0.	4.6
6b	0.2	8.6
6c	1.	7.5
6d	2.	7.2

Fig.6

(a)

(b)

(c)

(d)

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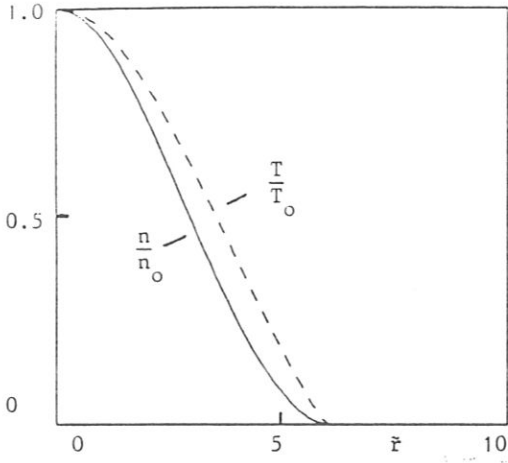
One also can define $\beta_p = 1$ in a more global way by requiring only $B_t(a) = B_t(0)$. For a certain class of toroidal fields $B_t(r)$ given by eqs.(A16,A17,A20,A22) one can allow $n(a) = 0$ also for $\alpha \neq 0$. In this case the relevant equations are (A23)-(A31) and $b_a = b_i$. We have used $\tilde{a} = 4$ as the plasma boundary if $n(\tilde{a}) = 0$ yields $\tilde{a} > 4$.

Figs.7-9 show $n(\tilde{r})$, $T(\tilde{r})$, $(B_t^2(\tilde{r}) - B_t^2(\tilde{a}))/(8\pi p_0)$

with $\tilde{r} \sim r$ defined in eqs.(A8) and (A27)

for the parameters listed in Table 2. This table again also contains the values of q_a/q_0. Note that

$\lambda = 0.5$ means $n \sim A^2$

$\lambda = 1$ means $n \sim A$

$\lambda = 2$ means $n \sim \sqrt{A}$.

The variation of $B_t(\tilde{r})$ is always modest, usually showing a small diamagnetic effect, if $\lambda + \alpha$ is not too small.

The case $\lambda = 1$ *and* $\alpha = 0$ is also treated analytically in the Appendix.

There is an interesting feature with these profiles that all the temperature and pressure profiles are essentially of the Gaussian type, except for $\tilde{r} > 3$, and therefore agree with the idea of profile consistency.

Table 2

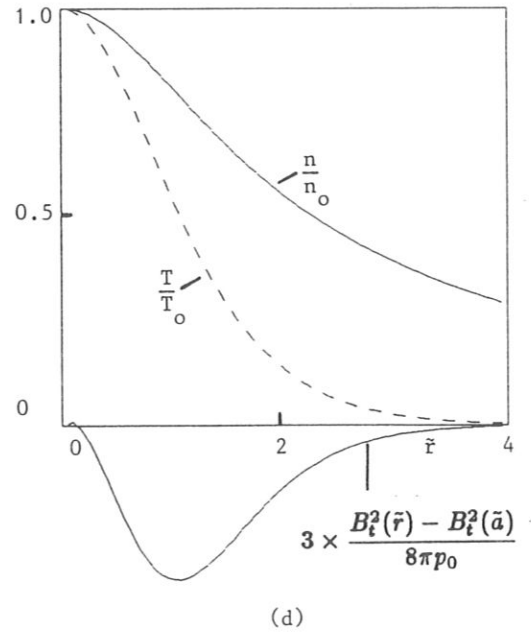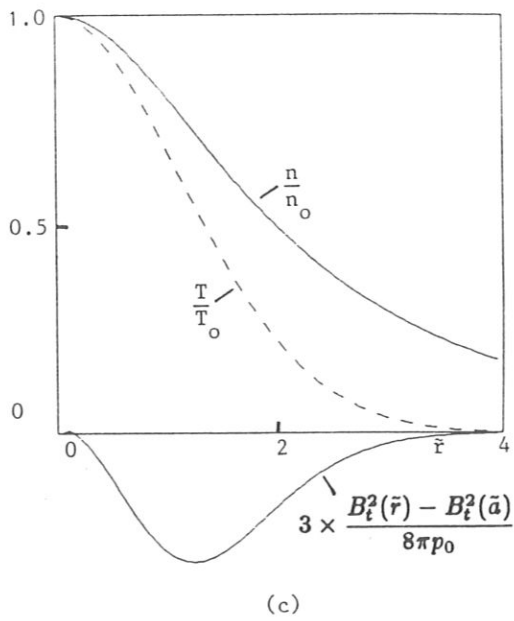| Fig	λ	α	q_a/q_0
7a	0.5	0.	4.3
7b	0.5	0.2	5.8
7c	0.5	1.	9.5
7d	0.5	2.	13.2
8a	1.	0.	2.3
8b	1.	0.2	3.2
8c	1.	1.	5.7
8d	1.	2.	7.6
9a	2.	0.	1.5
9b	2.	0.2	1.8
9c	2.	1.	3.2
9d	2.	2.	4.8

Fig.7

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(a)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(b)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(c)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(d)

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Fig.8

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$
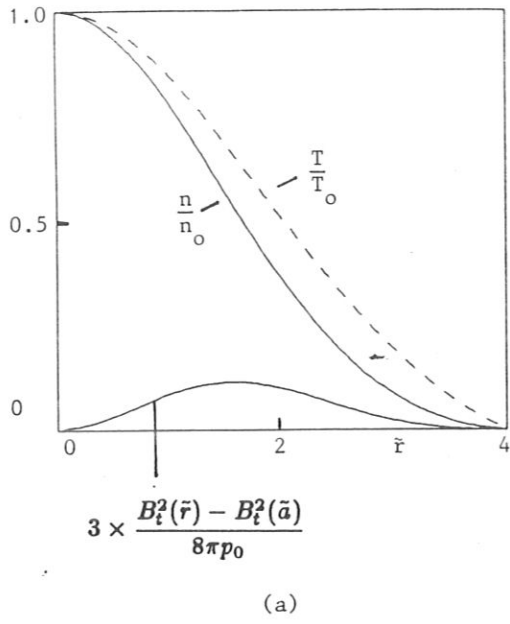
(a)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(b)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(c)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(d)

23

Fig.9

(a)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(b)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(c)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$

(d)

$$3 \times \frac{B_t^2(\tilde{r}) - B_t^2(\tilde{a})}{8\pi p_0}$$
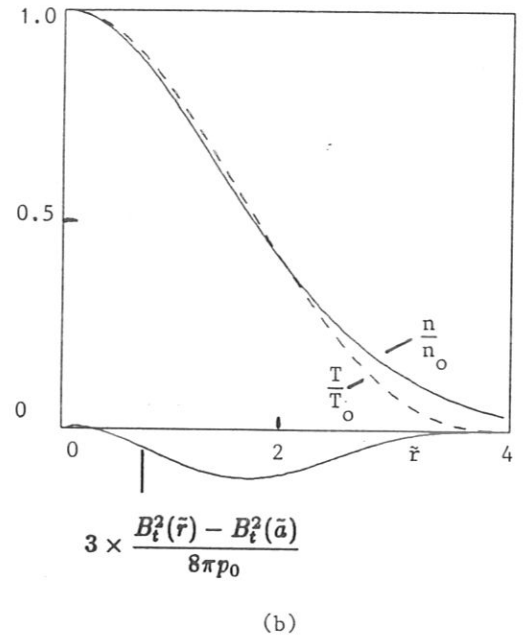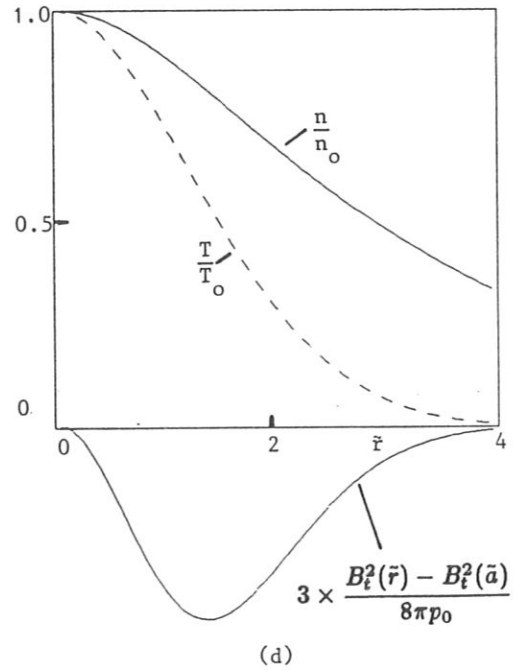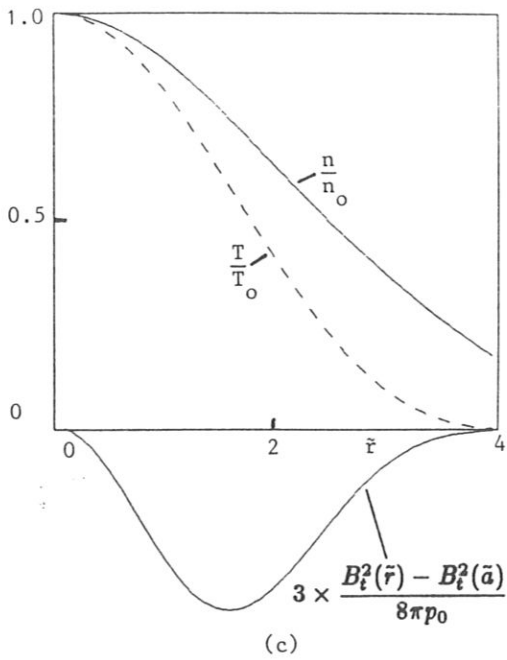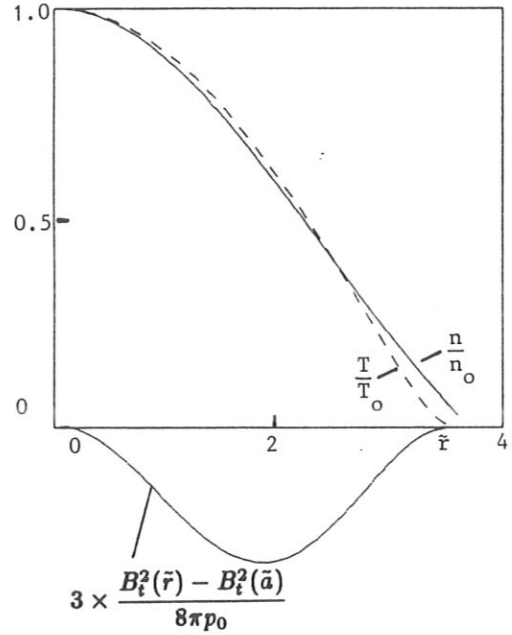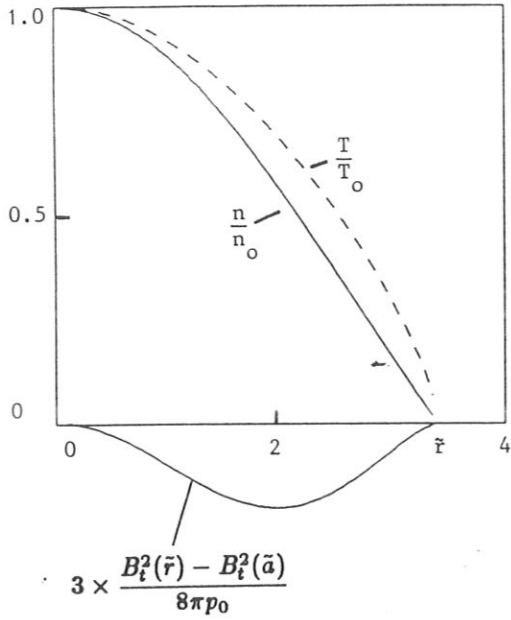
Summary

It has been shown that experimental temperature and density profiles in sufficiently relaxed tokamak discharges agree reasonably well with relation (1). Since this relation is obtained from a general principle, dubbed "entropy principle" here, one can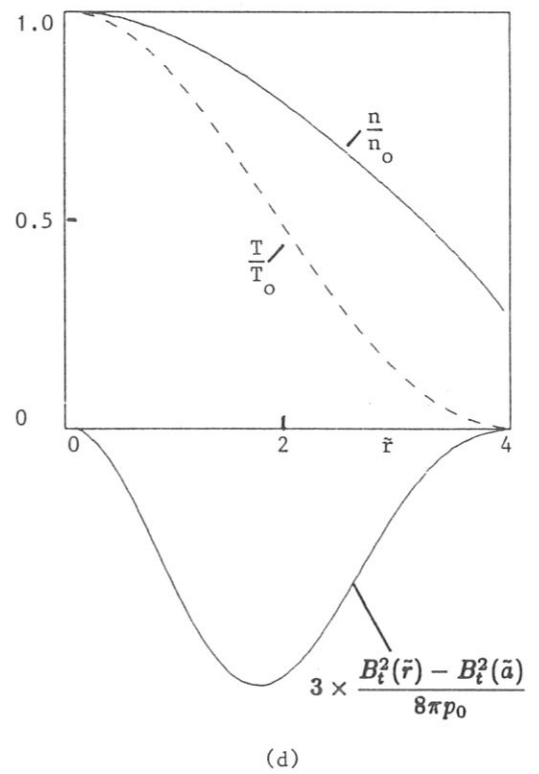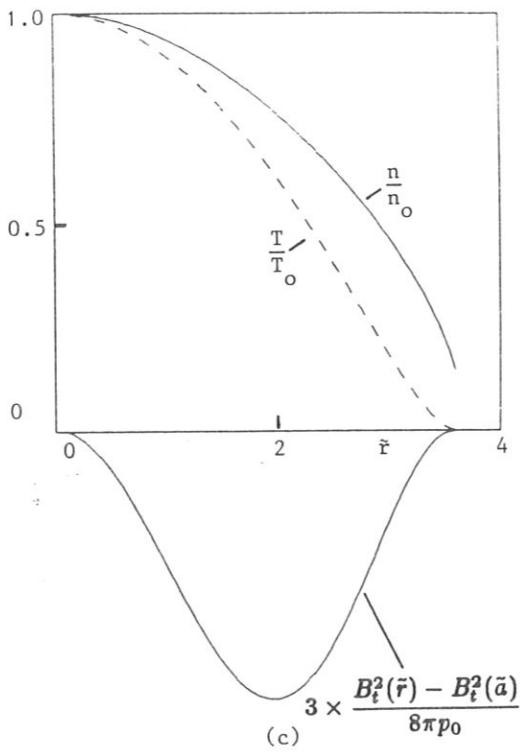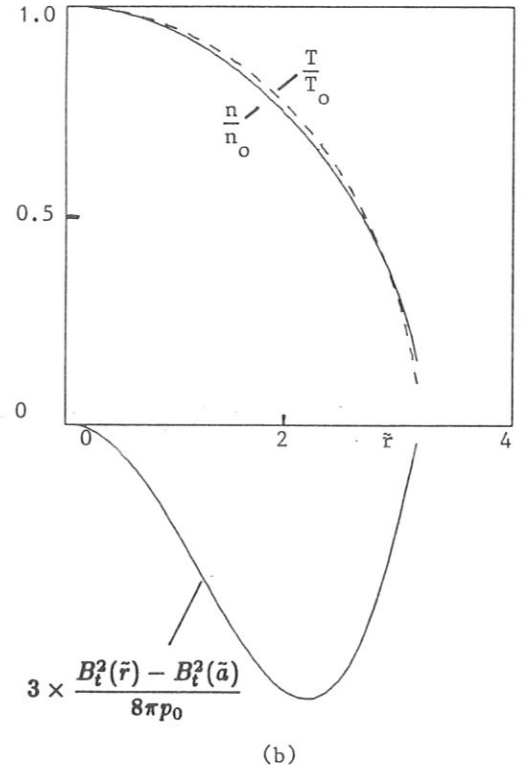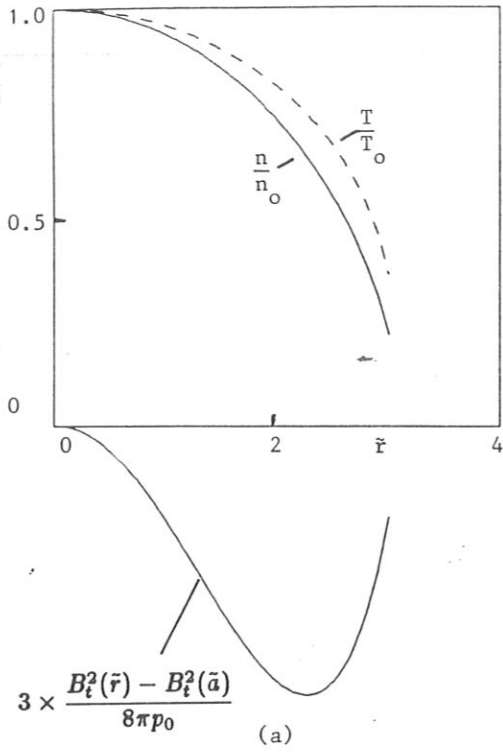 expect this relation to hold rather universally. It should therefore also be related to what is called "profile consistency". This is especially exhibited for $\beta_p = 1$ plasmas when eq.(1) is combined with OHM's law with SPITZER conductivity for the toroidal direction.

Acknowledgement

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        during Auxiliary Heating in ASDEX

        13th European Conference on Controlled Fusion and Plasma Heating

        Schliersee 1986, Paper P-Fr38

Appendix: Cylindrical Plasmas with Circular Cross-sections

In an r, ϕ, z cylindrical coordinate system the magnetic field consists of a "poloidal" field $B_p(r)$ in the $\phi-$ direction and a "toroidal" field $B_t(r)$ in the $z-$ direction:

$$\mathbf{B} = \left(0, B_p(r), B_t(r) \right) . \tag{A1}$$

It can be obtained from a vector potential

$$\mathbf{A} = \left(0, \int^r B_p(r') dr', A(r) \right) \tag{A2}$$

with

$$B_p(r) = - \frac{\partial A(r)}{\partial r} . \tag{A3}$$

In electrostatic cgs units, the current density is given by

$$\frac{4\pi}{c} \mathbf{j} = \left(0, -\frac{\partial B_t}{\partial r}, -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial A}{\partial r} \right) . \tag{A4}$$

With

$$B_t = B_t(A(r)), \qquad p = p(A(r))$$

the pressure balance equation yields a second relation for the toroidal component of the current density

$$j_t = c \frac{\partial}{\partial A} \left(p + \frac{B_t^2}{8\pi} \right) . \tag{A5}$$

We assume OHM's law to be valid with the electrical conductivity proportional to $T^{3/2}$ and

$$B_t \gg B_p . \tag{A6}$$

Hence one has

$$j_t = c \rho T^{3/2} . \tag{A7}$$

The constant ρ includes the external electric field necessary to drive the current and to make the system stationary. If one prescribes the total toroidal current, the quantity ρ

follows from the whole set of equations. The last equation we need is eq.(1) or its equivalent eq.(10) , which we repeat here:

$$\frac{p}{p_0} = \left(\frac{n}{n_0}\right)^\gamma exp(\ \alpha \cdot (1\ -\ \frac{n_0}{n})) \cdot \tag{10}$$

We introduce

a : plasma radius, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (A8)

$$B_{pa} = B_p|_{r=a} \qquad B_{ta} = B_t|_{r=a} \qquad\qquad B_{ti} = B_t|_{r=0}$$
$$b_a = B_{ta}^2/B_{pa}^2 \qquad b_i = B_{ti}^2/B_{pa}^2 \qquad\qquad \hat{p}_0 = \frac{p_0}{B_{pa}^2/8\pi}$$
$$\hat{p} = p/p_0 \qquad\qquad \hat{n} = n/n_0 \qquad\qquad\quad \hat{B}_p = B_p/B_{pa}$$
$$\hat{B}_t = B_t/B_{pa} \qquad\quad \hat{A} = \hat{\rho}\ A\ /\ (a\ B_{pa}) \qquad \hat{r} = \hat{\rho}\ r/a$$

$$\hat{\rho} = \rho\ \frac{8\pi a}{B_{pa}}\left(\frac{p_0}{n_0}\right)^{3/2} = \frac{4 j_0}{<j>}$$

$$<j> = \frac{2}{a^2}\int_0^a j\ r\ dr\ .$$

The following set of equations is then to be solved:

$$\hat{p}\ =\ \hat{n}^\gamma\ exp(\ \alpha \cdot (1 - \frac{1}{\hat{n}})) \tag{A9a}$$

$$\frac{\partial}{\partial\hat{A}}\left(\hat{p}\hat{p}_0\ +\ \hat{B}_t^2\right)\ =\ \left(\frac{\hat{p}}{\hat{n}}\right)^{3/2} \tag{A9b}$$

$$\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\ \frac{\partial\hat{A}}{\partial\hat{r}}\right)\ =\ -\frac{1}{2}\left(\frac{\hat{p}}{\hat{n}}\right)^{3/2} \cdot \tag{A9c}$$

We define the plasma boundary by $\hat{T} = \hat{T}_a$ which may be reached at a certain value $\hat{r} = \hat{r}_a$. In some cases $\hat{T}_a = 0$,

in other cases $\hat{T}_a$ could also be some other value $<< 1$.

In order to fulfill the boundary conditions

$$\frac{\partial\hat{A}}{\partial\hat{r}}|_{\hat{r}=\hat{r}_a}\ =\ -1\ \ ;\ \ \frac{\partial\hat{A}}{\partial\hat{r}}|_{\hat{r}=0}\ =\ 0\ , \tag{A9d}$$

the yet undetermined quantity $\hat{p}_0$ has to be chosen correspondingly. The value $\hat{r}_a$ itself gives us $\hat{\rho}$ as follows from the definition of $\hat{r}$ in eqs.(A8), namely

$$\hat{\rho}\ =\ \hat{r}_a\ \cdot \tag{A10}$$

Furthermore, we note that the safety factor ratio q_a/q_0 can be expressed as

$$\frac{q_a}{q_0} = -\hat{r}_a \sqrt{\frac{b_a}{b_i}} \left(\frac{1}{\hat{r}} \frac{\partial \hat{A}}{\partial \hat{r}} \right)_{\hat{r}=0}$$

$$= \frac{\hat{r}_a}{4} \sqrt{\frac{b_a}{b_i}} . \tag{A11}$$

We discuss separately the two cases a) B_t $const$ and b) B_t $variable$.

Constant Toroidal Field

In this case the toroidal field term in eq.(A10) is zero. We introduce

$$\tilde{A} = \hat{A}/\hat{p}_0 , \qquad\qquad \tilde{r} = \hat{r}/\sqrt{\hat{p}_0} , \tag{A12}$$

which transforms eqs.(A9a-d) into

$$\hat{p} = \hat{n}^{\gamma} exp(\alpha \cdot (1 - \frac{1}{\hat{n}})) \tag{A13a}$$

$$\frac{\partial \hat{p}}{\partial \tilde{A}} = \left(\frac{\hat{p}}{\hat{n}} \right)^{3/2} \tag{A13b}$$

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial \hat{A}}{\partial \tilde{r}} \right) = -\frac{1}{2} \left(\frac{\hat{p}}{\hat{n}} \right)^{3/2} . \tag{A13c}$$

If $\tilde{r}_a$ is the plasma boundary one finds $\hat{p}_0$ from

$$\sqrt{\hat{p}_0} \left(\frac{\partial \tilde{A}}{\partial \tilde{r}} \right)_{\hat{r}=\tilde{r}_a} = -1 \;\; ; \frac{\partial \tilde{A}}{\partial \tilde{r}}|_{\hat{r}=0} = 0 \tag{A13d}$$

In order to obtain information about the possible values of $\hat{T}_a$, we discuss eqs.(A9) and (A10) in the neighbourhood of the plasma boundary:

After multiplication by

$$\partial \hat{A}/\partial \hat{r} \approx -1$$

eq.(A9b) becomes

$$\hat{p}_0 \frac{\partial \hat{p}}{\partial \hat{r}} = \hat{p}_0 \frac{\partial \hat{p}}{\partial \hat{n}} \frac{\partial \hat{n}}{\partial \hat{r}}$$

$$= \hat{p}_0 \left(\frac{\gamma}{\hat{n}} + \frac{\alpha}{\hat{n}^2} \right) \hat{p} \frac{\partial \hat{n}}{\partial \hat{r}}$$

$$= - \left(\frac{\hat{p}}{\hat{n}} \right)^{3/2} \tag{A14}$$

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We integrate this equation between the radii $\hat{r}_0 \approx \hat{r}_a$ and $\hat{r}_a$ and obtain

$$\hat{r}_a - \hat{r}_0 = \hat{p}_0 \cdot \int_{\hat{n}(\hat{r}_a)}^{\hat{n}(\hat{r}_0)} d\hat{n} \; \hat{p}^{-1/2} \left(\gamma \, \hat{n}^{1/2} + \alpha \, \hat{n}^{-1/2} \right) . \tag{A15}$$

From this relation it follows that $\hat{n}(\hat{r}_a) = 0$ yields a finite value of $\hat{r}_a$ only for $\alpha = 0$.

Variable Toroidal Field

This case means a wide field. We restrict attention here to a simple case that allows $\hat{n}(\hat{r}_a) = 0$ also for $\alpha \neq 0$. The $\hat{A}$ dependence of $\hat{B}_t$ is expressed via an $\hat{n}$ dependence:

$$\hat{B}_t^2(\hat{A}) = F(\hat{n}). \tag{A16}$$

The function F must fulfill

$$F \geq 0 , \; F(0) = b_a , \; F(1) = b_i . \tag{A17}$$

In order to obtain simple relations, we define a function $G(\hat{n})$ by

$$\frac{\partial}{\partial \hat{n}} \left(\hat{p}\hat{p}_0 + F \right) = \hat{p}^{3/2} \, G(\hat{n}) \, \hat{n}^{-3/2}. \tag{A18}$$

This yields on the one hand

$$\int_0^{\hat{n}} G(n) \, dn = \hat{A} \tag{A19}$$

and on the other side

$$F = - \hat{p}\hat{p}_0 + \int_0^{\hat{n}} \hat{p}^{3/2} \, G(\hat{n}) \, \hat{n}^{-3/2} \, d\hat{n} + b_a. \tag{A20}$$

Interesting simple equilibria are obtained for

$$\hat{A} = g \, \hat{n}^\lambda , \; g = const \geq 0 , \; \lambda = const > 0 , \tag{A21}$$

which means

$$G = g \, \lambda \, \hat{n}^{\lambda - 1} . \tag{A22}$$

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The integral in eq.(A20) exists for all allowed α 's. The positivity of F is not guaranteed in general. But since we are interested in $b_a \gg 1$ whereas $\hat{p}_0$ should be of order 1, there should be no problem with $F > 0$. Furthermore, from eqs.(A24) and (10) or (1) we find

$$\hat{n} = \left(\frac{\hat{A}}{g}\right)^{1/\lambda} \tag{A23}$$

$$\hat{p} = \left(\frac{\hat{A}}{g}\right)^{\gamma/\lambda} \cdot exp\left(\alpha \cdot [1 - \left(\frac{g}{\hat{A}}\right)^{1/\lambda}]\right) \tag{A24}$$

$$\hat{T} = \left(\frac{\hat{A}}{g}\right)^{(\gamma-1)/\lambda} \cdot exp\left(\alpha \cdot [1 - \left(\frac{g}{\hat{A}}\right)^{1/\lambda}]\right) . \tag{A25}$$

The equation for $\hat{A}$ becomes

$$\frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}}\left(\hat{r} \frac{\partial \hat{A}}{\partial \hat{r}}\right) = -\frac{1}{2}\left(\frac{\hat{A}}{g}\right)^{\frac{3(\gamma-1)}{2\lambda}} \cdot exp\left(\frac{3}{2}\alpha \cdot [1 - \left(\frac{g}{\hat{A}}\right)^{1/\lambda}]\right) . \tag{A26}$$

If we introduce

$$\tilde{A} = \hat{A}/g , \qquad \tilde{r} = \hat{r}/\sqrt{g} , \tag{A27}$$

we obtain

$$\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}}\left(\tilde{r} \frac{\partial \tilde{A}}{\partial \tilde{r}}\right) = -\frac{1}{2} \tilde{A}^{\frac{3(\gamma-1)}{2\lambda}} \cdot exp\left(\frac{3}{2}\alpha \cdot [1 - \tilde{A}^{-1/\lambda}]\right) , \tag{A28}$$

which is to be solved with the initial conditions

$$\tilde{A}(\tilde{r} = 0) = 1 , \quad \frac{\partial \tilde{A}}{\partial \tilde{r}}(\tilde{r} = 0) = 0 . \tag{A29}$$

From eqs.(A27) we have

$$\frac{\partial \hat{A}}{\partial \hat{r}} = \sqrt{g} \cdot \frac{\partial \tilde{A}}{\partial \tilde{r}} . \tag{A30}$$

Thus if $\tilde{A}(\tilde{r}_a) = 0$ then g is found from

$$\sqrt{g} \left.\frac{\partial \tilde{A}}{\partial \tilde{r}}\right|_{\tilde{r}=\tilde{r}_a} = -1 . \tag{A31}$$

From eq.(A20) with $\hat{n} = 1$ it follows that

$$\hat{p}_0 + b_i - b_a = g \int_0^1 \hat{p}^{3/2} \lambda \, \hat{n}^{\lambda - 5/2} \, d\hat{n} , \tag{A32}$$

which determines $p_0 + b_i$.

Analytical Example

For illustration we present here an analytical example:

$$\gamma = 5/3 \; ; \qquad \lambda = 1 \; ; \qquad \alpha = 0$$

$$\hat{n} = \tilde{A} \; ; \qquad \hat{p} = \tilde{A}^{5/3} \; ; \qquad \hat{T} = \tilde{A}^{2/3}$$

$$\frac{\partial^2}{\partial \tilde{r}^2} \tilde{A} + \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \tilde{A} = -\frac{1}{2} \tilde{A}.$$

This yields

$$\tilde{A} = J_0(\tilde{r}/\sqrt{2}) \, ,$$

which has the correct value 1 at $\tilde{r} = 0$. The first zero of J_0 is at

$$\tilde{r}/\sqrt{2} = 2.4048 \, ,$$

from which we find

$$\tilde{r}_a = \tilde{r}|_{J_0=0} = 2.4048 \cdot \sqrt{2} = 3.40 \, .$$

Furthermore, we have

$$\frac{\partial \tilde{A}}{\partial \tilde{r}} = -\frac{1}{\sqrt{2}} J_1(\tilde{r}/\sqrt{2})$$

with

$$J_1(\tilde{r}_a/\sqrt{2}) = 0.5191 \, .$$

It follows that

$$\frac{\partial \tilde{A}}{\partial \tilde{r}}\Big|_{\tilde{r}=\tilde{r}_a} = -0.367$$

and therefore

$$g = 7.42$$

$$\hat{r} = 2.725\tilde{r} \; ; \; \hat{r}_a = \hat{\rho} = 9.27$$

$$q_a/q_0 = 2.32 \sqrt{ba/bi} \, .$$

The square of the toroidal field becomes

$$B_t^2 = F = -\hat{p}_0 \, \hat{n}^{5/3} + 3.71 \cdot \hat{n}^2 + b_a \, .$$

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An overall $\beta_p = 1$ plasma, i.e. $b_i = b_a$, is obtained for

$$\hat{p}_0 \;=\; 3.71 \;.$$

In this case the minimum of F is reached where

$$\hat{n} \;=\; \hat{n}_m \;=\; (5/6)^3 \;=\; 0.5787 \;.$$

It has the value

$$F \;=\; F_m \;=\; b_a \;-\; 0.248 \;.$$

There is thus a small dip in the toroidal field.