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An Integrable System of two Nonlinear Oscillators as Attractor

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Abstract:

A system of two nonlinear oscillators is proved to be integrable by reducing it to elliptic integrals. This conservative system delivers the attractors of a driven damped system of a generalized Van der Pol type that was previously introduced by the author [1].

In a previous paper [1] the Lyapunov stability and the location of attractors for a special system of nonlinear oscillators were investigated. This peculiar system is of the form

$$\ddot{Y} + \left[(Y, AY)M + (\dot{Y}, B\dot{Y})N - P\right]\dot{Y} + CY = 0 , \tag{1}$$

where Y is a real vector of arbitrary length r, A, B and C are r×r positive definite symmetric matrices, and M, N and P are r×r matrices whose symmetric part is positive definite. It is shown in Ref. [1] that the Lyapunov stability of system (1) can be discussed in terms of $(\dot{Y}, \dot{Y})$ and (Y, CY), and that the attractors have to be located in a strip of the first quadrant as drawn in Fig. 1.

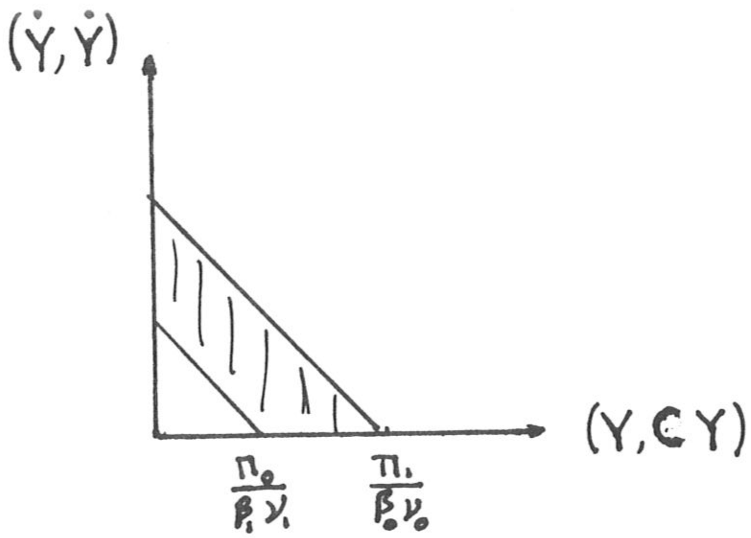

Fig. 1

β_1, ν_1, π_1 are the largest eigenvalues and β_o, ν_o, π_o the lowest eigenvalues of B, N_s and P_s, respectively, N_s and P_s being the symmetric parts of N and P.

It is reasonable to expect limit cycles when the strip of Fig. 1 goes to zero. An example which was considered in Ref. [1] was to take

$$\begin{aligned} A &= \alpha I , & B &= \beta I , \\ M &= \mu I , & N &= \nu I , \\ P &= \pi I , & C &= \frac{\alpha\mu}{\beta\nu} I = \varepsilon I , \end{aligned} \tag{2}$$

where I is the identity matrix. This leads [1] to a high-dimensional harmonic limit cycle. Notice that we can add to M, N and P antisymmetric parts M_a, N_a, P_a without altering the null thickness of the strip of Fig. 1. This brings the nonlinearities $(Y, AY)M_a$, $(\dot{Y}, B\dot{Y})N_a$ into play and the problem of testing for a limit cycle becomes very difficult.

In this note the problem is restricted to an r=2 system with the null thickness condition (see Ref. [1])

$$(\dot{Y},\dot{Y}) + \varepsilon (Y,Y) = \frac{\Pi}{\beta\nu} \tag{3}$$

$$M_a = \begin{pmatrix} 0 & m/\alpha \\ -m/\alpha & 0 \end{pmatrix}, \quad N_a = \begin{pmatrix} 0 & n/\beta \\ -n/\beta & 0 \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}. \tag{4}$$

This means that after the system has reached asymptotically the strip of zero thickness the equations of motion are given by

$$\begin{aligned} &\ddot{y}_1 + \left[m(y_1^2+y_2^2) + n(\dot{y}_1^2+\dot{y}_2^2) + p\right]\dot{y}_2 + \varepsilon y_1 = 0, \\ &\ddot{y}_2 - \left[m(y_1^2+y_2^2) + n(\dot{y}_1^2+\dot{y}_2^2) + p\right]\dot{y}_1 + \varepsilon y_2 = 0. \end{aligned} \tag{5}$$

If system (5) is integrable, this means that system (1) under conditions (3) and (4) has a stable four-dimensional attractor. We want to prove integrability. Let us set

$$z = y_1 + i y_2 \quad \text{and} \quad \bar{z} = y_1 - i y_2. \tag{6}$$

System (5) can then be written as

$$\ddot{z} - i\left[m z\bar{z} + n\dot{z}\dot{\bar{z}} + p\right]\dot{z} + \varepsilon z = 0. \tag{7}$$

The polar representation of z is

$$z = \rho(t)\, e^{i\,\theta(t)} . \tag{8}$$

Expression (8) is substituted in eq. (7) and real and imaginary parts are separated to yield

$$\ddot{\rho} - \rho\dot{\theta}^2 + m\rho^3\dot{\theta} + n\rho\dot{\theta}\left(\dot{\rho} + \rho^2\dot{\theta}^2\right) + \rho\mu\dot{\theta} + \varepsilon\rho = 0 , \tag{9}$$

$$\rho\ddot{\theta} + 2\dot{\theta}\dot{\rho} - m\rho^2\dot{\rho} - n\dot{\rho}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right) - \mu\dot{\rho} = 0 . \tag{10}$$

Multiply eq. (9) by ρ and eq. (10) by $\rho\dot{\theta}$ and add

$$\dot{\rho}\ddot{\rho} + \dot{\theta}^2\rho\dot{\rho} + \rho^2\dot{\theta}\ddot{\theta} + \varepsilon\rho\dot{\rho} = 0 . \tag{11}$$

Equation (11) can be integrated once to

$$\dot{\rho}^2 + \dot{\theta}^2\rho^2 + \varepsilon\rho^2 = h \tag{12}$$

with $h = \pi/\beta\nu$ according to eq. (3). The value of $\dot{\rho}^2 + \rho^2\dot{\theta}^2$ from eq. (12) is substituted in eqs. (9) and (10) to yield

$$\ddot{\rho} - \rho\dot{\theta}^2 + m\rho^3\dot{\theta} + n\rho\dot{\theta}\left(h - \varepsilon\rho^2\right) + \mu\rho\dot{\theta} + \varepsilon\rho = 0 , \tag{13}$$

$$\rho\ddot{\theta} + 2\dot{\theta}\dot{\rho} - m\rho^2\dot{\rho} - n\dot{\rho}\left(h - \varepsilon\rho^2\right) - \mu\dot{\rho} = 0 . \tag{14}$$

Multiply eq. (14) by ρ and integrate once:

$$\rho^2\dot{\theta} = \frac{m}{4}\rho^4 + \frac{nh}{2}\rho^2 - \frac{\varepsilon n}{4}\rho^4 + \frac{\mu}{2}\rho^2 + C , \tag{15}$$

where C is an integration constant. If we insert $\dot{\theta}$ from eq. (15) into eq. (13), we obtain an equation for ρ only:

$$\ddot{\rho} - \rho\left[\rho^2\left(\frac{m}{4} - \frac{\varepsilon n}{4}\right) + \frac{nh+p}{2} + \frac{C}{\rho^2}\right]^2 +$$

$$+\left[\rho^2\left(\frac{m-\varepsilon n}{4}\right) + \frac{(nh+p)}{2} + \frac{C}{\rho^2}\right]\left[m\rho^3 + n\rho(h-\varepsilon\rho^2) + p\rho\right] +$$

$$+ \quad \varepsilon\rho = 0, \tag{16}$$

or

$$\ddot{\rho} = \rho\left[\frac{C^2}{\rho^4} - \rho^2\frac{(m-\varepsilon n)(nh+p)}{2} - \frac{3}{16}\rho^4(m-\varepsilon n)^2\right.$$

$$\left. - \left(C\frac{(m-\varepsilon n)}{2} + \frac{(nh+p)^2}{4}\right)\right]. \tag{17}$$

Multiply eq. (17) by $\dot{\rho}$ and integrate once:

$$\dot{\rho}^2 = \left[-\frac{C^2}{\rho^2} - \rho^4\frac{(m-\varepsilon n)(nh+p)}{4} - \frac{\rho^6}{16}(m-\varepsilon n)^2\right.$$

$$\left. - \rho^2\left(\frac{C}{2}(m-\varepsilon n) + \left(\frac{nh+p}{2}\right)^2\right) + d\right], \tag{18}$$

where d is an integration constant.
Set $\rho^2 = x$ and multiply eq. (18) by $4x$. This yields

$$\dot{x} = \pm\left[-\left(\frac{m-\varepsilon n}{2}\right)^2 x^4 - (m-\varepsilon n)(nh+p)\,x^3\right.$$

$$\left. - \left((nh+p)^2 + 2C(m-\varepsilon n)\right)x^2 + 4dx - 4C^2\right]^{1/2}. \tag{19}$$

The integration of eq. (19) reduces [2] to elliptic integrals. The original system can now be integrated completely because eq. (15) needs an integration over a function of ρ. Of course, the explicit answer will be cumbersome in general. As an illustration consider here a rather simple case $m = \varepsilon,\ d > 0\ ,\ d^2 > \frac{c^2}{(nh+\mu)^2}$ Instead of an elliptic integral we get an arcsin, and the solution of (19) or (18) is

$$\rho = \sqrt{x} = \left[\frac{2d}{(nh+\mu)^2} + \frac{2}{nh+\mu}\left(\frac{d^2}{(nh+\mu)^2} - c^2\right)^{1/2} \sin(nh+\mu)t\right]^{1/2} . \tag{20}$$

Equation (15) is also easy to integrate in this case. We find

$$\theta = \frac{nh+\mu}{2} t + \operatorname{arctg}\left[\frac{d}{c(nh+\mu)} \operatorname{tg}\left(\frac{nh+\mu}{2} t\right) + \frac{1}{c}\left(\frac{d^2}{(nh+\mu)^2} - c^2\right)^{1/2}\right]. \tag{21}$$

Let us note finally that an equation of the type

$$\ddot{z} + i f(z\bar{z}, \dot{z}\dot{\bar{z}})\, \dot{z} + g(z\bar{z})\, z = 0$$

with f and g real can be proved to be integrable by a similar procedure but cannot be reduced to elliptic integrals.

References

[1] Tasso H. to be published,
see also IPP 6/259, March 1986

[2] Byrd P.F., Friedman M.D. "Handbook of Elliptic Integrals for Engineers and Scientists", Springer-Verlag Berlin, Heidelberg, New York, 1971.