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GARCHING BEI MÜNCHEN

Lyapunov Stability and Attractors of Some Systems of Nonlinear Oscillators

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IPP 6/259 March 1986

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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Abstract

Lyapunov functions valid in the greater part of phase space were found for a system of nonlinear oscillators of an extended Van der Pol type. They yield a good estimate of the location of attractors. For a particular single oscillator the appropriately modified Van der Pol equation (see eq. (16)) delivers an ellipse as limit cycle.

Zusammenfassung

Liapunov Funktionen gültig für den größten Teil des Phasenraums sind hier für ein System von nichtlinearen Oscillatoren von einem erweiterten Van der Pol Typ abgeleitet worden. Dies erlaubt eine gute Abschätzung der Lage der Attraktoren. Für einen speziellen Oscillator liefert die modifizierte Van der Pol Gleichung (siehe Gl. (16)) eine Ellipse als Grenzzyklus.

The Van der Pol equation is basic in the area of nonlinear oscillations (see, for example, /1/). It is unstable near the origin but possesses a stable limit cycle. Changing the sign in front of the first derivative makes the origin an attractor whose basin is bounded by an unstable limit cycle. We construct here arbitrarily large systems of an extended Van der Pol type for which Lyapunov functions leading to a good estimate of the location of the attractors can be found. Let us consider

$$\ddot{Y} + \varepsilon\left[(Y,AY)M + (\dot{Y},B\dot{Y})N - P\right]\dot{Y} + CY = 0, \tag{1}$$

where Y is a real vector of arbitrary length N. A, B, C, M, N and P are N x N real matrices, whose properties will be specified later. (...,...) denotes the scalar product and $\varepsilon = 1$ is taken first.

Assuming C to be symmetric and positive definite and taking the scalar product of eq. (1) with $\dot{Y}$, one obtains

$$\frac{1}{2}\frac{d}{dt}\left[(\dot{Y},\dot{Y}) + (Y,CY)\right] = -\left[(Y,AY)(\dot{Y},M\dot{Y}) + (\dot{Y},B\dot{Y})(\dot{Y},N\dot{Y}) - (\dot{Y},P\dot{Y})\right]. \tag{2}$$

Only the symmetric parts A_s, B_s, M_s, N_s, P_s of A, B, M, N and P appear in eq. (2). It is assumed that A_s, B_s, M_s, N_s and P_s are positive definite with highest eigenvalues α_1, β_1, μ_1, ν_1 and π_1 and lowest eigenvalues α_o, β_o, μ_o, ν_o and π_o. Let the highest and lowest eigenvalues of C be denoted by γ_1 and γ_o. Two basic inequalities can now be derived from eq. (2):

$$\frac{1}{2}\frac{d}{dt}\left[(\dot{Y},\dot{Y})+(Y,CY)\right] \leq -\beta_0\nu_0\left[(\dot{Y},\dot{Y})+(Y,CY)+\frac{\alpha_0\mu_0}{\beta_0\nu_0}(Y,Y)-(Y,CY)-\frac{\Pi_1}{\beta_0\nu_0}\right](\dot{Y},\dot{Y})\,, \quad (3)$$

$$\frac{1}{2}\frac{d}{dt}\left[(\dot{Y},\dot{Y})+(Y,CY)\right] \geq -\beta_1\nu_1\left[(\dot{Y},\dot{Y})+(Y,CY)+\frac{\alpha_1\mu_1}{\beta_1\nu_1}(Y,Y)-(Y,CY)-\frac{\Pi_0}{\beta_1\nu_1}\right](\dot{Y},\dot{Y})\,. \quad (4)$$

From inequality (4) we have instability around the origin and in the case

$$\gamma_0 \geq \frac{\alpha_1\mu_1}{\beta_1\nu_1} \quad (5)$$

the instability persists if

$$(\dot{Y},\dot{Y}) + (Y,CY) \leq \frac{\Pi_0}{\beta_1\nu_1}\,. \quad (6)$$

From inequality (3) and

$$\gamma_1 \leq \frac{\alpha_0\mu_0}{\beta_0\nu_0} \quad (7)$$

it can be seen that the system is stable if

$$(\dot{Y},\dot{Y}) + (Y,CY) \geq \frac{\Pi_1}{\beta_0\nu_0}\,. \quad (8)$$

If a plot is made in terms of $(\dot{Y},\dot{Y})$ and (Y,CY), we know that any attractor will have to be in a strip between the two lines obtained from eqs. (6) and (8) (see plot).

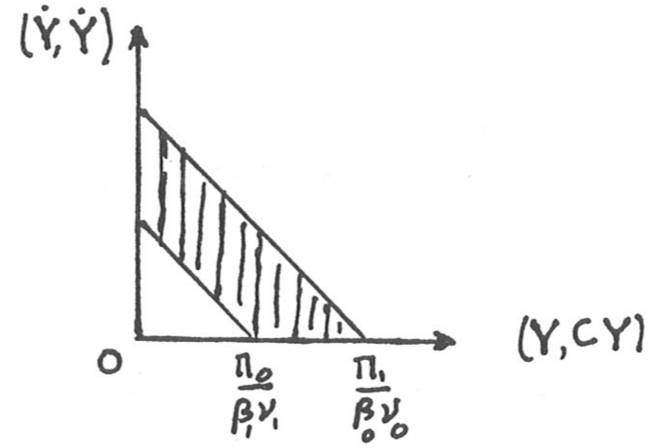

Fig 1: Location of attractors

An interesting case is when the strip goes to zero, i.e.

$$\frac{\Pi_0}{\beta_1\nu_1} = \frac{\Pi_1}{\beta_0\nu_0} . \tag{9}$$

According to the definitions eq. (9) can be verified only if

$$\Pi_0 = \Pi_1 = \Pi \qquad \text{and} \qquad \beta_1\nu_1 = \beta_0\nu_0 = \beta\nu . \tag{10}$$

From eq. (10) and the inequalities (7) and (5) we obtain

$$\frac{\alpha_1\mu_1}{\beta\nu} \leq \gamma_0 \leq \gamma_1 \leq \frac{\alpha_0\mu_0}{\beta\nu} . \tag{11}$$

Inequalities (11) can only be verified as equalities, which means

$$\alpha_0\mu_0 = \alpha_1\mu_1 = \alpha\mu \quad , \quad \gamma_0 = \gamma_1 = \gamma = \frac{\alpha\mu}{\beta\nu} . \tag{12}$$

Equations (10) and (12) are verified in particular if

$$\begin{aligned} A &= \alpha I, \qquad & B &= \beta I, \\ M &= \mu I, \qquad & N &= \nu I, \\ P &= \pi I, \qquad & C &= \frac{\alpha\mu}{\beta\nu} I , \end{aligned} \tag{13}$$

where I is the identity matrix. Equations (6) and (8) yield in that limit

$$(\dot{Y},\dot{Y}) + \frac{\alpha\mu}{\beta\nu}(Y,Y) = \frac{\pi}{\beta\nu} . \tag{14}$$

Relation (14) annihilates the coefficient of Y in eq. (1), which becomes

$$\ddot{Y} + \frac{\alpha\mu}{\beta\nu} Y = 0 . \tag{15}$$

Equations (14) and (15) are compatible and their common solutions yields a high-dimensional stable limit cycle. For N = 1 and conditions (13), eq. (1) becomes

$$\ddot{y} + (\alpha\mu y^2 + \beta\nu \dot{y}^2 - \pi)\dot{y} + \frac{\alpha\mu}{\beta\nu} y = 0 . \tag{16}$$

Equation (16) is an extension of the Van der Pol equation but has a much simpler limit cycle, i.e. an ellipse in the y, $\dot{y}$ plane. Note that the limit $\beta\nu \to o$ to the Van der Pol equation is singular.

In the general case there is no reason to expect a limit cycle as attractor. More plausible is something of the sort considered in Ref. /2/, sometimes called "strange" attractor. Numerical calculations in the strip in Fig 1 may help to identify the attractor. Let us finally mention that it is straightforward to go through the arguments for $\varepsilon = -1$. In essence, the stable regions become unstable and vice versa and the limit cycles become unstable. Note also that in essence all this theory still holds in case a positive monotonic nonlinearity $f((Y,HY))$, $f' > o$, $H > o$ is added in front of $\dot{Y}$ in eq. (1) if at the same time the operator $f'H$ is added in front of Y.

References

/1/ Hagedorn P. "Nichtlineare Schwingungen" Akademie Verlagsgesellschaft, Wiesbaden (1978)

/2/ Smale S. "Differentiable Dynamical Systems" Bull. Amer. Math. Soc. 73, 747, (1967)