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MAGNETOHYDRODYNAMIC EQUILIBRIUM CALCULATIONS USING MULTIGRID

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ABSTRACT

The multigrid method has been applied to the solution of the two-dimensional elliptic equation that governs axisymmetric ideal magnetohydrodynamic equilibrium. The possibility of applying multigrid to the computation of axisymmetric equilibria in the 'inverse coordinates' formulation and to three-dimensional equilibrium and evolution calculations is investigated.

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INTRODUCTION

In the magnetic confinement approach to controlled thermonuclear fusion, the hot plasma is prevented from escaping by the use of a strong magnetic field. The $\mathbf{v} \times \mathbf{B}$ force (velocity times magnetic field) inhibits motion of electrons and ions perpendicular to the magnetic field, and the plasma particles are constrained in lowest order to follow helical orbits 'tied' to a magnetic field line. In order to confine the plasma also with regard to its motion along the field, configurations are employed in which the magnetic field lines are enclosed in a toroidal region. The two main contenders in magnetic confinement research are the tokamak, which is an axisymmetric device, and the stellarator, which does not have a continuous symmetry. In either device, the magnetic field is generated by a combination of external currents and currents flowing in the plasma, making the configuration of plasma and magnetic field a nonlinear system.

We are concerned here with the application of multigrid methods to the computation of magnetic confinement configurations. In this context the plasma may be considered as an ideal magnetohydrodynamic (MHD) fluid, and the static equilibrium of plasma and field is described by the system of equations,

$$\nabla \cdot \mathbf{B} = 0, \tag{1a}$$

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p, \tag{1b}$$

where $\mathbf{B}$ is the magnetic field and p is the kinetic pressure. (Rationalized units in which $\mu_0 = 1$ are employed). Despite its simple appearance the system (1) is extremely difficult to solve numerically for general three-dimensional configurations, and a substantial fraction of the production time of the Cray-1 computer at Garching is spent on this problem. Typical calculations require $\sim 10^4$ iterations and take between 2 and 4 hours of CPU-time in order to compute a single equilibrium to acceptable accuracy. To find efficient methods for the solution of (1) is thus a major challenge for computational plasma physics. In the axisymmetric case the system (1) may be reduced to a single elliptic partial differential equation in the plane, which can be efficiently solved by a variety of methods.

Specifically, we consider the application of multigrid to three classes of problems in computational MHD: (1) Computation of axisymmetric equilibria on an Eulerian grid. (2) Axisymmetric equilibria in the inverse coordinates formulation. (3) Three-dimensional equilibrium and evolution calculations. Although fast numerical methods for these three problems are of considerable interest, no previous investigation into the use of multigrid for computations in magnetic confinement theory seems to exist.

Problem (1) requires the solution of an almost linear, uniformly elliptic equation (a nonlinearity occurs only in the right hand side), and application of multigrid is straightforward. Our code achieves full multigrid efficiency, and is several times faster than codes based on direct rapid elliptic solvers.

In regard to problem (2) a connection is noted between the inverse coordinates approach on one hand and grid generation through elliptic systems on the other. This leads to a novel formulation of the inverse coordinates equilibrium problem as a quasi-linear elliptic system, which is suitable for multigrid treatment. (A code has not been written).

The discussion of problem (3) is also of an analytical nature. Local mode analysis is employed in order to develop a relaxation procedure, rather than for a posteriori validation only. The best local relaxation scheme will still be slowly converging for two classes of disturbances, viz. the slow magnetosonic and the shear Alfvén modes, in both cases with the wavevector nearly transverse to the magnetic field. These are the lowest frequency modes in the MHD spectrum. A satisfactory treatment of these troublesome modes requires distributive line relaxation along the magnetic field and semi-coarsening on magnetic surfaces; this emphasizes the need for an adaptive, field-tied grid for 3-D MHD calculations. The proposed simple relaxation scheme can be suitable for implicit ideal MHD evolution calculations on the slowest timescale.

The reader is assumed to be familiar with multigrid methods, as presented in particular in [**1**] and [**2**]. Useful reviews of the relevant MHD theory may be found in [**3**]–[**6**].

1. A Multigrid Code for Axisymmetric Equilibrium

The Grad-Schlüter-Shafranov equation. A very substantial simplification in the system (1) is afforded by the assumption of axisymmetry. Under this assumption the equation $\nabla \cdot \mathbf{B} = 0$ may be solved by choosing the representation,

$$\mathbf{B} = F\nabla\phi + \nabla\psi \times \nabla\phi,$$

where ϕ is the ignorable angle in the cylindrical (r, ϕ, z) coordinate system, and F and ψ are axisymmetric scalar functions. A similar representation is found for the current,

$$\nabla \times \mathbf{B} = -\Delta^*\psi\nabla\phi + \nabla F \times \nabla\phi,$$

where the operator Δ^* is given by

$$\Delta^*\psi = r\frac{\partial}{\partial r}\left(r^{-1}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2}. \tag{2}$$

From the force balance equation (1^b) one may next derive $\nabla\psi \times \nabla p = \mathbf{0}$, $\nabla F \times \nabla p = \mathbf{0}$, and $\nabla\psi \times \nabla F = \mathbf{0}$. It is taken to follow that ψ, F, and p are functionally related, and one writes $F = F(\psi)$ and $p = p(\psi)$. (These must be understood as local relations in case a surface of constant ψ has disconnected parts). Finally, consideration of force balance along $\nabla\psi$ leads to the Grad-Schlüter-Shafranov equation [7]–[9],

$$\Delta^*\psi = -r^2\frac{dp}{d\psi} - F\frac{dF}{d\psi}. \tag{3}$$

This equation is the basis for the study of axisymmetric ideal MHD equilibrium. For given profiles $p(\psi)$ and $F(\psi)$ it is an almost linear elliptic p.d.e., the nonlinearity occurring only in the right hand side. The various methods (not including multigrid) that have been used in the past to solve Eq. (3) have been reviewed in [10] and [11].

Discretization scheme. Conventional second order accurate discretization methods for the equilibrium equation (3), written as $\Delta^*\psi = f(\mathbf{r}, \psi)$, are of the five point molecule form,

$$\begin{pmatrix} & * & \\ * & * & * \\ & * & \end{pmatrix} \psi = f.$$

All previous finite difference codes for the solution of (3) employ such a second order method. Better methods are available for smooth f, in particular a fourth order accurate 'compact' discretization of the shape,

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \psi = \begin{pmatrix} & * & \\ * & * & * \\ & * & \end{pmatrix} f.$$

Specifically: Consider a uniform rectangular mesh with spacing $\delta r = h$ and $\delta z = k$. Consider the natural splitting, $\Delta^* = \mathcal{L}_r + \mathcal{L}_z$, and define the second order accurate difference approximations $\mathcal{L}_r^h$ and $\mathcal{L}_z^k$ by,

$$[\mathcal{L}_r^h \psi](r,z) = \frac{r}{h^2}\Big(\frac{\psi(r+h,z)-\psi(r,z)}{r+h/2} - \frac{\psi(r,z)-\psi(r-h,z)}{r-h/2}\Big),$$

$$[\mathcal{L}_z^k \psi](r,z) = \frac{1}{k^2}(\psi(r,z+k) - 2\psi(r,z) + \psi(r,z-k)).$$

Then a fourth order discretization of $\Delta^*\psi = f$ is obtained from the identity,

$$\begin{aligned} \Big[\mathcal{L}_r^h + \mathcal{L}_z^k + \frac{1}{12}(h^2+k^2)\mathcal{L}_r^h\mathcal{L}_z^k + O(h^4+k^4)\Big]\psi \\ = \Big(1 + \frac{1}{12}h^2\mathcal{L}_r^h + \frac{1}{12}k^2\mathcal{L}_z^k\Big) f. \end{aligned} \tag{4}$$

A special treatment on the plasma–vacuum interface, on which f or its first order derivatives may be discontinuous, is required in order to gain full advantage of the higher accuracy obtained for the interior equations. (Such a treatment has not been implemented in our code).

Performance of multigrid. A demonstration code has been written, which employs multigrid relaxation to solve Eq. (3) on a rectangular region subject to Dirichlet boundary conditions. The mesh at level i has size $(2^i+1)\times(2^i+1)$, where $1 \le i$; the coarsest grid therefore contains only one single interior point. The cycling algorithm is of the full multigrid, full approximation storage variety, and employs adaptive switching. Full-weighting transfer is used for both the solution and the residuals, and bi-cubic interpolation is used for the corrections. Red-black point relaxation is employed on all grids except on the coarsest, where a nonlinear root-finding algorithm is employed. The special treatment on the coarsest grid is necessary because the equation generally admits multiple solutions; the algorithm that is employed on the coarsest grid is designed to find the interesting solution.

For an example calculation we assumed a right hand side in Eq. (3) of the form,

$$f(\mathbf{r},\psi) = \begin{cases} -r^2 g(\psi) - c, & \psi > 0 \\ 0, & \psi \leq 0 \end{cases}$$

in which g is a second-degree polynomial and c is a constant. The contour defined by $\psi = 0$ is the free boundary of the plasma. A calculation using 7 levels (the finest grid has size 129×129) required 120 msec on the Cray-1 to solve Eq. (3) to the level of the discretization error. The total number of passes over each of the levels 1–7 was 10, 19, 14, 8, 6, 4, and 2 respectively. The computing time was divided about evenly between evaluation of the r.h.s., evaluation of the residuals, bi-cubic interpolation, and all other chores.

Thus, full multigrid efficiency for the solution of the equation (3) has been achieved. For a linear problem this code is about 2–3 times slower than the well-optimized fast Buneman solver used at Garching, but for the more relevant nonlinear problems the codes based on a direct elliptic solver require Picard iteration, and multigrid relaxation is easily the fastest procedure available.

2. Axisymmetric Equilibrium in Inverse Coordinates

For stability and transport calculations related to axisymmetric configurations it is required to have an explicit representation of the magnetic surfaces of the equilibrium (the contours of constant ψ), which are in this case assumed to form a nested set that converges to a single 'magnetic axis'. Such a representation may be found by a numerical mapping after having computed the equilibrium on a fixed spatial grid. In recent years, however, a class of procedures has become popular in which the equilibrium is computed in a formulation which immediately gives the spatial coordinates r and z as functions of ψ and η, where η is some angular variable. This 'inverse variables' method has been employed in Refs. [**12**]–[**15**]. There is considerable freedom in the choice of the angular variable η. It is defined via orthogonality in [**12**], via a specification of the Jacobian in [**13**] and [**14**], and in [**15**] it is suggested to choose η such that lines of constant η are straight rays.

Surprisingly, in all these papers the authors fail to notice the immediate analogy between the inverse variables approach to MHD equilibrium calculations and the method of grid generation through elliptic equations [**16**], [**17**]. As a consequence they also miss the most natural formulation of the inverse equilibrium problem, in which the coordinate η is defined as the solution of an elliptic equation, resulting in a quasilinear elliptic system of equations for $r(\psi,\eta)$ and $z(\psi,\eta)$. This formulation is discussed below, as it is also the most convenient formulation for possible multigrid treatment.

Grid generation through elliptic equations. We first consider by way of example the problem of constructing a boundary-fitted curvilinear coordinate system (ξ^1,ξ^2) to cover the pseudo-rectangular region $G \in \mathbf{R}^2$. The Cartesian coordinates on $\mathbf{R}^2$ are (x_1,x_2). G is to be mapped to the unit square in the (ξ^1,ξ^2) plane, and the points A, B, C, and D on the boundary ∂G are to be mapped to $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. Using the method of grid generation through elliptic equations, the curvilinear coordinates ξ^i are defined by Poisson equations,

$$\Delta \xi^i = F^i, \tag{5}$$

subject to Dirichlet conditions on ∂G: $\xi^1 = 0$ on AB, $\xi^1 = 1$ on CD, etc. In the simplest case one chooses $F^i = 0$, but a nonzero right hand side in Eq. (5) may be used to obtain more control over the resulting mesh.

The expression for the Laplacian on the curvilinear coordinates ξ^i is,

$$\Delta u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \left(\sqrt{g}\, g^{ij} \frac{\partial u}{\partial \xi^j} \right). \tag{6}$$

(Summation over repeated indices is understood). Substituting ξ^k for u in the above equation and considering Eq. (5) one finds,

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} (\sqrt{g}\, g^{ij}) = F^j$$

and therefore,

$$\Delta u = g^{ij} \frac{\partial^2 u}{\partial \xi^i \partial \xi^j} + F^j \frac{\partial u}{\partial \xi^j} \tag{7}$$

To obtain numerically the transformation $\vec{x} \rightarrow \vec{\xi}$ it is convenient to solve in inverse coordinates, and obtain $\vec{x}$ as a function of $\vec{\xi}$ rather than $\vec{\xi}$ as a function of $\vec{x}$. (The whole point of the grid generation is that differential equations may be more easily solved on the transformed region). The components x_i are obtained by solving

$$\Delta x_i = 0, \quad i = 1, 2 \tag{8}$$

on the unit square in the (ξ^1, ξ^2) plane, again subject to Dirichlet boundary conditions. With Δ given by Eq. (7) this is a quasilinear elliptic system.

Application to axisymmetric equilibrium. The analogy between the above model problem of grid generation and the inverse coordinates approach to axisymmetric MHD equilibrium is quite obvious; one only has to replace Δ by Δ^* and change from a rectangular to a polar geometry, with the singularity at the magnetic axis. In a curvilinear coordinate system (ξ, η) the operator Δ^* has the representation,

$$\begin{aligned} \Delta^* u = |\nabla \xi|^2 \frac{\partial^2 u}{\partial \xi^2} + 2(\nabla \xi \cdot \nabla \eta) \frac{\partial^2 u}{\partial \xi \partial \eta} + |\nabla \eta|^2 \frac{\partial^2 u}{\partial \eta^2} \\ + \Delta^* \xi \frac{\partial u}{\partial \xi} + \Delta^* \eta \frac{\partial u}{\partial \eta}, \end{aligned} \tag{9}$$

which corresponds to Eq. (7). For the transformed coordinate ξ we may choose $\xi = \psi$, or any function of ψ alone, so that $\Delta^* \xi$ follows from the equilibrium equation (3), and may be assumed known. The natural choice of the differential equation for η is $\Delta^* \eta = 0$. From Eq. (2) it is seen that

$$\begin{cases} \Delta^* r = -r^{-1} \\ \Delta^* z = 0 \end{cases} \tag{10}$$

Considering r and z as functions of ξ and η, Eq. (10) naturally remains valid, but the operator Δ^* is given by Eq. (9) instead of by Eq. (2). This quasi-linear elliptic system of equations (10) governs the equilibrium in inverse coordinates. The boundary conditions require periodicity in η, specified r and z on the plasma boundary, and an appropriate expansion near the magnetic axis.

The system (10) appears suitable for multigrid treatment, although it is more complicated than the equilibrium equation in the form (3). The standard second order discretization has a symmetric nine-point stencil, and an incomplete Cholesky decomposition should probably be used for relaxation. (Alternating direction line relaxation is an alternative, but point relaxation must not be employed on a polar grid). Higher order discretizations would be of interest, in particular a spectral method in the angular coordinate.

3. Prospects for Three-Dimensional Calculations

In this Section an analytical study of the use of multigrid for the difficult and (at present) very expensive area of 3-D MHD computations is initiated. The system of equations (1) is equivalent to the system that governs steady, inviscid, incompressible flow, as can be seen by making the substitutions,

$$\mathbf{B} \rightarrow \mathbf{v}, \qquad p \rightarrow -\frac{p}{\rho} - \frac{v^2}{2}.$$

Progress in solving the corresponding hydrodynamic equations is therefore of immediate interest for magnetic confinement studies.

The Chodura and Schlüter approach. In attempting to develop a multigrid relaxation procedure for 3-D equilibrium I have found it convenient to take as the starting point the approach of Chodura and Schlüter [**18**], who employ a finite difference discretization on a fixed spatial mesh of the system (1) in primitive variables.

The solution procedure employed by Chodura and Schlüter is designed to find a constrained minimum of the potential energy, W, which is given by

$$W = \int_T \left(\frac{B^2}{2} + \frac{p}{\gamma - 1} \right) dV, \tag{11}$$

where T is the toroidal computational region, and γ is the adiabatic index. Minimization of W is performed through displacements of the form,

$$\begin{cases} \delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \\ \\ \delta \rho = -\nabla \cdot (\rho \boldsymbol{\xi}) \end{cases} \tag{12}$$

subject to $\boldsymbol{\xi} = \mathbf{0}$ on the boundary ∂T. The relation $p = \rho^\gamma$ is assumed (ρ corresponds to the mass density of the MHD fluid). Through these displacements, an arbitrary initial plasma and field configuration is transformed under the constraints of mass and flux conservation into a minimum energy state.

In leading order the change in energy due to the displacements (12) is given by

$$\delta W = -\int_T (\mathbf{F} \cdot \boldsymbol{\xi}) \, dV, \tag{13}$$

where $\mathbf{F} = (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$. Therefore, a state of minimum energy under the displacements (12) satisfies $\mathbf{F} = \mathbf{0}$, and is a solution to the force balance equation (1^b).

The equation (1^a) remains satisfied if it is satisfied initially. Eq. (13) also shows a possible route to the energy minimum, viz. to choose at each iteration $\boldsymbol{\xi} = \alpha \mathbf{F}$, where $\alpha > 0$ and α is sufficiently small to ensure stability, but this algorithm is prohibitively slow. Chodura and Schlüter employ various acceleration methods, with impressive results, but still require some 10^3–10^4 iterations for practical calculations.

A multigrid relaxation procedure. We now attempt to derive a relaxation procedure for the system (1) that effectively reduces short wavelength error components, and may therefore be suitable in the context of multigrid. A finite difference discretization of (1) on a staggered mesh is envisaged, but it turns out that the analysis of relaxation procedures can largely be carried out without reference to the discretized system of equations.

To satisfy flux conservation, $\nabla \cdot \mathbf{B} = 0$, is of course easy. At each relaxation sweep $\mathbf{B}$ may be updated by a distributive scheme of the form $\mathbf{B} \leftarrow \mathbf{B} + \delta\mathbf{B}$, where $\delta\mathbf{B} = \nabla\chi$. To satisfy exactly $\nabla \cdot \mathbf{B} = 0$ after the iteration sweep one would have to find χ as the solution to a Poisson equation, but here it suffices to approximate χ locally by any kind of relaxation prescription that is suitable for Poisson equations. Notice that the replacement $\mathbf{B} \leftarrow \mathbf{B} + \nabla\chi$ does not affect the force balance equation.

For the second equation, $\mathbf{F} = \mathbf{0}$, the work of Chodura and Schlüter suggests a relaxation scheme based on the coupled replacements $\mathbf{B} \leftarrow \mathbf{B} + \delta\mathbf{B}$ and $p \leftarrow p + \delta p$, where $\delta p = \gamma\,(p/\rho)\delta\rho$, and where $\delta\mathbf{B}$ and $\delta\rho$ are given by Eq. (12). These replacements do not affect $\nabla \cdot \mathbf{B} = 0$. The question is how to choose $\boldsymbol{\xi}$ in Eq. (12) as a function of the current residual $\mathbf{F}$.

To answer this question one must consider the principal terms of the change $\delta\mathbf{F}$ under the displacements (12), viz. those terms in which $\boldsymbol{\xi}$ is twice differentiated:

$$\begin{aligned}\delta\mathbf{F} \simeq\ & (\mathbf{B}\cdot\nabla)(\mathbf{B}\cdot\nabla)\boldsymbol{\xi} + (B^2 + \gamma p)\nabla(\nabla\cdot\boldsymbol{\xi}) \\ & - \mathbf{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi}) - (\mathbf{B}\cdot\nabla)\nabla(\mathbf{B}\cdot\boldsymbol{\xi})\end{aligned} \tag{14}$$

(The operator ∇ acts on $\boldsymbol{\xi}$ only). Fourier analysis transforms $\delta\mathbf{F}$ into $\delta\tilde{\mathbf{F}}$ and $\boldsymbol{\xi}$ into $\tilde{\boldsymbol{\xi}}$, related by $\delta\tilde{\mathbf{F}} \simeq \boldsymbol{A}\cdot\tilde{\boldsymbol{\xi}}$, where

$$\boldsymbol{A} = -B^2(k_\parallel^2 \boldsymbol{I} + (1+\beta)\mathbf{kk} - k_\parallel(\mathbf{bk} + \mathbf{kb})) \tag{15}$$

where $\mathbf{k}$ is the wavevector, $\beta = \gamma p/B^2$, $\mathbf{b} = \mathbf{B}/B$, and $k_\parallel = \mathbf{k}\cdot\mathbf{b}$. ($\beta$ is a small parameter for magnetic confinement).

The operator $\boldsymbol{A}$ may be inverted:

$$\boldsymbol{A}^{-1} = -(B^2 k_{\|}^2)^{-1}(\boldsymbol{I} + \beta^{-1}\mathbf{bb} - \mathbf{kk}/k^2), \tag{16}$$

and a desirable relaxation scheme must approximate $\tilde{\boldsymbol{\xi}} = -\boldsymbol{A}^{-1}\tilde{\mathbf{F}}$, at least for short wavelength components. Of course $\boldsymbol{A}^{-1}$ contains $\mathbf{k}$, and is therefore not a local linear operator. Various things can be tried, for instance to drop the term $\mathbf{kk}/k^2$ and to replace $k_{\|}^2$ by $2\omega^{-1}(h_x^{-2} + h_y^{-2} + h_z^{-2})$, where ω is a constant of order unity, and h_x, h_y, and h_z are the local mesh spacings. Via this prescription one obtains a relaxation procedure based on

$$\boldsymbol{\xi} = \boldsymbol{R} \cdot \mathbf{F}, \tag{17}$$

where $\boldsymbol{R}$ is the operator,

$$\boldsymbol{R} = \frac{\omega}{2} B^{-2}(h_x^{-2} + h_y^{-2} + h_z^{-2})^{-1}(\boldsymbol{I} + \beta^{-1}\mathbf{bb}). \tag{18}$$

The large coefficient on $\mathbf{bb}$ in this relaxation prescription is worthy of note.

Further analysis of the proposed procedure. The relaxation scheme (17) must now be analyzed in order to see whether all short wavelength error modes are effectively reduced. Continuing with the linear analysis, and still considering principal terms only, one finds,

$$\begin{aligned}\delta\tilde{\mathbf{F}} &= \boldsymbol{A} \cdot \boldsymbol{R} \cdot \tilde{\mathbf{F}} \\ &= -\frac{\omega}{2}(h_x^{-2} + h_y^{-2} + h_z^{-2})^{-1}(k_{\|}^2 \boldsymbol{I} + (1+\beta)\mathbf{kk} - k_{\|}\mathbf{bk}) \cdot \tilde{\mathbf{F}}.\end{aligned}$$

It may be seen that the scheme is not satisfactory, as those modes for which

$$\mathbf{k} \perp \mathbf{B} \quad \text{and} \quad \mathbf{k} \perp \tilde{\mathbf{F}}$$

(approximately) are not well eliminated. (In addition there may be problems related to the occurrence of different values of the grid spacing, but those difficulties are easily taken care of by line relaxation).

The troublesome modes are the slow magnetosonic mode, for which $\mathbf{F}$ and $\mathbf{B}$ are nearly parallel, and the shear Alfvén mode, for which $\mathbf{B}$, $\mathbf{F}$, and $\mathbf{k}$ form an orthogonal triad. These are the lowest frequency modes ($\nu \to 0$) in the MHD spectrum. The reason that these modes are not well eliminated can also be understood on physical grounds.

The perturbation related to the slow magnetosonic mode concerns the pressure only, and is characterized by a long wavelength along the magnetic field and a short wavelength across the field. As the restoring force for this perturbation acts along field lines, the relaxation procedure only becomes effective when the mesh spacing corresponds to the length scale along the field, but on such a mesh (assuming equal coarsening in all directions) the perturbation will be invisible due to the rapid variation across the field. Similarly, the restoring force for the shear Alfvén mode is located in a plane in which the mode has a long wavelength, but perpendicular to this plane there is a rapid variation.

Both the form of the operator $\boldsymbol{A}^{-1}$ in Eq. (16) and the physical picture outlined above point the way to a remedy. One needs line relaxation along the magnetic field (which allows to retain $k_{\parallel}$ in going from $\boldsymbol{A}^{-1}$ to $\boldsymbol{R}$) to eliminate effectively the slow magnetosonic mode, and either plane relaxation or (more likely) semi-coarsening within flux surfaces to deal with the shear Alfvén mode. As the magnetic field configuration is unknown a priori this requires an adaptive grid, approximately tied to the field. Development of adaptive grid methods for 3-D MHD is also important for reasons of numerical accuracy, but no satisfactory algorithm exists at present. Nevertheless, multigrid in conjunction with adaptive grid methods seems the most promising area of investigation towards efficient 3-D MHD equilibrium computations.

For time dependent three-dimensional calculations the scheme derived above may be more promising, as it would allow to follow accurately the evolution on the longest ideal MHD timescale, while eliminating efficiently the faster modes.

CONCLUSIONS

One objective in writing this paper has been to point out to both plasma physicists and multigrid experts that certain problems in computational MHD are of shared interest.

The axisymmetric equilibrium problem lends itself to a straightforward application of the multigrid procedure, and this has resulted in a code that is ~ 3 times faster than a code which uses a well optimized Buneman solver and Picard iteration. The main interest in very fast 2-D equilibrium calculations is for real-time data interpretation and control of an experiment, on a timescale of ~ 10 msec or less. Considering that in monitoring an experiment one is solving a chain of similar problems, and that a grid of modest size will suffice, our study has demonstrated at least the near-term feasibility of this application.

The problem of computing axisymmetric equilibrium in the inverse coordinates formulation is a more challenging (although hardly speculative) application of multigrid, for which furthermore the relative gain over competing methods would be much larger, as rapid direct solvers are not available. Previous formulations of the inverse equilibrium problem are not well suited for multigrid treatment, but the analogy with grid generation through elliptic equations shows the correct approach. In particular, any code for elliptic grid generation that can handle a polar geometry should almost immediately be applicable to the inverse coordinates MHD equilibrium problem.

The really difficult and expensive areas of work in computational MHD are the stability eigenvalue problem for axisymmetric equilibria (which has not been addressed in this paper), and the three-dimensional equilibrium and evolution problems. An impression of the complexity of the 2-D stability problem can be gained by noticing that it has required nearly a decade of work and the advent of the Cray-1 computer before the main result from the existing stability codes was obtained, viz. the Troyon scaling law, [**19**]. For three-dimensional equilibrium and evolution problems a multigrid approach has been initiated here, but a fully satisfactory procedure has not yet been obtained. The main outstanding problem for these 3-D computations is to develop adaptive methods, in which the grid is adjusted to the (unknown) magnetic configuration.

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