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B.J. Braams

Abstract

The analytical theory and the computational methods that are available for the determination of MHD equilibrium characteristics from magnetic measurements on axisymmetric systems are reviewed and developed. The interpretation of these measurements relies to a large extent on two classes of integral relations due to L.E. Zakharov and V.D. Shafranov. Following and extending their work we provide an inventory of useful integral relations, including the contributions due to pressure anisotropy and plasma rotation. Effective methods to evaluate the required integrals from imperfect measurements are considered. A full equilibrium analysis of the magnetic diagnostics implies a determination of the current profile, consistent with the equations of MHD equilibrium, aiming at an optimal fit between the corresponding calculated magnetic field and the measured data. Published approaches to this problem are evaluated, and a novel fast algorithm is proposed. Instead of the full equilibrium analysis several more limited problems are often considered, for which faster methods are available. The determination of only the plasma boundary requires the solution of a Cauchy problem, or similar, for an elliptic equation. The published approaches are compared. Analytical theory provides approximations that are suitable for the rapid estimation of characteristic parameters, related to the plasma current, position, shape of the cross-section, pressure and internal inductance. Very efficient algorithms may be obtained when this theory is employed in conjunction with the method of function parametrization. These algorithms are well suited to real-time control of the plasma.

Appendices present a discussion of the boundary conditions for the MHD equilibrium problem, a compendium of analytical solutions to the homogeneous equilibrium equation, and a re-examination of the possibility of determining the current distribution from knowledge of only the shape of the flux surfaces.

Contents

INTRODUCTION

The accurate and rapid determination of the magnetohydrodynamic (MHD) equilibrium configuration is of much importance for magnetic confinement experiments, be it for the purpose of feedback control of the plasma during machine operation, for the immediate on-line data analysis that takes place between successive discharges, or for the more extensive off-line analysis of an experiment. Measurements of the external magnetic field and flux provide the most basic information on the electromagnetic properties of the confined plasma, and are fundamental to the feedback control and to the further analysis of a discharge. A range of algorithms for the interpretation of these magnetic diagnostics is required, with different priorities as regards the speed vs. the accuracy or the scope of the computations. On the fastest timescale relevant to active control of a discharge (typically one to several milliseconds for tokamak operation, depending upon the skin-time of the vacuum vessel) one requires at least an estimate of the plasma position. Somewhat slower at present are the algorithms that determine accurately the location of the plasma boundary and compute estimates for such characteristic equilibrium parameters as the poloidal β and the internal inductance. The complete determination of the equilibrium configuration and its time evolution is not yet a matter of routine. For all these tasks it is fair to say that significant advances in the sophistication and/or the speed of the analysis would be most welcome.

The subject matter of this paper is the analytical theory and the computational methods that are available for the determination of MHD equilibrium characteristics from magnetic measurements on axisymmetric systems, in particular on tokamaks. The paper contains both a critical review of existing practices, and an exposition of some innovations in the analysis of magnetic measurements that have recently been developed by the author. In particular, it is shown that the accurate estimation of a wide set of characteristic equilibrium parameters can easily be done in the 1 millisecond timescale relevant to active control, and methods are proposed that will allow even a full 2-D MHD equilibrium analysis of the plasma to be performed in only a few tens of milliseconds on present computing equipment. The paper should be of interest not only to those plasma physicists who are directly involved with magnetic diagnostics on a tokamak, but also to those who are involved with the interpretation of other basic plasma diagnostics or with machine control, and to workers in computational MHD.

The paper is divided into two main parts. Sections 1–3 are of an analytical nature, and provide the fundamental equations that are required for the interpretation of magnetic measurements in the context of MHD equilibrium theory. Sections 4–7 are oriented towards computation, and discuss the various numerical approaches to the analysis of these diagnostics. The remainder of this introduction provides a more detailed outline of the paper. Specific references to the literature can be be found in the appropriate Sections, and are therefore omitted here.

In Section 1 the equations and boundary conditions that govern axisymmetric confinement are summarized, both for the case of static, ideal MHD, and in the presence of pressure anisotropy and plasma rotation. Some useful properties of the equations are listed. The discussion of the boundary conditions is to some extent original.

Section 2 is concerned with one of the two classes of integral relations that were first discussed by L.E. Zakharov and V.D. Shafranov. This class of integral relations relies on Maxwell's equations only, and relates measurements of the magnetic field and flux made outside the plasma to moments of the current distribution in the interior. These moments involve solutions of the homogeneous equilibrium equation. Several families of analytical solutions are exhibited, and the issue of their completeness is discussed.

Section 3 is concerned with the other class of integral relations of Zakharov and Shafranov. This class relies on an equation for MHD equilibrium in addition to Maxwell's equations, and relates the measured field and flux to moments of the energy density in the plasma. Following and extending the work of Cooper and Wootton, these relations are generalized to include the contributions due to pressure anisotropy and plasma rotation.

Section 4 contains an unconventional treatment of a rather elementary problem: the accurate approximation of the integrals of Sections 2 and 3 from imperfect measurements. The relevance of concepts from the numerical treatment of ill-conditioned equations and from statistical analysis is stressed.

Section 5 deals with methods for the difficult inverse problem of obtaining a complete solution to the equation for axisymmetric equilibrium, including a determination of free parameters that describe the current profile. We review the literature, and propose a novel algorithm to interleave the two iterative processes: the optimization of current profile parameters and the solution of the nonlinear equilibrium equation. In combination with a multigrid approach, this algorithm may yield a code that computes the complete equilibrium almost in real-time.

In Section 6 fast specialized methods for a more limited problem are discussed, viz. the determination of the plasma boundary contour and of the field on this contour from the external magnetic measurements. This involves a solution of one of the classical ill-posed problems of mathematical physics: the integration of an elliptic equation from Cauchy boundary data (or a similar problem). This problem, however, is well understood, and can easily be made well-posed in the sense of Tikhonov. The published approaches are compared.

Section 7 then deals with fast methods that seek to determine only a set of global parameters describing the plasma, such as the plasma current, position, shape, pressure and internal inductance. Existing methods for this problem generally require a preliminary identification of the plasma boundary, and then employ analytical approximations that have been derived on the basis of a large aspect ratio expansion and a specific model for the plasma current distribution. Our recent work has shown that Wind's method of function parametrization can provide simple and accurate expressions that are suitable for real-time control of the experiment.

There are three Appendices. Appendix A contains a discussion of the free-field boundary conditions and their reduction to an integral equation over a finite boundary, and also deals with the accurate discretization of all possible boundary conditions.

Appendix B provides several families of analytical solutions to the homogeneous equilibrium equation, appropriate to different coordinate systems. Appendix C re-examines the problem, first posed by Christiansen and Taylor, of the determination of the current profile from knowledge of only the shape of the flux surfaces.

1. Fundamental Relations for Axisymmetric Confinement

This Section serves to define some of the notation that will be employed throughout the paper, and to collect for later reference several important facts about axisymmetric magnetic fields and magnetohydrodynamic (MHD) equilibrium. For more information one is referred to the original literature, notably [1]–[8], and to standard texts and review papers [9]–[17].

Preliminaries. Throughout the paper, use is made of a righthanded cylindrical (r, ϕ, z) coordinate system, and where not noted otherwise, all occurring fields are assumed to be symmetric with respect to rotations about $r = 0$. An axisymmetric toroidal region T serves as the domain for the discussions. T should completely enclose the plasma, and may also contain a vacuum region and material regions. The cross-section of T in the poloidal (half-)plane ($\phi = 0$ and $r > 0$) is denoted as Ω, and ∂T and $\partial\Omega$ are the boundaries of T and Ω. dV is the volume element on T, dS the area element on Ω, dA the area element on ∂T, and ds the line element on $\partial\Omega$. ($dV = 2\pi r\, dS$ and $dA = 2\pi r\, ds$). The positive orientation on $\partial\Omega$ is such that Ω lies to the right. The normal and tangential derivatives on $\partial\Omega$ are denoted as $\partial/\partial n$ and $\partial/\partial s$. Ω is assumed to be bounded, and $\partial\Omega$ must be piecewise smooth. As a matter of convenience it will also be assumed that Ω is bounded away from $r = 0$, but in many cases this restriction can be removed with little effort. Ω need not be simply connected.

The general concern in this work will be with how to derive information on the magnetic field and the plasma in T (or Ω) from knowledge of the magnetic field on ∂T ($\partial\Omega$). In applications, ∂T will be a surface on or near which the magnetic probes are located, most often the inner or outer surface of the vacuum vessel.

In MHD confinement theory the magnetic permeability, μ_m, is usually assumed to be equal to the vacuum permeability, μ_0, throughout the domain of interest. This is appropriate for the plasma and vacuum regions (all plasma currents being written explicitely), but in the context of the interpretation of magnetic measurements one may be forced to consider the presence of other material media as well, such as the vacuum vessel and perhaps passive conductors located inside $\partial\Omega$, and we therefore generally allow a spatially varying permeability. Only linear magnetic material is considered in Ω; the presence of nonlinear media causes substantial computational difficulties. Furthermore, all nonconducting material has $\mu_m = \mu_0$ to sufficient accuracy for it to be treated here as if it were part of the vacuum region. Accordingly we assume a decomposition of Ω as $\Omega_p + \Omega_v + \Omega_c$; into a plasma region, a vacuum region, and a coil region, and in $\Omega_p + \Omega_v$ we assume $\mu_m = \mu_0$. The exterior region (the complement of Ω in the right half-plane) is denoted as T_e (Ω_e). About this region we assume only axisymmetry; it may carry any axisymmetric distribution of currents, and may also contain nonlinear magnetic material (i.e. iron).

The magnetic field. In the case of axial symmetry the divergence-free magnetic field, $\mathbf{B}$, may be represented as

$$\mathbf{B} = F\nabla\phi + \nabla\psi \times \nabla\phi, \tag{1.1}$$

in terms of two scalar functions, F and ψ. F is related to the toroidal field by $F = rB_t$, and ψ is related to the toroidal component of the vector potential by $\psi = rA_t$. From Ampère's law, $\nabla \times \mathbf{H} = \mathbf{J}$, where $\mathbf{H} = \mu_m^{-1}\mathbf{B}$, a similar representation for the current is obtained,

$$\mathbf{J} = -\mu_m^{-1}\mathcal{L}^*\psi\nabla\phi + \nabla(\mu_m^{-1}F)\times\nabla\phi, \tag{1.2}$$

where the operator $\mathcal{L}^*$, in a coordinate-invariant representation, is the following,

$$\mathcal{L}^*\psi = r^2\mu_m\nabla\cdot(r^{-2}\mu_m^{-1}\nabla\psi). \tag{1.3}$$

We consider $\mathcal{L}^*$ to be defined only for axisymmetric scalar fields. From Eq. (1.2) it follows that ψ satisfies the elliptic equation,

$$\mathcal{L}^*\psi = -\mu_m r j_t, \tag{1.4}$$

where j_t is the toroidal current density.

For the case of uniform permeability, $\mu_m = \mu_0$, $\mathcal{L}^*$ reduces to the operator Δ^*,

$$\begin{aligned}\Delta^*\psi &= r^2\nabla\cdot(r^{-2}\nabla\psi)\\ &= r\frac{\partial}{\partial r}\left(r^{-1}\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial z^2}.\end{aligned} \tag{1.5}$$

We also introduce the operator $\mathcal{L}$,

$$\mathcal{L}\psi = \mu_m^{-1}\nabla\cdot(\mu_m\nabla\psi), \tag{1.6}$$

which reduces to the Laplacian, Δ, for uniform permeability. In a current-free region one may employ the representation $\mathbf{H} = \nabla g$, and $\nabla\cdot\mathbf{B} = 0$ is then equivalent to $\mathcal{L}g = 0$.

Some useful identities. Green's first identity for $\mathcal{L}^*$ is the following:

$$\int_\Omega r^{-1}\mu_m^{-1}\psi\mathcal{L}^*\chi\,dS = \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\psi\frac{\partial\chi}{\partial n}\,ds - \int_\Omega r^{-1}\mu_m^{-1}\nabla\psi\cdot\nabla\chi\,dS, \tag{1.7}$$

and Green's second identity (Green's theorem) is:

$$\int_\Omega r^{-1}\mu_m^{-1}(\psi\mathcal{L}^*\chi - \chi\mathcal{L}^*\psi)\,dS = \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\left(\psi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\psi}{\partial n}\right)ds. \tag{1.8}$$

Here, $\partial/\partial n$ is the outward normal derivative on $\partial\Omega$. These identities are easily obtained by application of the divergence theorem to the appropriate expressions on the torus T. Similar relations, with a factor $r\mu_m$ instead of $r^{-1}\mu_m^{-1}$, hold for $\mathcal{L}$.

We now turn to Green's third identity for the operator $\mathcal{L}^*$. Let the function $G(\mathbf{r},\mathbf{r}')$ satisfy the equation $\mathcal{L}^*G = \mu_m r'\delta(\mathbf{r}-\mathbf{r}')$ in Ω, where G is considered as a function of $\mathbf{r}$ at fixed $\mathbf{r}'$. Boundary conditions on G are not specified, so this function is determined up to an arbitrary solution to the homogeneous equation. Then G is known as a

Green function for the operator $\mathcal{L}^*$, and Green's third identity (Green's representation theorem) holds:

$$\psi(\mathbf{r}') = -\int_{\Omega} G j_t \, dS + \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\left(\psi \frac{\partial G}{\partial n} - G\frac{\partial \psi}{\partial n}\right) ds. \tag{1.9}$$

Particularly useful specific Green's functions are obtained by imposing homogeneous boundary conditions (either Dirichlet, Neumann, or mixed conditions, as appropriate for the problem at hand), which are furthermore independent of $\mathbf{r}'$. Thanks to the factor $\mu_m r'$ in the right hand side of the equation $\mathcal{L}^* G = \mu_m r' \delta(\mathbf{r}-\mathbf{r}')$, these Green functions satisfy the symmetry property, $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$.

If the external region Ω_e also contains only linear magnetic material then Ω may be expanded to fill the right half plane, $\Omega + \Omega_e$, and if the specific Green function G_0 is then chosen to satisfy the free-field boundary conditions, $G_0(\mathbf{r},\mathbf{r}') \to 0$ as $|\mathbf{r}| \to \infty$ and as $r \to 0$, then the following simple representation is obtained from Eq. (1.9),

$$\psi(\mathbf{r}') = -\int_{\Omega+\Omega_e} G_0 j_t \, dS. \tag{1.10}$$

Thus, $G_0(\mathbf{r},\mathbf{r}')$ is the influence function, equal to flux at position $\mathbf{r}'$ due to a (negative) unit current at position $\mathbf{r}$. One is thereby led to define,

$$\begin{aligned} \psi^e(\mathbf{r}') &= -\int_{\Omega} G_0 j_t \, dS, \\ \psi^i(\mathbf{r}') &= \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\left(\psi \frac{\partial G_0}{\partial n} - G_0\frac{\partial \psi}{\partial n}\right) ds, \end{aligned} \tag{1.11}$$

so that $\psi = \psi^e + \psi^i$ according to Eq. (1.9). The function ψ^e is homogeneous on the exterior region Ω_e, and ψ^i is homogeneous in the interior of Ω. ψ^e may be understood as that part of the flux function that is due to the currents in Ω, while ψ^i is associated with currents in the exterior region.

An analytical expression for G_0 is available if $\mu_m = \mu_0$ everywhere:

$$\begin{aligned} G_0(\mathbf{r},\mathbf{r}') &= \frac{\mu_0}{k\pi}\sqrt{rr'}\Big(E(k^2) - (1-k^2/2)K(k^2)\Big) \\ &= \frac{\mu_0 k}{\pi}\sqrt{rr'}\left(\frac{1}{2}R_F(0,1-k^2,1) - \frac{1}{3}R_D(0,1-k^2,1)\right), \end{aligned} \tag{1.12}$$

where

$$k^2 = \frac{4rr'}{(r+r')^2 + (z-z')^2}.$$

$K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of the first and second kind, as in [**18**] and [**19**], whereas R_F and R_C are Carlson's forms of the elliptic integrals [**20**], which are used in [**21**, ch. S21]. The function G_0 defined in (1.12) is called the free space Green function. Even if magnetic material is present one may employ this free space Green function in Eqs. (1.10) and (1.11), provided that all magnetization currents are treated as true currents.

The following relations between Δ^* and Δ are sometimes useful [11]. For any sufficiently differentiable axisymmetric field ξ,

$$\Delta^*\Big(r\frac{\partial\xi}{\partial r}\Big) = r\frac{\partial}{\partial r}\Delta\xi, \tag{1.13}$$

and

$$r^{-1}\frac{\partial}{\partial r}\Delta^*\xi = \Delta\Big(r^{-1}\frac{\partial\xi}{\partial r}\Big). \tag{1.14}$$

Both Δ^* and Δ commute with $\partial/\partial z$.

Ideal MHD equilibrium. In ideal MHD equilibrium theory the magnetic field equations, $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mu_m^{-1}\mathbf{B}=\mathbf{J}$, are supplemented by a single equation of force balance, $\nabla p=\mathbf{J}\times\mathbf{B}$, where p is the kinetic pressure of the plasma. It is furthermore assumed that $\mu_m=\mu_0$. Employing the representation (1.1) there are three unknown scalar fields: ψ, F, and p. Invoking axisymmetry one derives the relations, $\nabla\psi\times\nabla p=\mathbf{0}$ (from $\mathbf{B}\cdot\nabla p=0$ and Eq. (1.1)), $\nabla F\times\nabla p=\mathbf{0}$ (from $\mathbf{J}\cdot\nabla p=0$ and Eq. (1.2)), and, for good measure, $\nabla\psi\times\nabla F=\mathbf{0}$ (from toroidal force balance). It is taken to follow that, locally, ψ, F, and p are functionally related, and one writes $F=F(\psi)$ and $p=p(\psi)$. (It bears saying that these relations need not hold globally in the case when a surface of constant ψ has disconnected parts, as occurs in configurations having a divertor or an internal separatrix.) Consideration of force balance along $\nabla\psi$ then leads to the following expression for the toroidal current in the plasma,

$$j_t = r\frac{dp}{d\psi} + r^{-1}\mu_0^{-1}F\frac{dF}{d\psi}, \tag{1.15}$$

and to what is commonly referred to as the Grad-Shafranov equation [1]–[3] for the flux function ψ,

$$\Delta^*\psi = -\mu_0 r^2\frac{dp}{d\psi} - F\frac{dF}{d\psi}. \tag{1.16}$$

This 'almost linear' elliptic equation is the basis for the study of axisymmetric ideal MHD. Notice that in the context of interpretation of experimental data, the functions $p(\psi)$ and $F(\psi)$ must be regarded as unknown.

The ideal MHD force balance equation can be written in conservation form [9], $\nabla\cdot\boldsymbol{T}=\mathbf{0}$, where the stress tensor $\boldsymbol{T}$ is

$$\boldsymbol{T} = (p + B^2/2\mu_0)\boldsymbol{I} - \mu_0^{-1}\mathbf{B}\mathbf{B}. \tag{1.17}$$

A convenient related form in terms of the flux functions is the following,

$$\nabla p + \frac{1}{2\mu_0}r^{-2}\nabla(F^2 - |\nabla\psi|^2) + \nabla\cdot\Big(\frac{1}{\mu_0}r^{-2}\nabla\psi\nabla\psi\Big) = \mathbf{0}, \tag{1.18}$$

as follows easily from Eq. (1.17).

Pressure anisotropy and plasma rotation. MHD equilibrium in the presence of anisotropy and flow is described in part by the following system of equations:

$$\begin{gathered}\nabla\cdot\mathbf{B}=0,\qquad \nabla\cdot\rho\mathbf{v}=0,\\ \rho\mathbf{v}\cdot\nabla\mathbf{v}=(\nabla\times\mu_0^{-1}\mathbf{B})\times\mathbf{B}-\nabla\cdot\boldsymbol{P},\\ \mathbf{v}\times\mathbf{B}=\nabla\Phi,\end{gathered}\tag{1.19}$$

where ρ is the mass density of the plasma, $\mathbf{v}$ is the flow velocity, $\boldsymbol{P}$ is the pressure tensor, and Φ is the electric potential. The general anisotropic pressure tensor has the form $\boldsymbol{P}=p_\perp\boldsymbol{I}+(p_\parallel-p_\perp)\mathbf{BB}/B^2$, where $p_\perp$ and $p_\parallel$ are the perpendicular and parallel pressures. Two additional thermodynamic equations are required for $p_\perp$ and $p_\parallel$, but the proper choice of these equations is open to dispute. One possibility is provided by the Chew-Goldberger-Low (CGL) equations [**4**], valid in the collisionless limit,

$$\mathbf{v}\cdot\nabla\Big(\frac{p_\perp B^2}{\rho^3}\Big)=0,\qquad \mathbf{v}\cdot\nabla\Big(\frac{p_\parallel}{\rho B}\Big)=0.\tag{1.20^a}$$

Alternatively it is possible to assume isotropy, $p_\perp=p_\parallel\ (=p)$, and then either adiabatic flow,

$$\mathbf{v}\cdot\nabla(p/\rho^\gamma)=0,\tag{1.20^b}$$

where γ is the adiabatic index ($\gamma=5/3$), or constant temperature on flux surfaces,

$$\mathbf{B}\cdot\nabla(p/\rho)=0.\tag{1.20^c}$$

We do not know of a reduction of this complete system of equations to a system consisting of a single elliptic equation and a family of free flux functions, as for the case of static, ideal MHD. (For ideal MHD with flow such a reduction is given in [**22**]–[**25**]). Here we only show the derivation of a simple relation between $\mathbf{v}$ and $\mathbf{B}$, which is independent of anisotropy.

First, $\nabla\cdot\mathbf{B}=0$ and $\nabla\cdot\rho\mathbf{v}=0$ are solved by introducing the representations,

$$\begin{aligned}\mathbf{B}&=F\nabla\phi+\nabla\psi\times\nabla\phi,\\ \rho\mathbf{v}&=G\nabla\phi+\nabla\omega\times\nabla\phi,\end{aligned}\tag{1.21}$$

so that there are eight unknown scalar fields: ψ, F, ω, G, ρ, $p_\perp$, $p_\parallel$, and Φ. From $\mathbf{v}\times\mathbf{B}=\nabla\Phi$, it follows, by taking the inner product with $\mathbf{B}$, $\mathbf{v}$, and $\nabla\phi$, that $\nabla\psi\times\nabla\Phi=\mathbf{0}$, $\nabla\omega\times\nabla\Phi=\mathbf{0}$, and $\nabla\psi\times\nabla\omega=\mathbf{0}$, so that ω and Φ are functions of ψ alone. Then from $(\mathbf{v}\times\mathbf{B})\cdot\nabla\psi=\nabla\Phi\cdot\nabla\psi$ it follows that

$$r^{-2}\rho^{-1}(G-\omega'F)=\Phi',\tag{1.22}$$

which is also a function of ψ alone. Thus, $\rho\mathbf{v}=\omega'\mathbf{B}+r^2\rho\Phi'\nabla\phi$. The mass flow can be decomposed on each individual flux surface into a divergence-free flow along $\mathbf{B}$ and a rigid toroidal rotation.

The conservation form of the force balance equation, $\nabla \cdot \mathcal{T} = \mathbf{0}$, is obtained with

$$\mathcal{T} = (p_\perp + B^2/2\mu_0)\mathcal{I} + \rho\mathbf{v}\mathbf{v} - \sigma\mu_0^{-1}\mathbf{B}\mathbf{B}, \tag{1.23}$$

where $\sigma = 1 + \mu_0(p_\perp - p_\parallel)/B^2$. The related form in terms of the scalar functions is,

$$\begin{aligned} &\frac{1}{2}\nabla(p_\perp + p_\parallel + r^{-2}\rho^{-1}(G^2 + |\nabla\omega|^2)) \\ &\quad + \frac{1}{2}r^{-2}\nabla(\sigma\mu_0^{-1}(F^2 - |\nabla\psi|^2) - \rho^{-1}(G^2 - |\nabla\omega|^2)) \\ &\quad + \nabla \cdot (r^{-2}\sigma\mu_0^{-1}\nabla\psi\nabla\psi - r^{-2}\rho^{-1}\nabla\omega\nabla\omega) = \mathbf{0}. \end{aligned} \tag{1.24}$$

Eqs. (1.23) and (1.24) will be useful in Section 3.

Boundary conditions and auxiliary equations. The MHD equilibrium problem is supplied with both external and internal boundary conditions, and furthermore is usually posed as a parameter estimation problem, thus requiring additional constraint equations. The external boundary conditions may be local conditions, or they may have the form of a boundary integral equation.

The familiar local external boundary conditions prescribe the value of $\alpha\psi + \beta\partial\psi/\partial n$ on $\partial\Omega$, for given functions α and β. Dirichlet and Neumann conditions arise as special cases. Such boundary conditions may be obtained by fitting a curve through a sufficiently dense set of local field and flux measurements made on $\partial\Omega$.

Alternatively the external boundary conditions may prescribe the behaviour of the solution on the axis ($r = 0$) and at infinity. Such conditions are obtained when the field due to currents in external coils, ψ_{ext}, is known. On the finite computational domain Ω, these free-field boundary conditions may be replaced by an integral equation relating ψ and $\partial\psi/\partial n$ on $\partial\Omega$,

$$\frac{\varphi(\mathbf{r}')}{2\pi}\psi(\mathbf{r}') + \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\psi\frac{\partial G}{\partial n}\,ds = \oint_{\partial\Omega} r^{-1}\mu_m^{-1}G\frac{\partial\psi}{\partial n}\,ds + \mu_m^{-1}\psi_{ext}(\mathbf{r}'), \tag{1.25}$$

for $\mathbf{r}' \in \partial\Omega$. Here, $\varphi(\mathbf{r}')$ is the exterior angle subtended by $\partial\Omega$ at the point $\mathbf{r}' \in \partial\Omega$, and $G(\mathbf{r},\mathbf{r}')$ is the Green function for the problem, defined by the equation $\mathcal{L}^*G = \mu_m r'\delta(\mathbf{r} - \mathbf{r}')$, subject to the boundary condition $G \to 0$ as $r \to 0$ or as $|\mathbf{r}| \to \infty$. Eq. (1.25) is a novel formulation of the free-field boundary conditions, although it is closely related to the formulation due to Von Hagenow and Lackner [**26**]. It is derived and discussed in Appendix A.

Nonlocal boundary conditions involving integrals of ψ and $\partial\psi/\partial n$ on $\partial\Omega$ can also arise from magnetic measurements using extended probes, or from measurements made not exactly on $\partial\Omega$.

The internal boundary conditions (interface conditions) require the continuity of the normal component of $\mathbf{B}$ and of the tangential component of $\mathbf{H}$ on all interfaces (in the absence of skin currents). Furthermore, in the free boundary problem, the plasma boundary contour $\partial\Omega_p$ is unknown a priori, and must be determined as part of the solution. The proper characterization of $\partial\Omega_p$ may vary somewhat from experiment to experiment, but a generally valid criterion is that Ω_p shall be the largest simply connected region that is bounded by an isocontour of ψ and that is wholly contained within a given limiting contour L. The plasma has a limiter geometry if $\partial\Omega_p$ and L have a point in common, otherwise it has a divertor geometry (and then $\partial\Omega_p$ passes through a saddle point of ψ). In the vacuum region the pressure must vanish and the toroidal magnetic field must have the form $B_{t0} = r^{-1}F_0$, for some constant F_0. If no singular current density is allowed on $\partial\Omega_p$ then p and F must satisfy the interface conditions $p = 0$ and $F = F_0$ on $\partial\Omega_p$.

Additional constraints are required when the plasma current profile (or the contribution of other currents) is given in parametric form, or is treated as functionally unknown. In particular, a formulation is common in which the plasma current profile is given only up to an undetermined constant factor, and in which the value of the total current provides the necessary additional constraint. The value, ψ_b, of the flux function on $\partial\Omega_p$ is also in almost all cases an unknown parameter in the current profile, to be determined as part of the solution. In the context of the interpretation of diagnostics there should further be some freedom in the shape of the current profile, and typically one to three additional free parameters are employed. A problem statement involving a functionally unknown current profile could be appropriate when a sufficiently sensitive set of diagnostics (more than just the external magnetic measurements) is to be interpreted. In a different context, the problem with a functionally unknown current profile arises when the q-profile is specified instead.

The various problem statements that are of interest to us may finally be classified as follows.

- $P0$: Fixed plasma boundary. The plasma fills the complete computational domain ($\Omega_p = \Omega$), and the boundary condition stipulates $\psi = \psi_b$ on $\partial\Omega$. (ψ_b constant).
- $P1$: Free plasma boundary, local boundary conditions. $\Omega_p \subset \Omega$, and boundary conditions that prescribe $\alpha\psi + \beta\partial\psi/\partial n$ on $\partial\Omega$. Interface conditions on the free contour $\partial\Omega_p$.
- $P2$: Free plasma boundary, nonlocal boundary conditions. $\Omega_p \subset \Omega$, and typically an integral equation boundary condition on $\partial\Omega$. Interface conditions on the free contour $\partial\Omega_p$.

In each case the problem may require also the estimation of one or more parameters. In addition the problem classes $P0'$, $P1'$ and $P2'$ are considered, in which the profile rj_t is regarded as functionally unknown.

2. Moments of the Toroidal Current Density

This Section is concerned with a class of integral relations that express moments of the toroidal current density in Ω as integrals of linear combinations of the poloidal magnetic field components on $\partial\Omega$. In the context of magnetic confinement theory these integral relations were first given by Zakharov and Shafranov [**27**], but they appear in potential theory as an immediate corollary of Green's theorem. The moments involve solutions to the homogeneous equilibrium equation, and various families of solutions are provided here and in Appendix B.

An integral relation. Let χ be an arbitrary function that satisfies the homogeneous equilibrium equation, $\mathcal{L}^*\chi = 0$ in Ω, and let ψ be the poloidal flux function, which satisfies $\mathcal{L}^*\psi = -\mu_m r j_t$. Then by application of Green's second identity for the operator $\mathcal{L}^*$, Eq. (1.8), to the pair (χ, ψ), one obtains the fundamental integral relation for the evaluation of moments of the toroidal current density,

$$\int_\Omega \chi j_t \, dS = \oint_{\partial\Omega} r^{-1}\mu_m^{-1} (\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n}) \, ds. \tag{2.1}$$

Notice that $\mathcal{L}^*\chi = 0$ implies $\oint r^{-1}\mu_m^{-1}(\partial\chi/\partial n)\, ds = 0$, so in Eq. (2.1) there is no dependence upon the choice of the arbitrary additive constant in the potential ψ. This can be made manifest by introducing together with χ also a conjugate function, ξ, according to the equation,

$$\nabla(r^{-1}\mu_m^{-1}\xi) = \mu_m^{-1}\nabla\chi \times \nabla\phi, \tag{2.2}$$

which indeed admits a solution subject to $\mathcal{L}^*\chi = 0$. This definition implies the identity $r^{-1}\mu_m^{-1}\partial\chi/\partial n = -\partial(r^{-1}\mu_m^{-1}\xi)/\partial s$, where $\partial/\partial s$ is the derivative along $\partial\Omega$ in the positive direction (clockwise on the outer boundary). By partial integration one may then eliminate ψ from Eq. (2.1) in favor of $\partial\psi/\partial s$ to obtain

$$\begin{aligned}\int_\Omega \chi j_t \, dS &= \oint_{\partial\Omega} r^{-1}\mu_m^{-1} \left(\xi \frac{\partial \psi}{\partial s} - \chi \frac{\partial \psi}{\partial n} \right) ds \\ &= \oint_{\partial\Omega} \mu_m^{-1}(\xi B_n + \chi B_s)\, ds \\ &= \oint_{\partial\Omega} (\xi H_n + \chi H_s)\, ds\end{aligned} \tag{2.3}$$

as an alternative form for Eq. (2.1).

In the usual cylindrical coordinates the relation between χ and ξ, Eq. (2.2), may be expressed as

$$\begin{cases} \dfrac{\partial}{\partial r}(r^{-1}\mu_m^{-1}\xi) = -r^{-1}\mu_m^{-1}\dfrac{\partial\chi}{\partial z}, \\ \dfrac{\partial}{\partial z}(r^{-1}\mu_m^{-1}\xi) = r^{-1}\mu_m^{-1}\dfrac{\partial\chi}{\partial r}. \end{cases} \tag{2.4}$$

The function ξ satisfies the equation $\mathcal{L}(r^{-1}\mu_m^{-1}\xi) = 0$, where $\mathcal{L}$ has been introduced in Eq. (1.6). Notice also that $\mathcal{L}(r^{-1}\mu_m^{-1}\chi) = \chi\mathcal{L}(r^{-1}\mu_m^{-1})$.

In [**27**] the following different derivation of the above integral relation was given. Let the fields $\mathbf{q}$ and g satisfy $\mu_m^{-1}\nabla\times\mathbf{q} = \nabla(\mu_m^{-1}g)$. Then,

$$\begin{aligned}\int_T \mathbf{q}\cdot\mathbf{J}\,dV &= \int_T \mathbf{q}\cdot(\nabla\times\mu_m^{-1}\mathbf{B})\,dV\\ &= \int_T (\nabla\cdot(\mu_m^{-1}\mathbf{B}\times\mathbf{q}) + \mu_m^{-1}\mathbf{B}\cdot(\nabla\times\mathbf{q}))\,dV\\ &= \int_T (\nabla\cdot(\mu_m^{-1}\mathbf{B}\times\mathbf{q}) + \mathbf{B}\cdot\nabla(\mu_m^{-1}g))\,dV\\ &= \oint_{\partial T} ((\mu_m^{-1}\mathbf{B}\times\mathbf{q})\cdot\mathbf{n} + g\mu_m^{-1}\mathbf{B}\cdot\mathbf{n})\,dA.\end{aligned}$$

These identities do not require the assumption of axisymmetry. The previous result for the axisymmetric case is obtained when one sets $\mathbf{q} = \chi\nabla\phi$ and $g = r^{-1}\xi$. In contrast to [**27**], the conjugate pair of functions (χ, ξ) has been defined in the present work in such a way that they have the same physical dimension. Our function χ corresponds to f of [**27**], and our ξ is rg in their notation.

Plasma current and position. Specific analytical instances of these integral relations can only be given for the case of constant permeability. Let us therefore temporarily assume that Ω contains only the plasma and vacuum regions, $\Omega = \Omega_p + \Omega_v$, so that $\mu_m = \mu_0$ and $\mathcal{L}^* = \Delta^*$. Four simple independent pairs of conjugate solutions (χ, ξ) to $\Delta^*\chi = 0$ and $\Delta(r^{-1}\xi) = 0$ are: $(\chi = 1, \xi = 0)$, $(\chi = 0, \xi = r)$, $(\chi = z, \xi = -r\log r)$, and $(\chi = r^2, \xi = 2rz)$. These moments lead to the following specific integral relations,

$$\int_\Omega j_t\,dS = \oint_{\partial\Omega} \mu_0^{-1}B_s\,ds, \tag{2.5}$$

$$0 = \oint_{\partial\Omega} \mu_0^{-1}rB_n\,ds, \tag{2.6}$$

$$\int_\Omega zj_t\,dS = \oint_{\partial\Omega} \mu_0^{-1}(-r\log r\,B_n + zB_s)\,ds, \tag{2.7}$$

$$\int_\Omega r^2j_t\,dS = \oint_{\partial\Omega} \mu_0^{-1}(2rzB_n + r^2B_s)\,ds. \tag{2.8}$$

The first two will be recognized as the integral forms of $\nabla\times\mathbf{H} = \mathbf{J}$ and $\nabla\cdot\mathbf{B} = 0$, but the higher moments would not have been evident a priori. These equations suggest a characterization of the plasma position, (r_c, z_c), according to the identities,

$$I_t = \int_\Omega j_t\,dS \tag{2.9}$$

$$z_cI_t = \int_\Omega zj_t\,dS \tag{2.10}$$

$$r_c^2I_t = \int_\Omega r^2j_t\,dS \tag{2.11}$$

The current center (r_c, z_c) defined here is an intrinsic plasma property (not dependent upon the position of the contour $\partial\Omega$ as long as it completely encloses the plasma and encloses no other currents), which furthermore can be rigorously evaluated from external magnetic measurements. Alternative characterizations of the plasma position are: a geometric center of plasma cross-section, or the position of the magnetic axis. These characterizations are more difficult to obtain from external measurements, and the plasma cross-section may not be a rigorously defined concept.

It is to be noted that Eqs. (2.10) and (2.11) are perfectly valid as *definition* of z_c and r_c, but that the corresponding relations (2.7) and (2.8) are not immediately suitable for the computation of these quantities. In particular the uncritical use of Eq. (2.8) to compute r_c is not to be recommended. Instead, after having obtained preliminary estimates r_0 and z_0 for the plasma position (e.g. as the center of the vacuum vessel), r_c and z_c should be computed from the relations,

$$(z_c - z_0)I_t = \oint_{\partial\Omega} \mu_0^{-1}\Big(-r\log\frac{r}{r_0}B_n + (z - z_0)B_s\Big)\, ds \tag{2.12}$$

$$(r_c^2 - r_0^2)I_t = \oint_{\partial\Omega} \mu_0^{-1}\Big(2r(z - z_0)B_n + (r^2 - r_0^2)B_s\Big)\, ds \tag{2.13}$$

This computation may be iterated in order to obtain (r_c, z_c) as that pair (r_0, z_0) that causes the vanishing of some reasonable discrete approximation (in terms of the magnetic measurements) to the right hand sides of (2.12) and (2.13).

Higher moments. Specification of ψ and $\partial\psi/\partial n$ on $\partial\Omega$ is equivalent to the specification of ψ on $\partial\Omega$ together with the moments of j_t with respect to a family of solutions $\{\chi_i\}_i$ to $\mathcal{L}^*\chi = 0$ that is complete on Ω: $q_i = \int_\Omega \chi_i j_t\, dS$. The second specification is very useful for the interpretation of magnetic measurements, as will be seen in Sections 5–7, and there is therefore an interest in (complete) families of higher moments of the toroidal current density.

Analytical families of conjugate pairs of solutions to $\Delta^*\chi = 0$ and $\Delta(r^{-1}\xi) = 0$ can be generated in a number of ways. Zakharov and Shafranov [**27**, Eq. (61)] provide the first few even terms in a sequence of homogeneous polynomial solutions (but beware of two errors in that equation). In our notation, and extended to all orders, these polynomial solutions are the family defined by,

$$\left\{\begin{aligned} \chi_n &= \sum_{k=0}^{\lfloor n/2\rfloor - 1} (-4)^{-k}\, \frac{(n-1)!\,/2}{k!\,(k+1)!\,(n-2k-2)!}\, r^{2k+2} z^{n-2k-2}, \\ \xi_n &= \sum_{k=0}^{\lceil n/2\rceil - 1} (-4)^{-k}\, \frac{(n-1)!}{(k!)^2\,(n-2k-1)!}\, r^{2k+1} z^{n-2k-1}. \end{aligned}\right. \tag{2.14}$$

for $n > 0$, together with the pair $(\chi_0 = 1, \xi_0 = 0)$. This family is not complete on any region that is of interest for tokamak studies. Polynomial solutions to the homogeneous equilibrium equation have also been discussed in Refs. [**28**]–[**30**].

Elementary solutions to the homogeneous equilibrium equation can also be found by allowing a factor $\ln r$ or a power of $\sqrt{r^2 + z^2}$. Further analytical solutions may be obtained through separation of variables, in either cylindrical, spherical, or toroidal coordinates. All these forms are provided in Appendix B.

Finally, a family of solutions to $\mathcal{L}^* \chi = 0$ may be generated by numerical solution of the elliptic equation for some family of boundary conditions for $\partial\chi/\partial n$ on $\partial\Omega$. This latter route is the only one available when the permeability in Ω is not constant, and is also the most suitable procedure for general shapes of the region Ω.

3. Moments Involving a Generalized Pressure

This Section is concerned with a class of integral relations that allow to express certain area (resp. volume) integrals of an energy density in Ω (or T) in terms of contour (surface) integrals of quadratic combinations of the magnetic field components on $\partial\Omega$ (∂T). In contrast to the relations of Section 2, which were derived using only the electromagnetic equations, $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{H}=\mathbf{J}$, the following relations are based also on an equation for the plasma equilibrium. In first instance the force balance equation for static, ideal MHD equilibrium will be assumed, viz. $\nabla p=\mathbf{J}\times\mathbf{B}$. The resulting class of integral relations was first given in general form by Zakharov and Shafranov [**27**], although two important special cases had been given earlier [**31**]. In second instance we will consider the modifications due to pressure anisotropy and plasma rotation, thereby extending the work of Cooper and Wootton [**32**], who considered only the two cases of [**31**].

Throughout this Section it will be assumed that Ω contains only a plasma and a vacuum region, $\Omega=\Omega_p+\Omega_v$, as clearly an MHD force balance equation should not be assumed to hold in the coil region, Ω_c. In fact, application of the following integral relations is usually preceded by an identification of the plasma boundary and of the magnetic field on this boundary, using methods that are discussed in Section 6, in which case one may identify $\Omega=\Omega_p$. By a generalized pressure we understand any local expression in terms of p and $\mathbf{B}$ (and $p_\parallel$, $p_\perp$, ρ and $\mathbf{v}$ in the nonideal case) that has the physical dimension of a pressure.

An integral relation for static, ideal MHD equilibrium. Consider the equilibrium equation in the form of Eq. (1.18). Taking the scalar product with an arbitrary axisymmetric poloidal vector field $\mathbf{Q}$ gives,

$$0=\mathbf{Q}\cdot\left[\nabla p+\frac{1}{2\mu_0}r^{-2}\nabla(F^2-F_0^2-|\nabla\psi|^2)+\nabla\cdot\left(\frac{1}{\mu_0}r^{-2}\nabla\psi\nabla\psi\right)\right].$$

The constant F_0 is the value of F on the plasma boundary and in the vacuum region (related to the vacuum toroidal field by $F_0=rB_{t0}$), which has been inserted here for convenience at a later stage. In order to arrive at a meaningful integral relation we rewrite this identity in a form that contains a total divergence,

$$\begin{aligned}&p\nabla\cdot\mathbf{Q}+\frac{1}{2\mu_0}(F^2-F_0^2)\nabla\cdot(r^{-2}\mathbf{Q})\\&\quad+\frac{1}{\mu_0}\nabla\psi\cdot\left(r^{-2}\nabla\mathbf{Q}-\frac{1}{2}\nabla\cdot(r^{-2}\mathbf{Q})\boldsymbol{I}\right)\cdot\nabla\psi\\&\quad=\nabla\cdot\left[p\mathbf{Q}+\frac{1}{2\mu_0}r^{-2}(F^2-F_0^2-|\nabla\psi|^2)\mathbf{Q}+\frac{1}{\mu_0}r^{-2}\nabla\psi\nabla\psi\cdot\mathbf{Q}\right].\end{aligned}$$

This identity is next integrated over the volume of the torus, the divergence term being expressed as a surface integral, and this surface integral is simplified by employing that

$p = 0$ and $F^2 - F_0^2 = 0$ on ∂T. We choose to re-express the ensueing integrals over T and ∂T as integrals over Ω and $\partial\Omega$, and arrive at the following identity,

$$\begin{aligned}\int_\Omega r \Big[p\nabla\cdot\mathbf{Q} + \frac{1}{2\mu_0}(F^2 - F_0^2)\nabla\cdot(r^{-2}\mathbf{Q}) \\ + \frac{1}{\mu_0}\nabla\psi\cdot\Big(r^{-2}\nabla\mathbf{Q} - \frac{1}{2}\nabla\cdot(r^{-2}\mathbf{Q})\boldsymbol{I}\Big)\cdot\nabla\psi\Big]\, dS \\ = \frac{1}{\mu_0}\oint_{\partial\Omega} r^{-1}\Big[(\mathbf{Q}\cdot\nabla\psi)(\nabla\psi\cdot\mathbf{n}) - \frac{1}{2}|\nabla\psi|^2(\mathbf{Q}\cdot\mathbf{n})\Big]\, ds.\end{aligned} \tag{3.1}$$

This is the desired integral relation. Specific choices of the vector field $\mathbf{Q}$ thus allow to express certain volume integrals of a generalized pressure in terms of surface integrals of quadratic combinations of the magnetic field components on $\partial\Omega$.

The integral in the left hand side of Eq. (3.1) is unfortunately not an invariant quantity, but depends on the choice of the region T (or Ω), because although p and $F^2 - F_0^2$ vanish outside the plasma, $\nabla\psi$ does not. In many applications of these relations $\partial\Omega$ is identified with the plasma boundary in order to obtain intrinsic plasma properties, but in other cases $\partial\Omega$ is taken as the measuring contour on the vacuum vessel.

Notice also the following closely related form,

$$\begin{aligned}\int_\Omega r \Big[p\nabla\cdot\mathbf{Q} + \frac{1}{2\mu_0}(F^2 - F_0^2)\nabla\cdot(r^{-2}\mathbf{Q})\Big]\, dS \\ = -\int_\Omega (\mathbf{Q}\cdot\nabla\psi) j_t\, dS,\end{aligned} \tag{3.2}$$

in which all integrals are invariant quantities, but in which the data on $\partial\Omega$ do not enter.

An alternative derivation. A related but different derivation suggested by the work of Cooper and Wootton [**32**] is also worth noting. Let $\mathbf{Q}$ be an arbitrary vector field. Starting with the force balance equation in conservation form, $\nabla\cdot\boldsymbol{T} = \mathbf{0}$, the following sequence of identities is derived:

$$\begin{aligned}0 &= \int_T \mathbf{Q}\cdot(\nabla\cdot\boldsymbol{T})\, dV \\ &= \int_T (\nabla\cdot(\boldsymbol{T}\cdot\mathbf{Q}) - \boldsymbol{T}:\nabla\mathbf{Q})\, dV \\ &= \oint_{\partial T} \mathbf{n}\cdot\boldsymbol{T}\cdot\mathbf{Q}\, dA - \int_T \boldsymbol{T}:\nabla\mathbf{Q}\, dV\end{aligned}$$

Inserting now $\boldsymbol{T}$ from Eq. (1.17), and using the fact that $p = 0$ on ∂T, it follows that

$$\begin{aligned}\int_T \Big[(p + \frac{1}{2\mu_0}B^2)\nabla\cdot\mathbf{Q} - \frac{1}{\mu_0}\mathbf{B}\cdot\nabla\mathbf{Q}\cdot\mathbf{B}\Big]\, dV \\ = \frac{1}{\mu_0}\oint_{\partial T}\Big[\frac{1}{2}B^2(\mathbf{Q}\cdot\mathbf{n}) - (\mathbf{B}\cdot\mathbf{Q})(\mathbf{B}\cdot\mathbf{n})\Big]\, dA\end{aligned} \tag{3.3}$$

This relation is valid independent of axisymmetry. In the axisymmetric case, the form Eq. (3.3) is equivalent to the earlier form, Eq. (3.1), as will be shown. First, without loss of generality $\mathbf{Q}$ may be required to be an axisymmetric vector field. Second, there is no reason to include a toroidal component in $\mathbf{Q}$. For suppose $\mathbf{Q} = \chi\nabla\phi$; then one obtains the integral relation,

$$\int_T rB_t\mathbf{B}\cdot\nabla(r^{-2}\chi)\,dV = \oint_{\partial T} r^{-1}B_tB_n\chi\,dS,$$

which is anyway trivial from $\mathbf{B}\cdot\nabla(rB_t) = 0$. Indeed, toroidal force balance was used earlier to prove that rB_t is constant on flux surfaces. Next, restricting $\mathbf{Q}$ to be an axisymmetric vector field without toroidal component, one may separate in Eq. (3.3) the contributions of the toroidal and the poloidal fields, and subtract from both sides the contribution due to the vacuum toroidal field. This vacuum field is $\mathbf{B}_{t0} = F_0\nabla\phi$, with F_0 constant, and one uses the identities

$$\frac{1}{2\mu_0}B_{t0}^2\nabla\cdot\mathbf{Q} - \frac{1}{\mu_0}\mathbf{B}_{t0}\cdot\nabla\mathbf{Q}\cdot\mathbf{B}_{t0} = \frac{1}{2\mu_0}B_{t0}^2r^2\nabla\cdot(r^{-2}\mathbf{Q})$$

and $\mathbf{B}_t\cdot\nabla\mathbf{Q}\cdot\mathbf{B}_p = 0$, and $\mathbf{B}_p\cdot\nabla\mathbf{Q}\cdot\mathbf{B}_t = 0$. The result is the following form of the integral relation:

$$\begin{aligned}\int_T\Big[p\nabla\cdot\mathbf{Q} &+ \frac{1}{2\mu_0}(B_t^2 - B_{t0}^2)r^2\nabla\cdot(r^{-2}\mathbf{Q}) \\ &- \frac{1}{\mu_0}\mathbf{B}_p\cdot\Big(\nabla\mathbf{Q} - \frac{1}{2}(\nabla\cdot\mathbf{Q})\mathbf{I}\Big)\cdot\mathbf{B}_p\Big]\,dV \\ &= \frac{1}{\mu_0}\oint_{\partial T}\Big[\frac{1}{2}B_p^2\,\mathbf{Q}\cdot\mathbf{n} - (\mathbf{Q}\cdot\mathbf{B}_p)(\mathbf{B}_p\cdot\mathbf{n})\Big]\,dA\end{aligned} \tag{3.4}$$

To simplify the right hand side it has been used that $B_t^2 - B_{t0}^2 = 0$ on ∂T. The final manipulations needed to demonstrate that this relation is equivalent to the earlier form, Eq. (3.1), may be left to the reader. We will henceforth work mainly with the last form, Eq. (3.4).

Pressure anisotropy and plasma rotation. It is a straightforward matter to repeat the derivation that lead to Eqs. (3.1) and (3.4), but starting from Eqs. (1.24) or (1.23). This leads to the identities,

$$\begin{aligned}\int_\Omega r\Big[\frac{1}{2}(p_\parallel &+ p_\perp + r^{-2}\rho^{-1}(|\nabla\chi|^2 + G^2))\,\nabla\cdot\mathbf{Q} \\ &+ \Big(\frac{\sigma F^2 - F_0^2}{2\mu_0} - \frac{1}{2}\rho^{-1}G^2\Big)\nabla\cdot(r^{-2}\mathbf{Q}) \\ &- \Big(\frac{\sigma}{\mu_0}\nabla\psi\nabla\psi - \rho^{-1}\nabla\chi\nabla\chi\Big) : \Big(r^{-2}\nabla\mathbf{Q} - \frac{1}{2}(\nabla\cdot(r^{-2}\mathbf{Q}))\mathbf{I}\Big)\Big]\,dS \\ &= \frac{1}{\mu_0}\oint_{\partial\Omega} r^{-1}\Big[(\mathbf{Q}\cdot\nabla\psi)(\nabla\psi\cdot\mathbf{n}) - \frac{1}{2}|\nabla\psi|^2(\mathbf{Q}\cdot\mathbf{n})\Big]\,ds,\end{aligned} \tag{3.5}$$

and equivalently,

$$\int_T \Big[\frac{1}{2}(p_\parallel + p_\perp + \rho v^2)\nabla\cdot\mathbf{Q} + \Big(\frac{\sigma B_t^2 - B_{t0}^2}{2\mu_0} - \frac{1}{2}\rho v_t^2\Big) r^2 \nabla\cdot(r^{-2}\mathbf{Q})$$
$$- \Big(\frac{\sigma}{\mu_0}\mathbf{B}_p\mathbf{B}_p - \rho\mathbf{v}_p\mathbf{v}_p\Big) : \Big(\nabla\mathbf{Q} - \frac{1}{2}(\nabla\cdot\mathbf{Q})\boldsymbol{I}\Big)\Big]\, dV \qquad (3.6)$$
$$= \frac{1}{\mu_0}\oint_{\partial T}\Big[\frac{1}{2}B_p^2\,\mathbf{Q}\cdot\mathbf{n} - (\mathbf{B}_p\cdot\mathbf{Q})\,(\mathbf{B}_p\cdot\mathbf{n})\Big]\, dA.$$

This equation is related to the static, ideal MHD relation through the substitutions,

$$\begin{aligned} p &\rightarrow \frac{1}{2}(p_\parallel + p_\perp + \rho v^2), \\ B_t^2 - B_{t0}^2 &\rightarrow \sigma B_t^2 - B_{t0}^2 - \mu_0\rho v_t^2, \\ \mathbf{B}_p\mathbf{B}_p &\rightarrow \sigma\mathbf{B}_p\mathbf{B}_p - \mu_0\rho\mathbf{v}_p\mathbf{v}_p, \end{aligned} \qquad (3.7)$$

and this indicates how pressure anisotropy and plasma rotation will affect the determination from magnetic measurements of the plasma parameters β_I, μ_I, and l_i.

Definition of the parameters β_I, μ_I, and l_i. A variety of definitions for these characteristic plasma parameters exists, and there would be good reason to avoid all of them and work directly with expressions for the energy content in the plasma: volume integrals of p, of $(B_t^2 - B_{t0}^2)/2\mu_0$, and of $B_p^2/2\mu_0$ (dotless multiplication taking precedence over division). We therefore define,

$$W_T = \int_{T_p} p\, dV, \quad W_M = \int_{T_p} \frac{B_t^2 - B_{t0}^2}{2\mu_0}\, dV, \quad W_L = \int_{T_p} \frac{B_p^2}{2\mu_0}\, dV, \qquad (3.8)$$

where T_p is the plasma volume. As p and $B_t^2 - B_{t0}^2$ vanish outside the plasma, the quantities W_T and W_M are completely unambiguous. W_L is dependent on which ψ contour is identified as the plasma boundary, and is therefore not such a good intrinsic plasma property. In order to obtain the most suitable and unambiguous dimensionless characterizations of the energy content of the plasma we propose the following definitions:

$$\begin{aligned} \beta_I &= \frac{4}{\mu_0 r_c I_t^2}\int_{T_p} p\, dV, \\ \mu_I &= -\frac{4}{\mu_0 r_c I_t^2}\int_{T_p} \frac{B_t^2 - B_{t0}^2}{2\mu_0}\, dV, \\ l_i &= \frac{4}{\mu_0 r_c I_t^2}\int_{T_p} \frac{B_p^2}{2\mu_0}\, dV, \end{aligned} \qquad (3.9)$$

where I_t is the toroidal plasma current and r_c is defined by Eq. (2.11).

The particular scaling factor $\frac{1}{4}\mu_0 r_c I_t^2$ has been chosen because it gives the customary cylindrical limit, is an intrinsic plasma property, and can be rigorously determined from the external measurements. Instead of the present normalizing energy (a) $\frac{1}{4}\mu_0 r_c I_t^2$ one also sees (b) $\frac{1}{4}\mu_0 R_0 I_t^2$, (c) $V\langle B_p^2\rangle/2\mu_0$, (d) $V\bar{B}_p^2/2\mu_0$, (e) $\mu_0 V I_t^2/8\pi S$, or (f) $\mu_0 V I_t^2/2s^2$, where R_0 is the major radius of the confinement vessel, V is the plasma volume, S is the area of the poloidal cross-section of the plasma, s is the cirumference of the poloidal cross-section, and

$$\langle B_p^2\rangle = \oint_{\partial\Omega_p} B_p\, ds \Big/ \oint_{\partial\Omega_p} B_p^{-1}\, ds$$
$$\bar{B}_p^2 = \frac{2}{1+\kappa^2}\left(\frac{\mu_0 I_t}{2\pi a}\right)^2 \tag{3.10}$$

in which a is the plasma minor radius and $\kappa = b/a$ the elongation. Occasionally one also sees definitions in which the integrals of p, $(B_t^2 - B_{t0}^2)/2\mu_0$, and $B_p^2/2\mu_0$ are taken over the poloidal cross-section instead of over the volume, and the normalizing constant is reduced by a factor $2\pi r_c$ or the equivalent.

In our opinion the choice (a) is preferable to any of these alternatives. (b) is not an intrinsic plasma property. (c) is singular for a plasma bounded by a magnetic separatrix, and is therefore suspect in all cases. (d) contains two geometric quantities (a and κ) of which the definition for unsymmetric configurations in particular is ambiguous. (e) and (f) are closest to our definition, and preferable to (b)–(d), but still have to make reference to the plasma boundary, thereby introducing an unnecessary ambiguity in the definitions of β_I and μ_I. Finally, those definitions in which the integrals are taken over the plasma cross-section instead of over the plasma volume lose the rigorous connection with the energy content in the plasma.

Some specific multipole moments. Three important instances from the general class of integral relations given by Eq. (3.4) follow. The integrals s_1, s_2, and s_3 can all be rigorously determined from knowledge of the magnetic field on $\partial\Omega$.

(1) Selecting $\mathbf{Q} = r\mathbf{e}_r + z\mathbf{e}_z$:

$$s_1 = \int_T \left(3p + \frac{B_t^2 - B_{t0}^2}{2\mu_0} + \frac{B_p^2}{2\mu_0}\right) dV \tag{3.11}$$

In the case when T is the plasma volume, s_1 is related to $3\beta_I - \mu_I + l_i$.

(2) Selecting $\mathbf{Q} = \mathbf{e}_r$:

$$s_2 = \int_T r^{-1}\left(p - \frac{B_t^2 - B_{t0}^2}{2\mu_0} + \frac{B_p^2}{2\mu_0}\right) dV \tag{3.12}$$

When T is the plasma volume, s_2 is related to $\beta_I + \mu_I + l_i$.

(3) Selecting $\mathbf{Q} = r\mathbf{e}_r$:

$$s_3 = \int_T \left(2p + \frac{B_z^2}{\mu_0}\right) dV \tag{3.13}$$

When T is the plasma volume, s_3 is related to $2\beta_I + l_i$.

The integrals s_1 and s_2 were given in [**31**]. They can be combined to eliminate one of the three quantities β_I, μ_I, and l_i; in particular, after eliminating μ_I they provide an estimate for $\beta_I + l_i/2$. This calculation does involve a large aspect ratio approximation, because the expression for s_2 has a factor r^{-1} in the integration, and that for s_1 does not. If an independent measurement of μ_I is available (a diamagnetic flux loop), then separate estimates can also be obtained for β_I and l_i. L.L. Lao [**33**] employs s_3 together with s_1 and s_2 in order to obtain a separate identification of β_I and l_i, without the use of a diamagnetic measurement. This approach relies on the volume average $\langle B_z^2/\mu_0\rangle$ being different from $\langle B_p^2/2\mu_0\rangle$. It provides analytical underpinning for the empirical observation (discussed further in Section 5) that full MHD equilibrium calculations allow a separate identification of β_I and l_i for sufficiently large deviations from circularity.

Systematic sets of moments. The class of all axisymmetric poloidal vector fields is still too large to deal with in a systematic manner. In order to see how much further $\mathbf{Q}$ may be restricted without loss of information it helps to consider the case where ∂T coincides with the plasma boundary. (No measurements further out can provide more information on the plasma). In this case the right hand side of Eq. (3.4) reduces to

$$\oint_{\partial T} \frac{1}{2\mu_0} B_p^2\, \mathbf{Q}\cdot\mathbf{n}\, dA$$

and it is seen that $\mathbf{Q}$ may be restricted to any class of axisymmetric, poloidal fields for which $(\mathbf{Q}\cdot\mathbf{n})$ generates a complete set of functions on $\partial\Omega$. Using this freedom, Eq. (3.4) may be simplified in several ways:

(a) Let $\mathbf{Q} = \nabla\chi$, where $\Delta^*\chi = 0$; this eliminates the toroidal field term. A complete basis of solutions to $\Delta^*\chi = 0$ can be obtained in a variety of ways, the most generally useful analytical approach being separation in toroidal coordinates. See Appendix B.

(b) Let $\mathbf{Q} = \nabla\xi$, where $\Delta\xi = 0$; this eliminates the pressure term. Separation in toroidal coordinates is again indicated.

(c) Let $Q_r + iQ_z = f(r+iz)$ for analytic f (and $i = \sqrt{-1}$); this simplifies the poloidal field term to a form that involves only B_p^2. Analytic function theory provides many different bases of solutions, of which the set of monomials, $f = w^m$ and $f = -iw^m$, for $w = (r-r_0) + i(z-z_0)$ and (r_0, z_0) an interior point of Ω, is the simplest.

The choices (a) and (b) do not quite provide a complete set of solutions for $\mathbf{Q}$, as the differential equation imposes a certain consistency constraint on the boundary values of $(\nabla\chi\cdot\mathbf{n})$; all functions $\mathbf{Q}$ constructed by method (a) satisfy $\oint r^{-1}(\mathbf{Q}\cdot\mathbf{n})\, ds = 0$, and those constructed by method (b) satisfy $\oint r(\mathbf{Q}\cdot\mathbf{n})\, ds = 0$. Therefore, to a complete set of solutions obtained from either $\Delta^*\chi = 0$ or from $\Delta\xi = 0$ one additional function $\mathbf{Q}$ must be added for which $(\mathbf{Q}\cdot\mathbf{n})$ does not satisfy the associated constraint. A function that is suitable for either case (a) or case (b) is $\mathbf{Q} = (r-r_0)\mathbf{e}_r + (z-z_0)\mathbf{e}_z$, where (r_0, z_0) is an interior point of Ω.

The first few integrals obtained by method (c) will now be shown explicitely. For that purpose we define the generalized pressures P_0 and P_1 according to,

$$\begin{aligned} P_0 &= p + \frac{1}{2\mu_0}(B_t^2 - B_{t0}^2), \\ P_1 &= p - \frac{1}{2\mu_0}(B_t^2 - B_{t0}^2) + \frac{1}{2\mu_0}B_p^2. \end{aligned} \tag{3.14}$$

Notice that P_0 vanishes outside the plasma, and therefore the contribution of P_0 to the integrals is an invariant quantity (in the sense introduced earlier, viz. independent of the precise location of the contour $\partial\Omega$). The contribution of P_1 does not have that pleasant property.

Then, selecting $f = 1$, $\mathbf{Q} = \mathbf{e}_r$:

$$\int_T r^{-1} P_1 \, dV$$

Selecting $f = -i$, $\mathbf{Q} = -\mathbf{e}_z$:

$$\int_T 0 \, dV$$

Selecting $f = w$, $\mathbf{Q} = x\mathbf{e}_r + y\mathbf{e}_z$:

$$\int_T (2P_0 + r^{-1} x P_1) \, dV$$

Selecting $f = -iw$, $\mathbf{Q} = y\mathbf{e}_r - x\mathbf{e}_z$:

$$\int_T r^{-1} y P_1 \, dV$$

Selecting $f = w^2$, $\mathbf{Q} = (x^2 - y^2)\mathbf{e}_r + 2xy\mathbf{e}_z$:

$$\int_T (4xP_0 + r^{-1}(x^2 - y^2)P_1) \, dV$$

Selecting $f = -iw^2$, $\mathbf{Q} = 2xy\mathbf{e}_r + (y^2 - x^2)\mathbf{e}_z$:

$$\int_T (4yP_0 + 2r^{-1} xy P_1) \, dV$$

All the above integrals can be rigorously evaluated from the external magnetic measurements. Notice that whenever both P_0 and P_1 appear in an integral, the contribution of P_1 is less by a factor involving the inverse aspect ratio. It appears that moments of P_0 may be of some special interest, considering in particular that these moments are invariant quantities.

4. Evaluation of the Current Moments from Measured Data

In this Section we are concerned with the problem of constructing numerical approximations in terms of the magnetic measurements for the boundary integrals that occurred in Sections 2 and 3. It is shown how methods that are familiar from the treatment of ill-posed linear equations and from multivariate statistical analysis can be used to obtain accurate discrete approximations, and an approach to the robust treatment of failing or wildly erroneous signals is initiated. The methods that are discussed in this Section are applicable in a wide variety of circumstances, and for many readers this may be the most valuable part of the paper. It should also serve as an introduction, in a linear context, to the method of function parametrization [**34**]–[**36**], to which we return in Section 7. For a general discussion of the physical characteristics of the various kinds of magnetic diagnostics we refer to [**37**] and [**38**], while [**39**] provides a detailed discussion of the engineering issues related to the implementation of these diagnostics on one particular machine (TFTR).

General considerations. In practice there may arise several complications when an integral such as $\oint_{\partial\Omega} \mu_m^{-1}(\xi B_n + \chi B_s)\, ds$, for given functions ξ and χ, is to be approximated from the magnetic measurements: (a) Only a finite set of measurements of B_n and B_s is made, and the precise nature of these measurements is dictated more by engineering considerations than by considerations from numerical analysis. (b) Actually, instead of a local B_n or B_s one often measures integrals of type $\oint_{\partial\Omega} w_i(s) B_n(s)\, ds$ or $\oint_{\partial\Omega} w_i(s) B_s(s)\, ds$. (c) B_n and B_s may be measured on different contours, or not even on smooth contours at all. (d) The measurements involve a random error. (e) Sometimes an individual signal may be completely wrong.

Complication (e) will be ignored initially, but we return to it in the final subsection. The discussion here will be restricted to the problem of the approximation of the integrals given in Section 2, which are linear in the magnetic field components, but the analogous treatment for the integrals of Section 3 can easily be developed by the reader.

Let numerical approximations in terms of m measurements, $\{q_i\}_{1\le i\le m}$, be wanted for a collection of n moments, $\{p_j\}_{1\le j\le n}$. Each of the q_i and p_j is defined by a linear expression in terms of the poloidal components of the magnetic field on $\partial\Omega$. For each moment one may therefore postulate a numerical approximation that is linear in the measurements: $p_j \simeq \sum_i c_{ji} q_i$, or $\mathbf{p} \simeq C^T \mathbf{q}$. ($C$ has size $m \times n$). The problem is to determine an optimal coefficient matrix C. Presumably, C, once determined, will be used many times, and we assume that efficiency in obtaining this matrix is not an issue.

A linear equation. By numerical simulation and/or in the course of calibrating the diagnostic system, one may obtain for a large number of current distributions (indexed by α, $1 \le \alpha \le N$) the values of the current moments, $\mathbf{p}_\alpha$, and of the corresponding measurements, $\mathbf{q}_\alpha$. One may then attempt to determine the matrix C by straightforward least squares optimization:

$$C \text{ to minimize} \qquad \sum\nolimits_\alpha w_\alpha \|\mathbf{p}_\alpha - C^T \mathbf{q}_\alpha\|^2 \Big/ \sum\nolimits_\alpha w_\alpha, \tag{4.1}$$

for given nonnegative weights w_α. The averaging over α that is implicit in Eq. (4.1) will henceforth be denoted by the bracket pair $\langle\cdot\rangle$, so the objective function above becomes $\langle\|\mathbf{p} - C^T\mathbf{q}\|^2\rangle$. The minimum is attained for C determined by

$$\langle\mathbf{q}\mathbf{q}^T\rangle \cdot C = \langle\mathbf{q}\mathbf{p}^T\rangle, \tag{4.2}$$

which has a unique solution provided that $\langle\mathbf{q}\mathbf{q}^T\rangle$ is not singular.

The obvious difficulty with this line of approach is that $\langle\mathbf{q}\mathbf{q}^T\rangle$ is likely to be very ill-conditioned. Routine methods are however available for dealing with such near-singularity in a system of linear equations in order to obtain an approximate solution that is stable with respect to small changes in the data. Numerical analysts refer to [**40**]–[**42**], and employ some form of quasi-inversion; either by selecting a least-squares solution in a subspace on which $\langle\mathbf{q}\mathbf{q}^T\rangle$ is well-conditioned ('truncation'), or through the introduction of a stabilizing functional ('damping'). Statisticians employ the same methods, but call these principal components regression and ridge regression respectively; see for instance [**43**, ch. 8], [**44**, ch. 8], and [**45**, ch. 6]. A quick review of these methods is given in the next subsection.

First, however, it is useful to provide a slight generalization of (4.1) and (4.2), namely to allow also a constant term in the linear relation between $\mathbf{p}$ and $\mathbf{q}$. Thus we seek to determine coefficients $\mathbf{p}_0$ and C so as to obtain in a stable manner an approximate minimum of the objective function,

$$I = \langle\|\mathbf{p} - \mathbf{p}_0 - C^T\mathbf{q}\|^2\rangle. \tag{4.3}$$

The minimum of this function is attained for C given by

$$\langle(\mathbf{q} - \bar{\mathbf{q}})(\mathbf{q} - \bar{\mathbf{q}})^T\rangle \cdot C = \langle(\mathbf{q} - \bar{\mathbf{q}})(\mathbf{p} - \bar{\mathbf{p}})^T\rangle, \tag{4.4}$$

and $\mathbf{p}_0 = \bar{\mathbf{p}} - C^T\bar{\mathbf{q}}$, in which $\bar{\mathbf{q}} = \langle\mathbf{q}\rangle$ and $\bar{\mathbf{p}} = \langle\mathbf{p}\rangle$. The matrix that occurs on the left hand side of Eq. (4.4) is the (sample) dispersion matrix associated with the data $(\mathbf{q}_\alpha)_\alpha$,

$$S = \langle(\mathbf{q} - \bar{\mathbf{q}})(\mathbf{q} - \bar{\mathbf{q}})^T\rangle. \tag{4.5}$$

The generalization to Eqs. (4.3) and (4.4) is needed already if all the magnetic measurements and moments are scaled to correspond to unit plasma current.

Stable solution methods. The dispersion matrix S is symmetric and positive semi-definite, and therefore has m eigenvalues, $\lambda_1^2 \geq \ldots \geq \lambda_m^2 \geq 0$, with corresponding orthonormal eigenvectors, $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Then,

$$S = \sum_{i=1}^{m} \lambda_i^2 \mathbf{a}_i \mathbf{a}_i^T, \qquad S^{-1} = \sum_{i=1}^{m} \lambda_i^{-2} \mathbf{a}_i \mathbf{a}_i^T \tag{4.6}$$

(the inverse existing only if all $\lambda_i^2 > 0$). Ill-conditioning of S is equivalent to $\lambda_m^2/\lambda_1^2 \ll 1$, and in order to make Eq. (4.4) well-posed it is necessary to reduce in some sense the influence of the smaller eigenvalues.

The first popular procedure for obtaining a stable approximate solution to the linear equation (4.4) is variously known as truncation, selection, quasi-inversion, or principal components regression. The method is simply to truncate the expansion for S^{-1} given in Eq. (4.6) at some index $m_0 \leq m$ (possibly $m_0 \ll m$), and thus to set

$$C = \sum_{i=1}^{m_0} \lambda_i^{-2} \mathbf{a}_i \mathbf{a}_i^T \langle (\mathbf{q} - \bar{\mathbf{q}})(\mathbf{p} - \bar{\mathbf{p}})^T \rangle. \tag{4.7}$$

The choice of the value of m_0 must depend on the accuracy with which the measurements are made. In particular, if the measurements $\mathbf{q}$ are assumed to suffer independent random errors coming from a normal distribution with mean 0 and width σ, then a value of m_0 should be chosen such that $\lambda_{m_0}^2 \simeq \sigma^2$. A preliminary transformation of the measurements in order to make the expected distribution of their errors equal and independent is therefore advisable.

The other popular procedure for obtaining a stable approximate solution to Eq. (4.4) is to employ damping (equivalently, to employ a stabilizing functional, ridge regression). Using that approach, S is replaced in Eq. (4.4) by $S + \sigma^2 I$, with σ^2 a (small) parameter, so that

$$C = \sum_{i=1}^{m} \frac{1}{\lambda_i^2 + \sigma^2} \mathbf{a}_i \mathbf{a}_i^T \langle (\mathbf{q} - \bar{\mathbf{q}})(\mathbf{p} - \bar{\mathbf{p}})^T \rangle. \tag{4.8}$$

More generally one can use $S + E$, for any well-conditioned positive definite E, and thus determine C from the equation

$$(S + E) \cdot C = \langle (\mathbf{q} - \bar{\mathbf{q}})(\mathbf{p} - \bar{\mathbf{p}})^T \rangle. \tag{4.9}$$

Increasing σ^2 or E makes the matrix equation better conditioned, but also increases the bias in the resulting coefficients C. In the present context a good case can be made for the damping method, when E is chosen to correspond to an estimate of the dispersion matrix for the random errors in the measurements. Then (assuming S was computed from idealized data) the coefficient matrix C will be optimal in a least squares sense for the actual measurements.

Further discussion. An important aspect of the above procedure is that it can work well with measurements that would be less suitable if analytical procedures were to be used to derive the approximations for $\mathbf{p}$ in terms of $\mathbf{q}$. For instance, returning to our original concern of obtaining the value of the integral $\oint_{\partial\Omega} \mu_m^{-1}(\xi B_n + \chi B_s)\, ds$ from measured data, the use of analytical approximations would favor equidistant point measurements of the magnetic field, whereas more accurate data are obtained with somewhat extended coils, and the distribution of the coils over the vacuum vessel will anyway be restricted by engineering considerations. Local measurements of the poloidal field in particular suffer from alignment errors and from perturbations due to (nonaxisymmetric) nearby eddy currents, and these measurements should almost certainly be abandoned in favor of measurements made using saddle loops and partial or variable-winding Rogowski coils.

The proposed procedure can easily be employed so as to make good use of redundant information, such as a mixture of point B_p measurements, partial or variable-winding Rogowski coils, full flux loops, saddle coils, and data on the currents in the external poloidal field generating coils. It is furthermore possible to combine this procedure with any standard method that yields a numerical approximation for the moments in terms of the measurements, namely by applying the presently described methods in a defect correction manner [**46**]. (A statistician would consider this to be a Bayesian procedure). This has the numerical advantage that a stronger stabilizing term can be employed to achieve comparable overall accuracy.

Treatment of erroneous measurements. As the magnetic signals are used both for machine control and for routine data analysis, it is particularly important to have an algorithm that will deal effectively and efficiently with failing or erroneous signals. On the common assumption of normally distributed errors it has been said that "everybody trusts it [...] for experimenters believe it is a mathematical theorem, whereas the mathematicians see it as an experimental fact." (see [**47**, p. 2]). We consider now how to deal with departures from normality.

Let us define for any measurement vector $\mathbf{q}$ the transformed measurement vector $\mathbf{x} = \boldsymbol{A}^T(\mathbf{q}-\bar{\mathbf{q}})$, where $\boldsymbol{A}$ is the matrix that has as columns the eigenvectors $\mathbf{a}_i$ $(1 \leq i \leq m)$ of the dispersion matrix. The original simulated measurements, $\mathbf{q}_\alpha$, shall have been scaled (and perhaps transformed) in such a way that they are assumed in the experiment to suffer independent random errors coming from a normal distribution having mean 0 and width σ. We introduce the functional $J(\mathbf{q};\sigma)$,

$$J(\mathbf{q};\sigma) = \sum_{i=1}^{m} \frac{x_i^2}{\lambda_i^2 + \sigma^2}. \tag{4.10}$$

Then $\langle J \rangle \simeq m$, and measured values $\mathbf{q}^{exp}$ for which $J(\mathbf{q}^{exp};\sigma) \gg m$ are suspect.

Consider now an actual measurement, $\mathbf{q}^{exp}$. If it is known that one or more specific components q_i^{exp} are in error, then these components can be restored to that set of values by which the quadratic form J is minimized. This is a simple and valuable procedure, immediately available as a by-product of an eigenanalysis on $\boldsymbol{S}$, and requires only the solution of a system of linear equations having dimension equal to the number of failing signals.

It is however far from easy to design a procedure that will decide effectively and efficiently whether one (or more!) signals really are wrong. The preferred approach for related problems in statistical analysis is to use a *robust* method [**47**], [**48**], viz. a method that is not overly sensitive to outlying data without requiring their explicit identification. (The simplest example is the use of the median rather than the mean for estimating a location).

One robust procedure to deal with possibly corrupt data $\mathbf{q}^{exp}$ would be the following: routinely perform the subsequent data analysis in terms of a vector $\mathbf{q} = \mathbf{q}^{exp} + \mathbf{h}$, where $\mathbf{h}$ is such that

$$\begin{cases} J(\mathbf{q}^{exp} + \mathbf{h}; \sigma) \leq m \\ ||\mathbf{h}||_1 \text{ is minimal} \end{cases} \tag{4.11}$$

where $||\cdot||_1$ is the l_1 norm. In fact there is a choice of other norms that are preferable to l_1 from a computational point of view and that also lead to a robust procedure; see [**47**, ch. 6]. An algorithm as outlined above must necessarily be nonlinear, and some further investigation will be required in order to develop a fully satisfactory procedure. Nevertheless, the simple structure of the objective function J provides confidence that such a procedure can be developed.

5. Full Equilibrium Determination From Magnetic Measurements

The problem that is considered in this Section is the complete solution of the equation for axisymmetric ideal MHD equilibrium, including the approximate determination of the profiles $\mu_0 p'(\psi)$ and $FF'(\psi)$, aiming at an optimal fit to the external magnetic measurements. We review the published studies in this area, initially concentrating on the physical content of the work, and then comparing the various numerical methods that have been employed. We also propose a novel fast algorithm for the current profile determination.

The discussion will be restricted to isotropic, static equilibria, for which the representation $j_t = r\,dp/d\psi + r^{-1}\mu_0^{-1} F dF/d\psi$ holds in Ω_p, with a parametrization $p = p(\psi, \vec{\alpha})$ and $F = F(\psi, \vec{\alpha})$. The problem is then to determine the parameter vector $\vec{\alpha}$, the value of the flux function at the plasma boundary, ψ_b, the plasma boundary contour $\partial\Omega_p$, and the solution $\psi(r, z)$.

General considerations. In order to allow a determination of the profiles $\mu_0 p'$ and FF' the available measurements must provide sufficient redundancy beyond what is required to solve the elliptic boundary value problem for ψ with a known current profile, Eq. (1.16). For instance one may have data for both ψ and $\partial\psi/\partial n$ on $\partial\Omega$, or alternatively one may know the contribution to the field due to the currents in external coils and in addition have some local measurements of the total magnetic field. In either case the determination of $\mu_0 p'$ and FF' from the external field measurements is a typical difficult 'inverse' problem, and is certainly ill-posed [**40**]–[**42**], if no further restrictions on these profiles are given. The fundamental task for the numerical analyst is thus to find a suitable method of quasi-inversion or stabilization for this problem.

There is, however, remarkably little mathematical understanding. Even in straight geometry (the limit of infinite aspect ratio), in which the equilibrium is governed by a Poisson equation with only one unknown profile function, $\Delta\psi = -f(\psi)$, the basic questions of the existence and uniqueness of solutions to the inverse problem are unanswered. A special case that is understood arises in straight geometry, when in addition the measurements are consistent with a solution that has concentric circular flux surfaces; in that case the measurements are automatically consistent with any profile function that gives the correct total current, and there is therefore an infinite degeneracy. In toroidal geometry there exists also a family of equilibria, all having the same flux surfaces, that is degenerate with respect to the interpretation of the magnetic measurements [**49**]–[**52**]. This is discussed further in Appendix C. It is not known whether any degeneracy remains in case the measurements do not correspond to one of these special cases, but it is clear that the problem remains ill-posed. The fundamental difficulty is that the magnetic field measurements are sensitive only to poloidal variations in the current profile, whereas $f(\psi)$ in straight geometry, or $\mu_0 p'(\psi)$ and $FF'(\psi)$ in toroidal geometry, primarily influence the radial distribution of the current.

The studies of Luxon and Brown. The first published extensive numerical studies that involved a current profile optimization aiming to fit a set of magnetic measurements were done for the Doublet IIa and Doublet III experiments, and were presented in [**53**]. This paper gave clear indications about the possibilities and limitations of the magnetics analysis, and we review it here in some detail.

The Doublet III studies reported in [**53**] are based on a system of magnetic diagnostics consisting of 24 one-turn loops, measuring the poloidal flux near each of the 24 poloidal field shaping coils, and 12 partial Rogowski coils, measuring the average poloidal field over a segment spanning two field-shaping coils in the poloidal direction. A plasma current measurement obtained from a full Rogowski coil is used for comparison purposes only, and there is no diamagnetic measurement. The arrangement of these diagnostics and of the poloidal field shaping coils on Doublet III is illustrated in Fig. 1. (This Figure also shows the location of 11 point magnetic field probes, which have been used in different work).

Luxon and Brown employ a variety of current profiles, of which the following (with unimportant change in notation) is illustrative,

$$j_t = \begin{cases} \alpha\left(\beta\dfrac{r}{R_0} + (1-\beta)\dfrac{R_0}{r}\right) g(\widetilde{\psi};\gamma) & \text{in } \Omega_p \\ 0 & \text{in } \Omega_v \end{cases} \tag{5.1}$$

where

$$g(\widetilde{\psi};\gamma) = \exp(-\gamma^2(1-\widetilde{\psi})^2). \tag{5.2}$$

The normalized flux function $\widetilde{\psi}$ is defined by $\widetilde{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b)$, where ψ_a is the value of ψ on the magnetic axis and ψ_b is the value on the plasma boundary (these are not known a priori). The quantity R_0 is a characteristic radius of the machine.

The free parameters, α, β, and γ, are selected to minimize the (chi-squared) cost function,

$$J = \sum_i \frac{(B_i - \hat{B}_i)^2}{\sigma_i^2} \tag{5.3}$$

where B_i and $\hat{B}_i$ are the measured and the calculated values of some component of the poloidal field at position i, and σ_i is the standard error of the measurement. The $\hat{B}_i$ are calculated as functions of (α, β, γ) by solving the equilibrium equation subject to boundary conditions obtained from the measurements of the poloidal flux ψ. (These boundary conditions are imposed in an indirect manner. The 'infinite domain' Green function equilibrium solver GAQ [**54**] is employed, and the currents in the external coils are adjusted in order to let the computed solution match the boundary data for ψ). Minimization of J as a function of (α, β, γ) is carried out using a standard library routine.

By comparison of Eq. (5.1) with the equilibrium relation, Eq. (1.16), one sees that the term containing r/R_0 corresponds to the contribution of $\mu_0 p'$, and the term containing

R_0/r corresponds to FF'. The parameters α, β and γ may be seen to be related roughly to the toroidal current, the poloidal β, and the internal inductance (or the peakedness of the current profile). Notice that the parametrization in Eq. (5.1) assigns the same shape, $g(\widetilde{\psi};\gamma)$, but independent weighting factors to the contributions from $\mu_0 p'$ and from FF' in the current density.

Luxon and Brown give contour plots of J as a function of β and γ, for fixed, optimal α. A well-defined minimum generally exists for non-circular equilibria, but for near-circular equilibria the contours become very elongated ellipses, and a separate identification of β and γ is no longer possible. This is understood to correspond to the impossibility of separately determining β_I and $l_i/2$ for circular cross-section. They proceed to study different expressions for the current profile, including some with four free parameters instead of three, but conclude that three parameters, equivalent to I_t, β_I, and l_i, are adequate to fit the magnetic measurements. Determination of a fourth parameter becomes marginally possible only at the most highly shaped equilibria.

Approximately the same minimum value of J, and near-identical values for I_t, β_I, and l_i at the optimum, are obtained for a variety of mathematical parametrizations, indicating that (for noncircular cross-section) these three physical parameters are indeed well-determined by the external magnetic measurements. For circular cross-section one is only able to determine the two parameters I_t and $\beta_I + l_i/2$. Other properties of the plasma which are reported to be accurately determined by the magnetic analysis are the location of the plasma boundary (and as a consequence also the value of the safety factor at the boundary), and the position of the magnetic axis. For the plasma boundary this comes as no surprise; we will see in Section 6 that the plasma boundary is well determined even without the need for a determination of the current profile in the plasma. As regards the location of the magnetic axis the result is less transparent, as analytical approximations for the magnitude of the shift of the magnetic axis with respect to the geometric center of the cross-section have to rely on a specific model for the current distribution in the interior of the plasma. Apparently this shift is not too sensitive to the actual current distribution.

As the determination of a fourth parameter in the current profile is at best marginally possible, it follows that the magnetic measurements alone do not provide sufficient information to determine separately the shape of the $\mu_0 p'$ and FF' terms. Additional information, in particular a measurement of the location of the $q = 1$ surface, is employed in the analysis of D-III data in order to obtain a more accurate current profile.

MHD equilibrium determination on JET. Descriptions of the experimental system and of the various codes employed for magnetic data analysis on the JET tokamak have been given in [**55**] and [**56**]. Here we are concerned with the methods for full MHD equilibrium analysis, developed by J. Blum and co-workers, and described in [**57**]–[**60**]. The physics content of these studies is very similar to the work of [**53**], but the numerical methods employed are entirely different.

On JET the poloidal flux function is measured at 14 locations on the outside surface of the vacuum vessel, using 8 full flux loops and 14 saddle coils. The component of the poloidal field tangential to the vacuum vessel is measured by a system of 18 local magnetic probes, mounted on the inside of the vessel. An independent measurement of the plasma current is available, but not used in the work described here, and no mention is made of a diamagnetic flux measurement. Fig. 2 shows the layout of the JET magnetic diagnostics.

For the JET studies the current profile is parametrized as,

$$j_t = \begin{cases} \alpha\Big(\beta\frac{r}{R_0}g(\widetilde{\psi};\gamma_1) + (1-\beta)\frac{R_0}{r}g(\widetilde{\psi};\gamma_2)\Big) & \text{in } \Omega_p \\ 0 & \text{in } \Omega_v \end{cases} \tag{5.4}$$

where, as in [**53**], $\widetilde{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b)$ (in which ψ_a and ψ_b are the value of ψ on the magnetic axis and on the boundary), and R_0 is a characteristic radius of the device. For the profile function $g(\widetilde{\psi};\gamma)$ either a polynomial

$$g(\widetilde{\psi};\gamma) = \widetilde{\psi} + \gamma\widetilde{\psi}^2 \tag{5.5^a}$$

or a power function

$$g(\widetilde{\psi};\gamma) = \widetilde{\psi}^{\gamma} \tag{5.5^b}$$

is selected. There are at most four free parameters, α, β, γ_1 and γ_2, but one or two of these may be fixed in advance, or it may be required that $\gamma_1 = \gamma_2$.

As in the work of [**53**], the optimization criterion is minimization of a cost function, J, defined in terms of the poloidal field measurements, Eq. (5.3), whereas the measured flux values provide the boundary conditions for the equilibrium solver. Information on the the currents in external coils is not needed as input, nor is it obtained from the analysis. The equilibrium solvers used at JET are the IDENTB and IDENTC codes, which are related to the SCED code of J. Blum [**58**]. These codes employ a finite element discretization together with a Newton iteration scheme, as described in detail in [**60**].

The JET studies show that from the magnetic measurements alone, two or three parameters can be determined: I_t and $\beta_I + l_i/2$ for low-β, near-circular plasmas, and I_t, β_I, and l_i for non-circular plasma or at high β. The minimum elongation for which (at low β, and with the magnetic diagnostics available) the parameters β_I and l_i can be separated is reported to lie around $b/a \simeq 1.25$. An indication of the minimum $\beta_I + l_i/2$ for which β_I and l_i can be separated at circular cross-section is not available.

The IDENTC code allows specification of the value of the pressure on axis, or of the radial profile of the pressure, in addition to the magnetic data. With this additional information one more parameter in the current profile can be determined, separating β_I and l_i in the circular case, and providing both the coefficients γ_1 and γ_2 for elongated cross-sections. This experience is consistent with the results obtained in the D-III modelling.

A study for ASDEX. Winter and Albert present in [**61**] a method for determining the separatrix location from magnetic measurements, with application to the ASDEX tokamak. The next Section will discuss specialized methods for plasma boundary identification, but as Winter and Albert rely on a solution of the MHD equilibrium equation, their work is discussed here.

Only a limited number of local magnetic measurements is available on ASDEX: the toroidal plasma current, and two flux- and two field measurements (one of each on the outer and on the inner side of the cross-section), and these are reduced further to three signals by considering only I_t and the difference signals $\delta\psi$ and δB_p. However, in [**61**] the field due to the external currents is also assumed to be known.

Their current profile parametrization has the form,

$$j_t = \alpha\Big(\beta\frac{r}{R_0} + (1-\beta)\frac{R_0}{r}\Big)(\tilde{\psi} + \gamma\tilde{\psi}^2) \tag{5.6}$$

in the plasma region Ω_p, and $j_t = 0$ in Ω_v. In the present case, R_0 is the major radius of the geometric center of the plasma boundary, and $\tilde{\psi} = \psi - \psi_b$.

As ASDEX has a near-circular cross-section, a separate identification of the parameter β is not expected to be feasable (this is confirmed by the analysis). Winter and Albert therefore supply an estimate of β based on other information, and perform the minimization of the cost function over the parameters α and γ only. The Garching free boundary equilibrium code [**26**], which solves the equilibrium equation for given external field, is employed, and the three measured signals mentioned before are all available for the the profile determination.

The procedure has been tested on numerically generated magnetic field data, which may correspond to a different functional form for the current profile than the one used in the reconstruction, Eq. (5.6). Good agreement between the reconstructed magnetic measurements and the input data is obtained at the optimum (α, γ) for a range of estimated β, and over this range the computed separatrix location remains relatively immobile. This confirms that β is not well determined by the magnetic analysis, and shows that the separatrix location is well determined. Winter and Albert report that $\beta_I + l_i/2$ is also accurately determined.

An interesting limitation on the performance of the reconstruction algorithm is noted in [**61**]. A current distribution according to Eq. (5.6) has l_i values in the range $0.8 < l_i < 2.2$ (they claim). If the guessed input value of β is such that the correct value of $\beta_I + l_i/2$ cannot be reproduced using this current distribution, then it turns out that there may also be a large error in the computed separatrix location. This is to be taken as an injunction to employ a current profile parametrization that admits a sufficiently large range of l_i values.

Recent work for Doublet III. L.L. Lao et al. describe in [**62**] a new code for MHD equilibrium determination, EFIT, which is significantly faster than the code used by Luxon [**53**], and has been employed for routine analysis of the D-III magnetic measurements.

The diagnostics used in EFIT include the 24 flux loops and 12 partial Rogowski coils that were used also in [**53**], and furthermore include 11 local magnetic field probes, one full Rogowski loop, and optionally a diamagnetic flux loop. The layout of these diagnostics on D-III has been shown in Fig. 1. In addition the value of the safety factor on axis, q_a, may be specified as input to EFIT.

The two terms in the plasma current profile, corresponding to $\mu_0 p'$ and to FF', are parametrized using a polynomial model, constrained by $j_t = 0$ on the plasma boundary. Good results are obtained by using a third degree polynomial for $\mu_0 p'$ and a linear function for FF', whence

$$j_t = (\alpha_1 \widetilde{\psi} + \alpha_2 \widetilde{\psi}^2 + \alpha_3 \widetilde{\psi}^3) r + \beta_1 \widetilde{\psi} r^{-1} \tag{5.7}$$

in Ω_p, and $j_t = 0$ in Ω_v, where again $\widetilde{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b)$. Alternatively a second degree polynomial model for both $\mu_0 p'$ and FF' has been used.

An 'infinite domain' equilibrium solver is employed in EFIT, and the vector of unknowns contains the profile parameters $(\alpha_1, \alpha_2, \alpha_3, \beta_1)$ as well as the values of the currents in the external coils. The optimization criterion is a cost function defined in terms of all the measurements and any constraints between the profile parameters:

$$J = \sum_{i=1}^{N_m} \frac{(M_i - \hat{M}_i)^2}{\sigma_i^2} + \sum_{i=1}^{N_c} \frac{(H_i - \hat{H}_i)^2}{\varsigma_i^2} \tag{5.8}$$

where N_m and N_c denote the number of measurements and the number of constraints, M_i, $\hat{M}_i$, and σ_i denote the measured value, the computed value, and the error associated with the i-th measurement, and H_i, $\hat{H}_i$, and ς_i denote the given value, the computed value, and the uncertainty associated with the i-th constraint.

The results described in [**62**] are consistent with those obtained by Luxon and Brown [**53**], and in the JET studies [**56**]. Without a specification of q_a or a diamagnetic measurement, two independent parameters can be determined for approximately circular plasmas, and three for elongated configurations. Assuming isotropic pressure, a diamagnetic measurement allows to separate β_I and l_i in the circular case. Specification of q_a allows to determine a fourth parameter.

Good agreement is found between the diamagnetic β_I and the β_I obtained from poloidal field and flux measurements for elongated plasma ($b/a > 1.15$) "covering a wide range of plasma operating conditions". Unfortunately there is no quantitative discussion in [**62**] of the influence of high power neutral injection (which will cause the pressure to become anisotropic) on the difference between these two methods for determining β_I.

Recent work for Tuman 3. A recent code for full equilibrium determination from magnetic measurements on the Tuman-3 tokamak was described in [**63**]. The diagnostics used in this work include measurements of the poloidal flux function along two different contours L_1 and L_2 encircling the plasma column, an independent measurement of the toroidal plasma current, and a diamagnetic measurement. The parametrization for the plasma current is

$$j_t = \alpha_1(\alpha_2\widetilde{\psi} + (1-\alpha_2)\widetilde{\psi}^2)\frac{r}{R_0} + \beta_1(\beta_2\widetilde{\psi} + (1-\beta_2)\widetilde{\psi}^2)\frac{R_0}{r} \tag{5.9}$$

in Ω_p, $j_t = 0$ in Ω_v, where again $\widetilde{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b)$.

A Green function method is employed to solve the equilibrium equation (1.16) for given parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, subject to Dirichlet boundary conditions obtained from the flux measurements on the contour L_1 (presumably L_1 is located outside L_2). The parameters are determined in a two stage procedure. For given values α_2 and β_2 the parameters α_1 and β_1 are determined from the condition of reproducing exactly the measured toroidal current and diamagnetic flux, while α_2 and β_2 themselves are determined from the condition of chi-squared minimization of the error in the measurements made on L_2, viz. minimization of the cost function,

$$J = \sum_{i=1}^{N} \frac{(\psi_i - \hat{\psi}_i)^2}{\sigma_i^2}. \tag{5.10}$$

Here, N is the number of measurements made on L_2, ψ_i is the measured value and $\hat{\psi}_i$ the computed value of the flux function at location i, and σ_i is the standard error of the i-th measurement.

Consistent with all the studies described previously, the Tuman-3 studies show that only three independent parameters can be determined using this set of diagnostics; the parameters α_2 and β_2 cannot be separated. Along each line of constant J in the (α_2, β_2) plane, the resulting current profile shows very little change. Ref. [**63**] suggests that beyond the plasma current and the diamagnetic flux also the internal inductance l_i, the poloidal beta β_p, the safety factor on the plasma boundary q_b, and the safety factor on axis, q_a, are well determined by the magnetic analysis. As regards q_a a more specific investigation seems desirable, as all other work suggests that the Tuman-3 diagnostics would not suffice for the accurate determination of this parameter.

Possibilities and limitations of magnetic analysis. All the studies described above show that the external magnetic measurements, even if these include a diamagnetic flux loop, provide only limited information on the interior structure of the plasma current profile. One obtains the plasma current I_t, the parameters β_I and l_i, the plasma boundary $\partial\Omega_p$, the current center (r_c, z_c), the position of the magnetic axis (r_a, z_a), and the safety factor on the boundary, q_b. The field due to the external currents is accurately obtained even if these currents are not measured, and with that information included, only three of the characteristic plasma parameters (namely I_t, β_I, and l_i), can be considered to be genuinely independent.

In Sections 6 and 7 we will discuss fast specialized methods for the identification of the plasma boundary and for the determination of characteristic parameters of the equilibrium, including all the parameters listed above. It will be seen that these methods can be both very efficient and accurate, and the full equilibrium analysis as described in this Section is therefore not required for routine data analysis, but only to provide a standard of comparison for the more rapid and specialized methods. In fact it must be recognized that the full equilibrium analysis over-fits the data, by producing a complete solution to the equilibrium equation in the interior of the plasma, although the measurements only warrant a specification of some integral characteristics. The danger exists that those physicists who are not familiar with the limitations of the magnetics analysis will hold for real also those details of the 'solution' that are strongly dependent upon the specific mathematical parametrization that is employed.

An important virtue of the studies described above is that they have demonstrated the limitations to the analysis of magnetic measurements alone. Clearly it remains a desirable objective to be able to determine on a routine basis the complete MHD equilibrium configuration and its time evolution throughout a discharge, and the methods used in the above studies will remain relevant for equilibrium determination from an appropriate extended set of diagnostics. The possible rôle of information on the safety factor on axis, q_a, or the location of the $q = 1$ surface (if it exists) has already been demonstrated. Significant additional information on the current profile in the interior of the plasma could come from Faraday rotation measurements [**64**]–[**67**], where it would be important to analyze these measurements in conjunction with the magnetic diagnostics, and not in isolation. Further diagnostic input could be in the form of purely geometric information on the shape of the flux surfaces (as available from electron temperature measurements in particular). The use of such geometric information for current profile determination was proposed by Christiansen and Taylor [**68**], and is developed further in Appendix C.

It should be pointed out that some codes exist for equilibrium determination from a consistent analysis of a wide range of diagnostic systems, notably the ZORNOC code [**69**], [**70**], developed for the analysis of ISX-B data, and the TRANSP code [**71**], developed at Princeton, but neither of these codes appears suitable for routine analysis of many time slices for a single discharge. Development of procedures for efficient MHD equilibrium analysis based on a range of diagnostic systems remains an open challenge.

Comparison of numerical methods. So far in this Section the review of published work has concentrated on the physics content of the studies, without more than a brief mention of the numerical procedures employed. We now turn to that issue.

First it must be pointed out that each of the groups whose work was discussed above has used a different equilibrium solver; we have seen finite difference methods involving a Buneman rapid solver, a finite element method employing Newton iteration, a 'moments' method, and even the Green function method (which must compete with unaccelerated relaxation and Gaussian elimination for being the worst possible procedure). Another, more fundamental, distinction is that between the use of a 'finite domain' or an 'infinite domain' equilibrium solver, corresponding respectively to fitting only to local measurements of the field and flux, and to using also information on the currents in external coils. These distinctions are not the subject of this paper, but they should be noticed as a warning against facile comparisons between the the work of the different groups.

The optimization problem, determining the current profile that fits best to the magnetic measurements, contains two different sources of nonlinearity. First there is the complicated dependence of the cost function, J, on the unknown parameters, while for fixed parameter values the solution of the p.d.e., Eq. (1.16), is also in general a nonlinear problem. One would like to have a reasonably efficient numerical scheme for the combined problem.

In the work of Luxon and Brown [**53**] the iterative procedures for these two nonlinearities are nested: an outer iteration varies the estimated values of the parameters, and for each parameter set the cost function is evaluated from a fully converged solution to the differential equation. In fact, the Jacobian of J with respect to variations in the parameters is also evaluated repeatedly, and this has to be done numerically, so that the p.d.e. must be solved even more often. This treatment of the outer nonlinearity, together with the use of an inefficient basic equilibrium solver, explains the extremely long running times of the algorithm of [**53**].

In Blum's work [**57**]-[**60**] the equilibrium equation is discretized by a finite element method, which is solved using Newton iteration [**58**]. For the combined parameter estimation and equilibrium problem again a Newton iteration scheme is employed, as described in detail in the forthcoming book [**60**]. This leads to an efficient algorithm, with running time quoted as several seconds on a Cray-1.

In the work of Winter and Albert [**61**] (following a suggestion of K. Lackner), and also in work of Lao et al. [**62**], a functional form for the current density is employed that is linear in the unknown parameters: $\mu_0 r j_t = \sum_i \alpha_i g_i(r,\psi)$. This makes it possible to interleave the two iterative procedures in a relatively straightforward manner. At stage n of the procedure one has the approximate parameter vector $\vec{\alpha}^{(n)}$ and the approximate solution $\psi^{(n)}$. Then the following linear problems are solved by a suitable direct method: for each i the inhomogeneous equation, $\Delta^*\psi_i^{(n+1)} = -g_i(r,\psi^{(n)})$ in Ω, subject to homogeneous boundary conditions ($\psi_i^{(n+1)} = 0$ on $\partial\Omega$), and in addition the homogeneous equation, $\Delta^*\chi^{(n+1)} = 0$, with the correct inhomogeneous boundary conditions. (In the standard case the boundary conditions are linear, and χ has to be computed only once). Next $\vec{\alpha}^{(n+1)}$ and $\psi^{(n+1)}$ are computed from the solution to a *linear* least squares problem; with

$$\psi = \chi^{(n+1)} + \sum_i \alpha_i \psi_i^{(n+1)} \tag{5.11}$$

$\vec{\alpha}^{(n+1)}$ is that parameter vector $\vec{\alpha}$ by which the cost function $J(\psi)$ is minimized, and $\psi^{(n+1)}$ is the corresponding minimizing function ψ.

A widely used method to solve the equilibrium problem with known current profile is to employ Picard iteration, solving at each stage $\mathcal{L}^*\psi^{(n+1)} = -\mu_m r j_t(r,\psi^{(n)})$ by the use of a rapid direct solver [**26**]. The work involved in the procedure for equilibrium determination that is employed in [**61**] and [**62**] is seen to be a small multiple (corresponding to the number of free parameters in the current profile) of the work involved in the solution of the problem with known current profile via Picard iteration. A significant limitation is that the form of the profile is restricted to be linear in the unknown parameters.

Fast optimization of parameters. It appears possible to construct a considerably faster algorithm for the optimization (with respect to the measured data) of the parameters describing the current profile, and this without requiring that the parameters enter linearly in the profile; it may be assumed that $rj_t = f(r, \psi, \vec{\alpha})$ in the plasma, without specific restrictions on the functional form of f. The proposal is to determine the parameters $\vec{\alpha}$ not from the requirement of obtaining a best fit to the external magnetic measurements directly, but rather from the requirement of obtaining a best fit to a set of moments of the current, as obtained from these measurements according to the theory of Section 2.

We take as the starting point any efficient algorithm for solution of the equilibrium equation with known current profile (nonlinear in ψ); our favorite method, and the fastest available, is multigrid relaxation [**72**]–[**75**], but some form of Picard iteration based on a rapid direct solver [**26**] or a Newton iteration scheme as used by Blum [**58**] is also suitable. Next, in between the iterations of this basic equilibrium solver we interleave the parameter optimization procedure, correcting the parameters $\vec{\alpha}$ in order to improve the fit between the moments derived from the measured data and those computed from $rj_t(\psi, \vec{\alpha})$. In this way the correction to the parameters is found by purely algebraic methods, and does not require the solution of any auxiliary p.d.e. as does the method of [**61**] and [**62**].

Notice that this proposed procedure is closely related to the usual way in which a total current constraint is imposed on the equilibrium problem. In that case one is asked to determine ψ and λ such that $\Delta^*\psi = -\lambda\mu_0 rf(r, \psi)$ subject to the integral constraint $\lambda \int_\Omega f\, dS = I_t$, for given current shape function f and total current I_t. No one writing a standard equilibrium code would seek to impose this constraint in the manner of [**53**]. Instead, after each iteration of the equilibrium solver, λ is adjusted in order to obtain the desired total current. This standard procedure may obviously be extended to more than one constraint, and also to an overdetermined system, to be solved in a least-squares sense.

The author's present equilibrium code [**74**], [**75**] was written purely to provide a 'proof of principle' for multigrid as a method for computing MHD equilibrium. The code solves equation (1.16) subject to Dirichlet boundary conditions on a rectangle, which is of no practical interest. However, as a demonstration code it has been entirely successful, achieving full multigrid efficiency, and solving the equilibrium equation (for nonlinear right hand side) on a 128×128 grid in $\simeq 120$ msec on the Cray-1. In the context of the interpretation of experimental data a much coarser grid will be adequate, and it becomes realistic to strive for full MHD equilibrium determination in ~ 20 msec on a Cray-1. This is in fact close to the timescale that is relevant for active control of an experiment.

6. Fast Identification of the Plasma Boundary

In this Section we consider fast specialized methods for the determination of the plasma boundary contour $\partial\Omega_p$ and of the magnetic field on $\partial\Omega_p$ from the external magnetic measurements. Knowledge of the location of the plasma boundary is important for control of the experiment, and if in addition the field on the plasma boundary is known, then the theory presented in Section 3 can be used to obtain estimates for a number of internal plasma parameters.

General considerations. As for the more general profile determination problem discussed in the previous Section, the measurements may provide data about both ψ and $\partial\psi/\partial n$ on $\partial\Omega$, or they may provide a specification of the field due to the external currents, together with some local field measurements. The basis for all fast specialized methods for plasma boundary identification, and the reason that this problem can be solved without the need for a complete determination of the plasma equilibrium, is that in the vacuum region, Ω_v, bounded by $\partial\Omega_p$ and $\partial\Omega$, the flux function ψ satisfies the homogeneous equation, $\mathcal{L}^*\psi = 0$. A solution to this equation that is valid in a region including Ω_v, and that agrees with the given boundary conditions, therefore suffices in principle to determine the plasma boundary, which will be taken to be the largest closed flux surface inside a given limiting contour. Thus, if both ψ and $\partial\psi/\partial n$ are given on $\partial\Omega$, the plasma boundary identification problem appears as one of the classical ill-posed problems (in the sense of Hadamard): the integration of an elliptic equation from Cauchy boundary data. A similar ill-posed type of problem arises in case the boundary conditions include a specification of the field produced by the external currents.

Fortunately, the Cauchy problem for elliptic equations is well-understood [**40**]–[**42**]. Stable solution methods may be obtained either by restricting the class of allowed solutions ψ on Ω_v to an appropriate finite-dimensional space (truncation, quasi-inversion), or through the introduction of a stabilizing functional (damping), or through a combination of these two methods. Most of the successful procedures for fast plasma boundary identification therefore rely on an approximation of the flux function ψ by a finite series in terms of solutions to the homogeneous equilibrium equation,

$$\hat{\psi} = \psi^0 + \sum_{j=1}^{N} \hat{c}_j \chi_j, \tag{6.1}$$

in which ψ^0 represents any known contribution to the flux function (this term may be absent), and in which the basis functions χ_j all satisfy the homogeneous equation, $\mathcal{L}^*\chi_j = 0$, on some (annular) region Ω_0 that is known to include the vacuum region Ω_v. The coefficients $\hat{c}_j$ are to be determined from the measured data.

Let us denote the relevant actual measurements by y_i, where $1 \le i \le M$. The expectation values of the measurements depend linearly on the magnetic field and flux, and there exists therefore a response matrix $\boldsymbol{Q}$ such that

$$\hat{y}_i = \hat{y}_i^0 + \sum_{j=1}^{N} Q_{ij}\hat{c}_j, \qquad 1 \le i \le M \tag{6.2}$$

where $\hat{y}_i$ is the expectation value of the i-th measurement when the flux function ψ is given by Eq. (6.1). $\hat{y}_i^0$ is associated with ψ^0 and the matrix element Q_{ij} is associated with χ_j. The response matrix Q will be assumed to be known exactly.

The usual least squares approach to determining the coefficients $\hat{c}_j$ is to minimize the (chi-squared) cost function,

$$J = \sum_{i=1}^{M} \frac{(y_i - \hat{y}_i)^2}{\sigma_i^2}, \tag{6.3}$$

where σ_i is the standard error of the i-th measurement, and the relation (6.2) is assumed. This minimization criterion gives rise to a linear representation for the coefficients $(\hat{c}_j)_j$ in terms of the measurements $(y_i)_i$. Whether this procedure is stable depends on the choice of the basis functions χ_j. A more general approach is to replace the cost function J of (6.3) by a function J_ϵ of the form,

$$J_\epsilon = \sum_{i=1}^{M} \frac{(y_i - \hat{y}_i)^2}{\sigma_i^2} + \epsilon \sum_{j=1}^{N} \frac{(c_j - \hat{c}_j)^2}{\eta_j^2}, \tag{6.4}$$

where ϵ, $(c_j)_j$, and $(\eta_j)_j$ are constants. The second term in J_ϵ is intended to provide numerical stabilization, and reflects a priori knowledge about the range of values for the coefficients $\hat{c}_j$. One may set c_j to some 'average' value of $\hat{c}_j$ over all possible states of the system, and set η_j to a measure for the dispersion of the values of $\hat{c}_j$. Then ϵ is a tuning constant of order unity.

With the minimization criterion (6.4) the coefficients $(\hat{c}_j)_j$ are still linearly related to the measurements $(y_i)_i$, and the matrix elements in this linear relation can be pre-computed. Specifically, these matrix elements are obtained by inversion of the relation,

$$(Q^T D_\sigma^{-1} Q + \epsilon D_\eta^{-1})\hat{\mathbf{c}} = Q^T D_\sigma^{-1}(\mathbf{y} - \hat{\mathbf{y}}^0) + \epsilon D_\eta^{-1}\mathbf{c}, \tag{6.5}$$

where $D_\sigma = \mathrm{diag}(\sigma_i^2)$ and $D_\eta = \mathrm{diag}(\eta_j^2)$. In this way a stable approximation $\hat{\psi}$ to the true flux function ψ in the vacuum region Ω_v is rapidly computed, after which the plasma boundary $\partial\Omega_p$ is obtained by finding the largest closed flux surface inside the given limiting contour. A stable approximation to the magnetic field on $\partial\Omega_p$ is obtained at the same time. Inside the plasma region Ω_p the function $\hat{\psi}$ must not be considered an approximation to the true flux function ψ.

In most cases the series (6.1) splits naturally into two parts: an interior solution ψ^i, which is regular throughout Ω and is associated with a current distribution outside Ω, and an exterior solution ψ^e, which is associated with the plasma currents. The treatment of the exterior solution is critical for the plasma boundary identification problem, as this is the part that causes the problem to be ill-posed. The interior solution may be known from measurements of the currents in external coils, or it may be computed directly from the magnetic measurements via Green's representation theorem in the form of Eq. (1.11), or it may be found together with the exterior solution through optimization of Eq. (6.3) or (6.4). In any case, the computation of the interior solution is a stable process.

A variety of different representations for the field due to the plasma current has been used; the most important ones being an expansion of the field in toroidal eigenfunctions [**76**]–[**79**], a discrete current filament model [**80**]–[**83**], [**33**], [**62**], and a representation via a single layer potential on a control surface [**57**], [**60**], [**84**]. The field due to the external currents is sometimes assumed known [**80**]–[**83**]. It has also been represented by an expansion having undetermined coefficients, either based on a filament model [**33**], [**62**], or on toroidal eigenfunctions [**76**]–[**79**]. A Green function or single layer potential approach has been used in Refs. [**57**], [**60**], and [**84**]. The following three subsections review in more detail the studies based on an expansion in toroidal harmonics, on a filamentary current model, and on a single layer potential.

Expansion in toroidal harmonics. One well established stable method to integrate the equation $\Delta^*\psi = 0$ inwards from the boundary data on $\partial\Omega$, is to employ a truncated series expansion in toroidal eigenfunctions of the homogeneous equation. The toroidal (ς, η) coordinate system about the point $(r = r_0, z = z_0)$, where $r_0 > 0$, is defined by

$$\begin{cases} r = r_0 \sinh\varsigma/(\cosh\varsigma - \cos\eta) \\ z - z_0 = r_0 \sin\eta/(\cosh\varsigma - \cos\eta) \end{cases} \tag{6.6}$$

where $\varsigma > 0$, $0 \le \eta \le 2\pi$, and the ignorable coordinate ϕ is ignored. The contour defined by $\varsigma = \varsigma_0$ is a circle in the right half-plane, having radius $r_0 \sinh\varsigma_0$ and center at $(r = r_0 \coth\varsigma_0, z = z_0)$. The corresponding torus has aspect ratio $\cosh\varsigma_0$. The circle degenerates to the singular point $(r = r_0, z = z_0)$ for $\varsigma_0 \to \infty$, and to the axis of symmetry, $r = 0$, for $\varsigma_0 \to 0$. The metric coefficients of the coordinate system are given by $h_\varsigma = h_\eta = r_0/(\cosh\varsigma - \cos\eta)$ and $h_\phi = r_0 \sinh\varsigma/(\cosh\varsigma - \cos\eta)$.

The most general solution to $\Delta^*\psi = 0$ outside the singular point is given by $\psi = \psi^i + \psi^e$, where

$$\psi^i = \frac{r_0 \sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}} \Big[\sum_{n=0}^{\infty} a_n^i Q^1_{n-\frac{1}{2}}(\cosh\varsigma)\cos(n\eta) + \sum_{n=1}^{\infty} b_n^i Q^1_{n-\frac{1}{2}}(\cosh\varsigma)\sin(n\eta) \Big], \tag{6.7}$$

and

$$\psi^e = \frac{r_0 \sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}} \Big[\sum_{n=0}^{\infty} a_n^e P^1_{n-\frac{1}{2}}(\cosh\varsigma)\cos(n\eta) + \sum_{n=1}^{\infty} b_n^e P^1_{n-\frac{1}{2}}(\cosh\varsigma)\sin(n\eta) \Big]. \tag{6.8}$$

P_ν^m and Q_ν^m are Legendre functions, for which we follow the notation of [**18**] and [**19**] (See also Appendix C). The interior solution ψ^i is finite throughout the right half plane, and corresponds to a field due to currents located on the axis $r = 0$ and at ∞, while the exterior solution ψ^e has a singularity at (r_0, z_0), and corresponds to the field of a multipole current distribution at the singular point.

Similar series representations may be derived for the magnetic field components,

$$B_\eta = -\frac{1}{rh_\varsigma}\frac{\partial\psi}{\partial\varsigma}, \qquad B_\varsigma = \frac{1}{rh_\eta}\frac{\partial\psi}{\partial\eta}, \tag{6.9}$$

It will be noticed that (having to deal with orthogonal coordinate systems only) we do not use tensor notation here or elsewhere in the paper, but we have defined $B_\varsigma = \mathbf{B}\cdot\nabla\varsigma/|\nabla\varsigma|$ and $B_\eta = \mathbf{B}\cdot\nabla\eta/|\nabla\eta|$. The explicit representations follow easily from the relations,

$$\begin{aligned}
&-\frac{1}{rh_\varsigma}\frac{\partial}{\partial\varsigma}\Big(\frac{r_0\sinh\varsigma}{\sqrt{\cosh\varsigma-\cos\eta}}Q^1_{n-\frac{1}{2}}(\cosh\varsigma)\Big)\\
&=\frac{\sqrt{\cosh\varsigma-\cos\eta}}{r_0}(n+\frac{1}{2})\Big((n-\frac{1}{2})\cos\eta\,Q^0_{n-\frac{1}{2}}-n\cosh\varsigma\,Q^0_{n-\frac{1}{2}}+\frac{1}{2}Q^0_{n+\frac{1}{2}}\Big)\\
&=\frac{\sqrt{\cosh\varsigma-\cos\eta}}{r_0}(n-\frac{1}{2})\Big((n+\frac{1}{2})\cos\eta\,Q^0_{n-\frac{1}{2}}-n\cosh\varsigma\,Q^0_{n-\frac{1}{2}}-\frac{1}{2}Q^0_{n-\frac{3}{2}}\Big)
\end{aligned}\tag{6.10}$$

and

$$\begin{aligned}
&-\frac{1}{rh_\varsigma}\frac{\partial}{\partial\varsigma}\Big(\frac{r_0\sinh\varsigma}{\sqrt{\cosh\varsigma-\cos\eta}}P^1_{n-\frac{1}{2}}(\cosh\varsigma)\Big)\\
&=\frac{\sqrt{\cosh\varsigma-\cos\eta}}{r_0}(n+\frac{1}{2})\Big((n-\frac{1}{2})\cos\eta\,P^0_{n-\frac{1}{2}}-n\cosh\varsigma\,P^0_{n-\frac{1}{2}}+\frac{1}{2}P^0_{n+\frac{1}{2}}\Big)\\
&=\frac{\sqrt{\cosh\varsigma-\cos\eta}}{r_0}(n-\frac{1}{2})\Big((n+\frac{1}{2})\cos\eta\,P^0_{n-\frac{1}{2}}-n\cosh\varsigma\,P^0_{n-\frac{1}{2}}-\frac{1}{2}P^0_{n-\frac{3}{2}}\Big)
\end{aligned}\tag{6.11}$$

and

$$\begin{aligned}
&\frac{1}{rh_\eta}\frac{\partial}{\partial\eta}\Big(\frac{r_0\sinh\varsigma}{\sqrt{\cosh\varsigma-\cos\eta}}\cos(n\eta)\Big)\\
&=\frac{\sqrt{\cosh\varsigma-\cos\eta}}{r_0}\Big(\frac{n+1}{2}\sin((n-1)\eta)-n\cosh\varsigma\sin(n\eta)+\frac{n-1}{2}\sin((n+1)\eta)\Big)
\end{aligned}\tag{6.12}$$

and

$$\begin{aligned}
&\frac{1}{rh_\eta}\frac{\partial}{\partial\eta}\Big(\frac{r_0\sinh\varsigma}{\sqrt{\cosh\varsigma-\cos\eta}}\sin(n\eta)\Big)\\
&=-\frac{\sqrt{\cosh\varsigma-\cos\eta}}{r_0}\Big(\frac{n+1}{2}\cos((n-1)\eta)-n\cosh\varsigma\cos(n\eta)+\frac{n-1}{2}\cos((n+1)\eta)\Big)
\end{aligned}\tag{6.13}$$

By truncating the series for ψ and for the magnetic field components, and demanding that the corresponding field and flux on $\partial\Omega$ provide a least squares fit to the measured data, a linear algebraic system for the coefficients, $\hat{a}_n^{e,i}$ and $\hat{b}_n^{e,i}$, of an approximate solution $\hat{\psi}$ is obtained. The condition number of this system will depend on the number of terms retained in the series, and a damping term may be added in order to provide further stabilization.

After having obtained in this way a stable approximate solution $\hat{\psi} = \hat{\psi}^i + \hat{\psi}^e$ to the ill-posed boundary value problem, the plasma boundary $\partial\Omega_p$ is identified as the largest closed flux surface inside the given limiter contour. For this procedure to provide a meaningful result it is necessary that the point (r_0, z_0) shall have been chosen to lie inside the plasma region (which is unknown a priori), and preferably not too close to the boundary. In the vacuum region Ω_v, $\hat{\psi}^i$ and $\hat{\psi}^e$ may be understood to correspond to the fields due to the external and to the internal currents respectively. For this interpretation, however, all magnetization currents must be considered as true currents.

The method of expansion in toroidal harmonics has been implemented by a number of authors.

Lee and Peng **[76]** describe a numerical study of this method for use on an idealization of the ISX-B tokamak. The vacuum chamber of ISX-B has a rectangular cross-section, and a system of magnetic probes provides point measurements of both B_r and B_z at 18 points along the circumference. The idealized device of **[76]** has the same rectangular cross-section, on which a numerical grid of 34×56 points is imposed, and measurements of the poloidal flux function are made at each of the 344 points in the outermost two layers of the grid. No reason is given for the replacement of the real ISX-B diagnostic system by this construction.

The singular point (r_0, z_0) is positioned at the center of the cross-section, and the following truncated series expansion is employed:

$$\hat{\psi} = \hat{a}_0 + \frac{\sinh\varsigma}{\sqrt{2(\cosh\varsigma - \cos\eta)}}\Bigg[\sum_{n=0}^{N_c}[\hat{a}_n^e P^1_{n-\frac{1}{2}}(\cosh\varsigma) + \hat{a}_n^i Q^1_{n-\frac{1}{2}}(\cosh\varsigma)]\cos(n\eta) + \sum_{n=1}^{N_s}[\hat{b}_n^e P^1_{n-\frac{1}{2}}(\cosh\varsigma) + \hat{b}_n^i Q^1_{n-\frac{1}{2}}(\cosh\varsigma)]\sin(n\eta)\Bigg]. \tag{6.14}$$

The difference of a factor $\sqrt{2}r_0$ compared to the series in Eqs. (6.7) and (6.8) is of course unimportant, but their inclusion of the constant term $\hat{a}_0$ is somewhat surprising, as it is not independent of the infinite series in Eq. (6.7). In fact

$$1 = \frac{\sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}}\frac{\sqrt{2}}{\pi}{\sum_{n=0}^{\infty}}' \frac{Q^1_{n-\frac{1}{2}}(\cosh\varsigma)\cos(n\eta)}{n^2 - \frac{1}{4}} \tag{6.15}$$

where the prime on the summation signifies that the term for $n = 0$ should be taken with a factor $\frac{1}{2}$. It is true that the truncated system (6.14) remains non-singular even with the term $\hat{a}_0$ included, but it will tend to be badly conditioned.

Two test cases are studied in **[76]**, one elongated D-shape and one circular plasma, both symmetric about the midplane. These shapes are well recovered by Eq. (6.14) for N_c in the range 1–3 and for $N_s = 1$, when the simulated ψ measurements are perturbed by a random error of not more than $\sim 5\%$. The maximum relative error in B_p is usually 25–35 times larger (this factor must be the ratio of the gradient scale length to their grid spacing) and Lee and Peng are led to conclude that the method of expansion in toroidal harmonics would be useful in general if the maximum relative random error of the poloidal **B** field data does not exceed 100%. This author is very sceptical.

Simultaneously with the work of Lee and Peng [**76**], the use of an expansion in toroidal harmonics for plasma boundary identification was also proposed, but not actually implemented, by Kuznetsov and Naboka [**77**]. Their proposal was further developed and studied numerically in [**78**], and actual results from an implementation for the Tuman-3 tokamak are reported in [**79**]. This last reference will be the basis for our discussion.

As described in [**79**], the system of magnetic diagnostics on Tuman-3 consists of 24 probes distributed uniformly along a circular contour encircling the vacuum vessel, with 12 probes measuring the tangential component and 12 measuring the normal component of the magnetic field. In addition there is an independent measurement of the plasma current, which is used to obtain an overall correction to the tangential field measurements after which a discretized Ampère's law holds exactly. In the same spirit, the measurements of the normal component of the field are adjusted in order to satisfy the integral form of $\nabla \cdot \mathbf{B} = 0$.

In the Tuman-3 study the expansions (6.7) and (6.8) are employed, all four series being truncated after the same index N. Thus there are $4N + 2$ unknown coefficients, which are determined from the condition of minimization of the cost function,

$$J_\epsilon = J + \epsilon \sum_{n=0}^{N} \Big((a_n^i)^2 + (b_n^i)^2 + (a_n^e)^2 + (b_n^e)^2 \Big). \tag{6.16}$$

Common sense dictates that J in the above expression should have the form (6.3), but whether this is what is intended by Eq. (11) of [**79**] is not clear. It is reported that $N \geq 3$ provides a good reconstruction of the plasma boundary.

The parameter ϵ is not a constant in [**79**], but is determined from the condition of minimization of the functional

$$\Phi(\epsilon) = \left\{ \epsilon^2 \sum_n \left[\left(\frac{da_n^i}{d\epsilon} \right)^2 + \left(\frac{db_n^i}{d\epsilon} \right)^2 + \left(\frac{da_n^e}{d\epsilon} \right)^2 + \left(\frac{db_n^e}{d\epsilon} \right)^2 \right] \right\}^{\frac{1}{2}}. \tag{6.17}$$

While this is an interesting mathematical procedure, it does have the drawback of requiring an iterative algorithm, for which furthermore the matrix elements of the inverse of Eq. (6.5) cannot be precomputed.

Filamentary current model. Another widely used expansion method for plasma boundary identification is based on a model of the plasma current as a finite set of current filaments,

$$j_t = \sum_{j=1}^{N} I_j \delta(\mathbf{r} - \mathbf{r}_j) \tag{6.18}$$

in Ω_p, where the coefficients I_j are to be determined. The fixed points $\mathbf{r}_j$ must lie inside the (unknown) plasma region. The field in Ω due to the external currents may be known from measurements of these currents, or it may be obtained by an application of Green's representation theorem employing the measured field and flux on $\partial\Omega$, or it may also be modelled in terms of current filaments of unknown strength. The filamentary current method has been in routine use on a number of experiments.

A.J. Wootton [**80**] describes a study, for the TOSCA tokamak, in which the discrete filamentary current model is employed to determine the shape of the plasma boundary. This is in fact the first published study on fast methods for plasma boundary identification. The currents in the external windings are known in [**80**], and plasma-induced currents external to the measuring contour are assumed to be negligible. Wootton has a measurement of the total current, and furthermore uses three variable winding Rogowski and three saddle coils to measure the first three harmonics of the tangential and normal components of the magnetic field on the vacuum vessel, viz. the coefficients $\hat{a}_j$ and $\hat{b}_j$ $(1 \leq j \leq 3)$ in the expansions,

$$B_\tau \simeq \frac{\mu_0 I_t}{2\pi d}\Big(1 + \sum_{j=1}^{3} \hat{a}_j \cos j\omega\Big)$$
$$B_n \simeq \frac{\mu_0 I_t}{2\pi d} \sum_{j=1}^{3} \hat{b}_j \sin j\omega \tag{6.19}$$

where ω is the angular variable in a local polar coordinate system, and d is the minor radius of the measuring contour. This expansion, and the further analysis in [**80**], assumes symmetry about the horizontal midplane. It is also clearly oriented towards large aspect ratio, as for finite aspect ratio the chosen representation of B_n is inconsistent with $\nabla \cdot \mathbf{B} = 0$.

Only three current filaments (apparently in carefully optimized locations) are employed to represent the plasma current. Within this framework the most natural computational procedure would seem to be to determine the 3 unknown coefficients from the requirement of obtaining a least squares fit to the magnetic measurements. Wootton instead uses the measurements to evaluate large aspect ratio approximations to 3 moments of the current density, and then determines the coefficients to obtain an exact fit to these moments. The results in [**80**] show that even strongly distorted shapes of the plasma boundary, as computed using an MHD equilibrium code, may be accurately retrieved by this analysis.

A filamentary current model was also employed by D.W. Swain and G.H. Neilson for plasma boundary identification on ISX-B [**81**], [**82**]. They have point measurements of the r and z components of the magnetic field in 18 locations distributed around the vacuum vessel, and also know the currents through each of the four groups of poloidal field generating coils (all coils within one group are connected in series). This diagnostic system and the field coils are illustrated in Fig. 3.

Due to the presence of an iron core in the main transformer coil of ISX-B, the free space Green function cannot be used to compute the field produced by a given filamentary current. Instead an analytically tractable model for the main transformer is employed in Refs. [**81**] and [**82**], in which it is represented as an infinitely permeable center post of the correct radius and infinite axial extent, with an annular iron cylinder modelling the return leg. Two free parameters in their model are the radius of the outer annulus, and the iron permeability. (The thickness of the annulus is specified by requiring that the cross-sectional area of the core and of the annulus be equal). These parameters are adjusted so as to give a good fit to the experimental data.

Using this model for the iron core, the response matrices $\boldsymbol{Q}^C$ and $\boldsymbol{Q}^P$ are calculated, which linearly relate the expectation values of the magnetic measurements to the currents in the external coils and in the model plasma filaments:

$$\hat{y}_i = \sum_{j=1}^{N_C} Q^C_{ij} I^C_j + \sum_{j=1}^{N_P} Q^P_{ij} I^P_j, \qquad 1 \le i \le M. \tag{6.20}$$

Here, N_C, N_P, and M are the number of groups of poloidal field generating coils, the number of plasma current filaments, and the number of measurements ($N_C = 4$, $N_P = 6$ typically, and $M = 36$); I^C_j is the current through the j-th group of poloidal field coils, I^P_j is the j-th filamentary current, and $\hat{y}_i$ is the expectation value of the i-th magnetic field measurement corresponding to this set of currents.

Given actual magnetic field measurements y_i and coil currents I^C_j, the filamentary currents I^P_j are determined from the requirement of obtaining a least squares fit in Eq. (6.20), viz. from minimization of a cost function as given in Eq. (6.3). The resulting expansion is employed to trace out the plasma boundary surface and to determine the field on the plasma boundary. The lowest order moments of the plasma current distribution are also evaluated. Comparison tests using a free boundary equilibrium code show that the plasma boundary and the dipole and quadrupole moments of the current distribution are accurately recovered.

Lao et al. [**33**], [**62**] have implemented the filamentary current model in a code (MFIT) for the determination of plasma shape on Doublet III. The diagnostics used in these calculations consist of 24 flux loops, 12 partial Rogowski coils, 11 local magnetic probes, and one full Rogowski coil, as illustrated in Fig. 1.

The plasma current is modelled using 6 filaments in fixed positions, while the field due to the external currents is also considered unknown, and is modelled using 24 filaments in the positions of the field shaping coils. Doublet III does not have an iron core transformer, and the response matrix in Eq. (6.20) is calculated by using the free space Green function.

The currents I^C_j and I^P_j are determined from the requirement of minimization of the following cost function,

$$J_\epsilon = \sum_{i=1}^{M} \frac{(y_i - \hat{y}_i)^2}{\sigma_i^2} + \epsilon\Big(\sum_{j=1}^{N_C} (I^C_j)^2 + \sum_{j=1}^{N_P} (I^P_j)^2\Big) \tag{6.21}$$

where ϵ is a constant.

It is mentioned in [**62**] that for large sized plasmas, which simultaneously touch the top, the inner, and the outer limiters of D-III, the filamentary current method becomes inaccurate, also in comparison with the results obtained using the full equilibrium determination code EFIT. This is quite interesting, as at first sight there is no reason to suspect a deterioration of the filamentary current method for large sized plasmas. Two possible explanations which occur to this author are: (1) For large plasma the boundary is located close to the many magnetic diagnostics of D-III, hence it is in principle very accurately determined, and 6 current filaments are too few to obtain full accuracy. (2) For large plasma the boundary is located much closer to the external current filaments than to the internal filaments, and it becomes improper to use the same value of ϵ in Eq. (6.21) to dampen both the term related to the external currents and the one related to the internal currents. The internal currents should be damped less strongly.

Single layer potential methods. The two approaches to be described in this subsection both employ a *control surface* C_0, which is a (smooth) closed contour in the poloidal plane, located in a position where it is guaranteed to lie inside the plasma region. The function $\hat{\psi}$ is determined as a solution to the homogeneous equilibrium equation on the region Ω_0, bounded by C_0 and $\partial\Omega$. The methods that rely on a control surface turn out to be closely related to the single layer approach in potential theory.

Feneberg, Lackner and Martin [**84**] have developed a fast boundary identification code (FASTB) for JET using such an approach. We recall the system of magnetic diagnostics on JET (Fig. 2), which consists of 14 flux loops on the outside of the vacuum vessel, and 18 tangential field probes on the inside.

In FASTB the unknown plasma current is represented by an expansion in N Fourier modes of a singular current density on the control surface C_0. The external currents are similarly represented by N' Fourier modes on the surface $\partial\Omega$, which is coincident with the contour on which the poloidal flux function is measured, and therefore lies just outside the contour on which measurements of the poloidal field are made. The free space Green function is employed to compute the field and flux due to the basic current distributions on $\partial\Omega$, while the field and flux due to the currents on C_0 are computed subject to the homogeneous Dirichlet condition on $\partial\Omega$. This leads to an approximate flux function of the form

$$\hat{\psi} = \sum_{n=1}^{N} \hat{c}_n^e \chi_n^e + \sum_{n=1}^{N'} \hat{c}_n^i \chi_n^i, \tag{6.22}$$

where the basis functions χ_n^e vanish on $\partial\Omega$.

In [**84**], first the coefficients $(c_n^i)_n$ are computed from the requirement of obtaining a least squares fit to the flux measurements on $\partial\Omega$, and afterwards the coefficients c_n^e follow from the requirement of a least squares fit to the poloidal field measurements. Typically, $N = 6$ and $N' = 14$ (so the flux measurements are fitted exactly).

FASTB is used routinely for the analysis of JET discharges. After determining the plasma boundary and the field on the boundary, integral relations are employed to calculate characteristic parameters of the configuration.

Another single layer potential approach was followed by J. Blum in an early study for plasma boundary identification on JET [**57**], [**60**]. In his work the problem $\mathcal{L}^*\hat{\psi} = 0$ is solved in the region Ω_0, bounded by C_0 and $\partial\Omega$, subject to the boundary conditions $\hat{\psi} = f$ on $\partial\Omega$ and $\hat{\psi} = v$ on C_0, where f is obtained from the poloidal flux measurements, and v is unknown. The function v has to be determined from the condition of minimization of (a discrete approximation to) the cost function,

$$J_\epsilon(v) = \sum_j \left(\frac{1}{r_j}\frac{\partial\hat{\psi}}{\partial n}(\mathbf{r}_j) - g_j\right)^2 + \epsilon \oint_{C_0} \left(\frac{1}{r}\frac{\partial\hat{\psi}}{\partial n}\right)^2 ds, \tag{6.23}$$

where the g_j are the poloidal field measurements, and the second term is included in order to stabilize the problem.

Using a finite element discretization for v, and also for the solution $\hat{\psi}$, the minimization criterion (6.23) is translated into a linear equation giving v in terms of the flux and field measurements. The matrix elements of this equation may be pre-computed.

Irregular methods. The various methods for plasma boundary identification that have been discussed above, whether based on an expansion in toroidal eigenfunctions [**76**]–[**79**], on a filamentary current model [**80**]–[**83**], [**33**], [**62**], or on an expansion employing functions on a control surface [**84**], [**57**], [**60**], are rather alike in their basic structure. In each case the unknown flux function ψ in the vacuum region Ω_v is expanded in a (small) series of mutually independent solutions to the homogeneous equation, giving rise to a system of linear equations, which may be further stabilized by the inclusion of a regularizing functional. This is correct mathematical procedure. Several nonstandard treatments of the plasma boundary identification problem have also appeared in the literature.

One such method was described by Lee and Peng [**76**] under the name of "local fitting". (The method of expansion in toroidal harmonics is proposed in the same paper, and referred to as global fitting). In this method of local fitting, a large number of overlapping circular regions is placed on the domain Ω, and in each of these regions a local Taylor series representation for the flux function (satisfying the homogeneous equation) is found by making a fit to the measured boundary values that are located inside the region and to any interior values that have already been included in other regions (and that are not inside the plasma). The value of the flux function at any point is then computed as the average of the values given by all local Taylor series representations that cover that point. See [**76**] for the details. The method of local fitting appears to be without virtue.

Another nonstandard procedure was presented in [**78**] as the "integral method". (These authors too discuss the method of expansion in toroidal harmonics, which they call the differential method). This integral method has apparently been motivated by the desire to obtain a smooth two-dimensional current distribution to fit the measurements, rather than something as patently unphysical as a multipole distribution concentrated at one point, a set of wire currents, or a singular current layer.

In the integral method of [**78**] the discrete vector of unknowns contains the currents in all the points of a two-dimensional mesh. The associated matrix equation obtained from the requirement of chi-squared minimization with respect to the measurements is then extremely ill-conditioned, but a regularizing functional is employed. This regularizing functional is defined as the sum of squares of the mesh currents.

In this way the equation is successfully stabilized. Nevertheless it is hard to find anything useful in this integral method. The resulting matrix equation is much larger than the one associated with any of the standard methods, and the resulting current distribution may be smooth, but does not satisfy the equilibrium equation, and therefore still has no meaning inside the plasma.

The third of the nonstandard procedures which we wish to mention is a method employed by Ida and Toyama [**85**] for boundary identification on **TNT-A** (Tokyo Noncircular Tokamak). Their model flux function does not satisfy the homogeneous equilibrium equation, but satisfies the two-dimensional Laplace equation instead. Some local fitting procedure is employed to integrate the Laplace equation inwards from the boundary data. The statement that this procedure is 50 times as fast as a filamentary current method cannot be based on a reasonable implementation of the latter method.

7. Fast Determination of Characteristic Parameters

In this Section we consider fast specialized methods for the approximation of characteristic parameters of the MHD equilibrium configuration. After providing an overview of the available analytical approaches, this Section will be devoted mainly to recent studies [**36**], [**86**], [**87**], which have demonstrated the method of function parametrization [**34**], [**35**] as an extremely effective procedure for obtaining a wide variety of characteristic parameters from the magnetic measurements.

Some important parameters. The following is a (not exhaustive) list of parameters that one may wish to determine. They are not all independent, even if it is generally impossible to provide explicit connecting formulae.

I_t	toroidal plasma current
(r_c, z_c)	position of the current center, Eqs. (2.10) and (2.11)
(r_a, z_a)	position of the magnetic axis
(r_g, z_g)	position of a geometric center of Ω_p
a, b	horizontal and vertical minor radius of Ω_p
ϑ, δ, ...	triangularity, indentation, ..., of Ω_p
S	area of Ω_p
V	plasma volume
A	plasma surface area
(r_x, z_x)	position of some saddle point
$\psi_a - \psi_b$	flux difference between the magnetic axis and $\partial\Omega_p$
$\psi_b - \psi_r$	flux difference between $\partial\Omega_p$ and a reference location
q_a	safety factor on axis
q_b	safety factor on $\partial\Omega_p$
β_I	poloidal β
l_i	internal inductance
μ_I	toroidal diamagnetism parameter
$\beta_I + l_i/2$	Shafranov parameter
...	etc.

(Recall that Ω_p denotes the poloidal cross-section of the plasma).

The important question is to what extent these parameters may be determined (semi-)directly from the magnetic measurements, without requiring a complete solution of the equilibrium equation. In this context a method will be called *direct* when it provides a physical parameter as an explicit function of the measured data, and *semi-direct* when it relies on a preliminary identification of the plasma boundary.

Review of analytical methods. A direct and rigorous determination of I_t, r_c, and z_c is possible using the theory of Section 2. After identification of the plasma boundary, with the aid of one of the methods described in Section 6, all the geometric quantities (r_g, z_g, a, b, ϑ, δ, S, V, A, r_x, z_x), as well as $\psi_b - \psi_r$ and q_b, are rigorously determined.

All the fast procedures for plasma boundary identification also provide the magnetic field on the boundary. The theory of Section 3 then provides approximate expressions for $\beta_I - \mu_I$ and $\beta_I + l_i/2$, and if a diamagnetic measurement is available or if the plasma is sufficiently shaped, also for β_I, l_i, and μ_I separately. These expressions are exact only in the limit of large aspect ratio, but a good approximation to $\beta_I + l_i/2$ is generally obtained. The approximations derived by Cordey, Lazzaro et al. [**88**], [**56**] and by Lao et al. [**62**] in order to separate β_I and l_i without the use of a diamagnetic measurement rely on a specific model of the equilibrium current distribution, but appear to work well in the application to JET and D-III data.

An analytical formula for the radial displacement of the magnetic axis with respect to the geometric center of the plasma, $r_a - r_g$, was given by Shafranov [**9**, Eq. (6.33)],

$$r_a - r_g = \frac{a^2}{2r_0}(\beta_I + l_i/2), \tag{7.1}$$

where a is the plasma minor radius and r_0 is a characteristic major radius. The derivation of this relation relies on the assumptions of up-down symmetry, circular cross-section and on a specific model for the current distribution, but in fact equilibrium calculations for ASDEX have shown it to be accurate over a wide range of current distributions.

Finally, no satisfactory approximations in terms of the magnetic measurements alone exist for the parameters $\psi_a - \psi_b$ and q_a, and the studies described in Section 5 tend to show that the determination of these parameters requires additional information.

Throughout this paper we have emphasized those methods that make as few assumptions as possible about the shape of the plasma cross-section and about the current distribution in the plasma. In this case, for almost all of the parameters listed at the beginning of this Section it is necessary to rely on one of the methods for fast plasma boundary equation, followed by application of the integral relations of Section 3. In fact, it is only with the advent of the recent generation of devices with shaped cross-section, and in particular D-III, JET, ISX-B and Tuman-3 that application of these fairly general methods has become routine. For configurations having approximately circular cross-section, direct approximations are available for many of the characteristic parameters, and these are used in practice.

The most widely used of these direct approximations are again due to Shafranov [**8**], [**9**, Section 6], [**10**, Section 4.7] and provide the radial plasma position and $\beta_I + l_i/2$ in terms of the magnetic measurements. A local polar coordinate system (ϱ, ω) is employed, centered at a fixed position $(r_0, 0)$. $r = r_0 + \varrho\cos\omega$ and $z = \varrho\sin\omega$. Then Shafranov's approximations for the flux function ψ and the poloidal field components B_ω and B_ϱ outside the plasma, but inside any other conducting surface, are

$$\begin{aligned}\psi(\varrho,\omega) \simeq &-\frac{\mu_0 r_0 I_t}{2\pi}\left(\ln\frac{\varrho}{8r_0} + 2\right) \\ &- \frac{\mu_0 \varrho I_t}{4\pi}\left[\left(1 - \frac{a^2}{\varrho^2}\right)\left(\beta_I + \frac{l_i - 1}{2}\right) + \ln\frac{\varrho}{a} - \frac{2r_0\Delta}{\varrho^2}\right]\cos\omega\end{aligned} \tag{7.2}$$

$$B_\omega(\varrho,\omega) \simeq -\frac{\mu_0 I_t}{2\pi\varrho} - \frac{\mu_0 I_t}{4\pi r_0}\Big[\Big(1+\frac{a^2}{\varrho^2}\Big)\Big(\beta_I + \frac{l_i - 1}{2}\Big) + \ln\frac{\varrho}{a} - 1 + \frac{2r_0\Delta}{\varrho^2}\Big]\cos\omega \qquad (7.3)$$

$$B_\varrho(\varrho,\omega) \simeq -\frac{\mu_0 I_t}{4\pi r_0}\Big[\Big(1-\frac{a^2}{\varrho^2}\Big)\Big(\beta_I + \frac{l_i - 1}{2}\Big) + \ln\frac{\varrho}{a} - \frac{2r_0\Delta}{\varrho^2}\Big]\sin\omega \qquad (7.4)$$

where $\Delta = r_g - r_0$. Assume that the plasma current I_t is measured and that an estimate of the minor radius a is available. Then two flux measurements and two measurements of B_ω, preferable made on a contour of constant ϱ, suffice to determine the coefficients of $\cos\omega$ in Eqs. (7.2) and (7.3), and thereby to determine $\beta_I + l_i/2$ and Δ. As a straightforward variation on this procedure, the flux measurements may be replaced by measurements of the other component of the poloidal magnetic field, B_ϱ, and if more measurements are available, then a least squares fit can be employed to determine $\beta_I + l_i/2$ and Δ.

Function parametrization. We turn now to a discussion of a radically different method for estimating the physical parameters from magnetic measurements. Function parametrization was originally developed by H. Wind (CERN) for fast momentum determination from spark chamber data in high energy physics experiments [**34**], [**35**], but it has a much wider range of applicability, and can be considered whenever many measurements are to be made with the same diagnostic setup. In a recent report [**36**] we have presented function parametrization in the context of controlled fusion research, in particular with a view towards the interpretation of magnetic diagnostics. An application to the magnetic data analysis on the ASDEX experiment has been described in [**86**] and [**87**], and is summarized below.

The method relies on an analysis of a large data set of simulated experiments, aiming to obtain an optimal representation of some simple functional form for intrinsic physical parameters of a system in terms of the values of the measurements. Statistical techniques for dimension reduction and multiple regression are used in the analysis. The resulting function may be chosen to involve only low-order polynomials in only a few linear combinations of the original measurements; this function can therefore be evaluated very rapidly, and needs only minimal hardware facilities.

Three steps have to be made for experimental data evaluation based on function parametrization. (1) A numerical model of the experiment and of the relevant diagnostics is used to generate a data base of simulated states of the physical system, in which each state is represented by the values of the relevant physical parameters and of the associated measurements. (2) This data base is made the object of a statistical analysis, with the aim to provide a relatively simple function that expresses the physical parameters in terms of the measurements. (3) The resulting function is then employed for the fast interpretation of real measurements.

Mathematical description. The statistical analysis involves established methods for dimension reduction and linear regression, which are briefly discussed here.

A classical physical system is considered, of which $\mathcal{P}$ denotes a typical state. The system may have any number of degrees of freedom, but interest will be restricted to a (partial) characterization by n intrinsic real parameters, represented collectively by a point $\mathbf{p} \in \mathbf{R}^n$. In the experimental situation $\mathbf{p}$ is to be estimated from the readings of m measurements, represented by a point $\mathbf{q} \in \mathbf{R}^m$.

The aim of the function parametrization is to obtain some relatively simple function $\mathbf{F} : \mathbf{R}^m \rightarrow \mathbf{R}^n$, such that for any state $\mathcal{P}$ the associated $\mathbf{p}(\mathcal{P})$ and $\mathbf{q}(\mathcal{P})$ satisfy $\mathbf{p} = \mathbf{F}(\mathbf{q}) + \mathbf{e}$ for a sufficiently small error term $\mathbf{e}$. The functional form of $\mathbf{F}$ may typically be chosen as a low-order polynomial in only a few linear combinations of the components of $\mathbf{q}$. The unknown coefficients in $\mathbf{F}$ are then determined by analysis of a data base containing the values of the parameters $\mathbf{p}_\alpha$ and of the measurements $\mathbf{q}_\alpha$ corresponding to N simulated states $\mathcal{P}_\alpha$ $(1 \leq \alpha \leq N)$. This is a problem for which techniques from multivariate statistical analysis are appropriate.

Since the dimensionality m of the space of the measurements lies between several tens and several hundred in many cases, the dimensionality of the space of trial functions with which the parameters are to be fitted can be very large. A polynomial model of degree l in all variables, for instance, has $\sim m^l/l!$ degrees of freedom for each physical parameter. It is therefore desirable to first reduce the number of independent variables (the components of $\mathbf{q}$) by means of a transformation to a lower-dimensional space. A second, and also very important, aim for this transformation of variables must be to eliminate or reduce multicollinearity (near linear dependencies) between the data points, and thus to improve the conditioning of the regression problem. Multicollinearity is expected to be present whenever the number of measurements is much larger than the number of independently determinable physical parameters.

A common statistical technique to find a lower-dimensional space in which to represent the measurements is based on principal component analysis. From the N suitably scaled pseudo measurements, $\{\mathbf{q}_\alpha\}_{1 \leq \alpha \leq N}$, the sample mean, $\bar{\mathbf{q}} = N^{-1} \sum_\alpha \mathbf{q}_\alpha$, and the $m \times m$ sample dispersion matrix,

$$S = \frac{1}{N} \sum_{\alpha=1}^{N} (\mathbf{q}_\alpha - \bar{\mathbf{q}})(\mathbf{q}_\alpha - \bar{\mathbf{q}})^T, \tag{7.5}$$

are calculated. S is symmetric and positive semi-definite. An eigenanalysis will yield m eigenvalues, $\lambda_1^2 \geq \ldots \geq \lambda_m^2 \geq 0$, with corresponding orthonormal eigenvectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Any measurement vector $\mathbf{q}$ may be resolved along these eigenvectors to obtain a set of transformed measurements, $x_i = \mathbf{a}_i \cdot (\mathbf{q} - \bar{\mathbf{q}})$. The sample variance of the component x_i is given by λ_i. Now the assumption is made that the most significant information will be contained in those transformed measurements that show the largest variation over the simulated data, viz. in the components $(x_i)_{1 \leq i \leq m_0}$ where $m_0 \leq m$, and preferably $m_0 \ll m$. These m_0 components are called the 'significant components', and the associated first m_0 eigenvectors $\mathbf{a}_i$ are the 'significant variables'. The desired dimension reduction is thus achieved through the transformation $\mathbf{R}^m \rightarrow \mathbf{R}^{m_0}$ defined by $\mathbf{x} = \mathbf{A}^T \cdot (\mathbf{q} - \bar{\mathbf{q}})$, where $\mathbf{A}$ is the matrix that has columns $\mathbf{a}_i$ $(1 \leq i \leq m_0)$.

Having obtained the preliminary linear transformation $\mathbf{q} \to \mathbf{x}$ it is next necessary to face the task of fitting the, in general nonlinear, relation between $\mathbf{x}$ and $\mathbf{p}$. It is desired to find for each component p_j $(1 \leq j \leq n)$ a regression, $p_j = f_j(\mathbf{x}) + \epsilon_j$, to fit the data $(\mathbf{x}_\alpha, \mathbf{p}_\alpha)_{1 \leq \alpha \leq N}$. A polynomial model, of the form

$$p_j = \sum_{\mathbf{k}} c_{\mathbf{k}j} \cdot \prod_{i=1}^{m_0} \phi_{k_i}(x_i/r_i) + \epsilon_j, \tag{7.6}$$

is suitable. Here, the multi-index $\mathbf{k}$ has m_0 components $k_1, \ldots, k_{m_0}$ in the nonnegative integers, the $c_{\mathbf{k}j}$ are the unknown regression coefficients, which are determined by a least-squares fitting procedure over the data base, $(\phi_k)_{k \geq 0}$ is some family of polynomials (Chebyshev, Hermite, Legendre, or monomials), r_i is a suitable scaling factor for the component x_i, and ϵ_j is the error term. An upper bound on some norm of $\mathbf{k}$ must be supplied in order to make the model finite, and in addition it is possible to employ with the above model some form of subset regression, the objective being to retain in the final expression only the terms which make a significant contribution to the goodness-of-fit.

Function parametrization thus leads to simple explicit approximations for the physical parameters in terms of the measurements. Although a significant effort may be involved in generating and analyzing the data base, the evaluation of the final function — and this is the operation that has to be performed many times — is almost trivial.

Application to magnetic data analysis. As an initial study we applied function parametrization to the determination of a limited set of characteristic equilibrium parameters for the ASDEX experiment. The relevant measurements consist of three differential flux measurements, four field measurements, the current through the multipole shaping coils, and the plasma current. However, the plasma current can be scaled out of the problem, so that 8 independent measurements remain. The physical parameters to be determined include the position of the magnetic axis, the geometric center of the cross-section, the current center, the horizontal and vertical minor radius, $\beta_I + l_i/2$, a normalized q-value at the separatrix, the flux difference between the separatrix and a reference position, the position of the lower and upper saddle points, and the point of intersection of the separatrix with each of the four divertor plates.

The details of this study are given in [**86**] and [**87**], and are highly encouraging. Second order polynomial fits involving 4 to 6 principal components are found to provide good accuracy also in the presence of realistic measurement errors. For instance the fits for the plasma position are accurate to ~ 5 mm, while $\beta_I + l_i/2$ is fitted to an accuracy of ~ 0.05. This application has established function parametrization as a straightforward and effective way in which to obtain numerical approximations for a variety of characteristic equilibrium parameters in terms of the magnetic measurements. These approximations are not only extremely easy to evaluate, but are also more accurate than the analytic approximations that are now in common use. The procedure does not require very specific assumptions about the MHD equilibrium, and is also well suited to a consistent analysis of a system consisting of several different diagnostics.

It can be expected that in the future function parametrization will have an important rôle both for on-line data analysis and for real-time plasma control. To the latter point, we estimate in [**87**] that a small processor, having a few kilowords of 32-bit store and a speed of $\sim$ 5 Mflops/sec for modest size matrix-vector multiplication, will suffice to evaluate the approximations to a set of $\sim$ 25 interesting physical parameters within 1 msec, which is the timescale that is relevant for active control.

Conclusions

We have reviewed the analytical theory that is available for MHD equilibrium determination from magnetic measurements on axisymmetric systems, emphasizing the utility of the two classes of integral relations due to Zakharov and Shafranov. These relations have been extended to take full account of pressure anisotropy and plasma rotation. The main emphasis in this work, however, has been placed on a reconsideration and development of the relevant numerical methods.

For the interpretation of magnetic signals we indicated a target code performance of < 1 msec for the estimation of a set of global parameters characterizing at least the plasma position, shape, pressure and internal inductance, and of ~ 20 msec for a full equilibrium determination, The 1 msec timescale is below the skin time of the vacuum vessel in present experiments (e.g. 8 msec for Asdex, 3 msec for JET), and is therefore relevant to active feedback control of the plasma. As most tokamak experiments in operation strive to operate near the beta limit, or instability threshold, the need for accurate plasma control is particularly accute.

The ~ 20 msec timescale which we believe is needed for a full 2-D equilibrium determination is still above what is required for active position control, but may be relevant to the programming of auxiliary heating, and of course to rapid inter-shot analysis. However, a full equilibrium analysis can only provide genuine additional information over that delivered by the fast specialized methods if additional diagnostics are taken into account, and the actual codes for such a rapid analysis of a spectrum of diagnostic systems remain to be developed. The immediate aim of such an effort should be to allow the routine determination after every shot of the complete time-evolution of the 2-D equilibrium configuration throughout the discharge, and this aim appears realistic using presently available super-mini computers.

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Appendix A. Transformation of Free-Field Boundary Conditions

In the case when the field due to the currents in external coils is specified, the proper boundary conditions for the equilibrium problem demand regularity as $r \to 0$ and as $|\mathbf{r}| \to \infty$ for that part of the flux function that is due to currents in the plasma. On a finite computational domain, Ω, which must completely enclose the plasma, these conditions may be replaced by an integral equation relating ψ and $\partial\psi/\partial n$ on the boundary $\partial\Omega$. A method of this nature has been developed by Von Hagenow and Lackner [**26**], but in their formulation it is required to solve an auxiliary problem having homogeneous boundary conditions. This Appendix shows a somewhat different approach.

A boundary integral equation. Let ψ_{ext} be the (known) flux function due to the currents in the external coils, and define $\psi_{pl} = \psi - \psi_{ext}$. Let $G(\mathbf{r}, \mathbf{r}')$ be the Green function for the problem in the infinite domain: $\mathcal{L}^* G = \mu_m r' \delta(\mathbf{r} - \mathbf{r}')$, where $\mathbf{r}'$ is fixed, and G vanishes as $r \to 0$ and as $|\mathbf{r}| \to \infty$. We recall Green's second identity for $\mathcal{L}^*$ on Ω, Eq. (1.8), and note that a similar relation, with only a change of sign for the normal derivative on $\partial\Omega$, holds for $\mathcal{L}^*$ on the exterior region Ω_e. By application of Green's second identity to the functions ψ_{pl} and $G(\mathbf{r}, \mathbf{r}')$ on Ω_e, on which $\mathcal{L}^* \psi_{pl} = 0$, one obtains,

$$c(\mathbf{r}')\psi_{pl}(\mathbf{r}') + \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\psi_{pl}\frac{\partial G}{\partial n}\,ds = \oint_{\partial\Omega} r^{-1}\mu_m^{-1} G \frac{\partial \psi_{pl}}{\partial n}\,ds, \tag{A.1}$$

where

$$c(\mathbf{r}') = \begin{cases} 0, & \text{if } \mathbf{r}' \text{ is an interior point of } \Omega, \\ \dfrac{\varphi(\mathbf{r}')}{2\pi}, & \text{if } \mathbf{r}' \in \partial\Omega, \\ 1, & \text{if } \mathbf{r}' \text{ is an interior point of } \Omega_e, \end{cases} \tag{A.2}$$

and where $\varphi(\mathbf{r}')$ is the exterior angle subtended by $\partial\Omega$ at the point $\mathbf{r}' \in \partial\Omega$. Similarly, by application of Green's second identity to the functions ψ_{ext} and $G(\mathbf{r}, \mathbf{r}')$ on Ω, one finds,

$$-(1 - c(\mathbf{r}'))\psi_{ext}(\mathbf{r}') + \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\psi_{ext}\frac{\partial G}{\partial n}\,ds = \oint_{\partial\Omega} r^{-1}\mu_m^{-1} G \frac{\partial \psi_{ext}}{\partial n}\,ds, \tag{A.3}$$

and combining these two relations,

$$c(\mathbf{r}')\psi(\mathbf{r}') + \oint_{\partial\Omega} r^{-1}\mu_m^{-1}\psi\frac{\partial G}{\partial n}\,ds = \oint_{\partial\Omega} r^{-1}\mu_m^{-1} G \frac{\partial \psi}{\partial n}\,ds + \psi_{ext}(\mathbf{r}'), \tag{A.4}$$

which, when specialized to the case $\mathbf{r}' \in \partial\Omega$, is the desired integral equation to connect ψ and $\partial\psi/\partial n$ on $\partial\Omega$. Alternatively, if ψ_{pl} instead of ψ is selected as the unknown field, then Eq. (A.1) specialized to $\mathbf{r}' \in \partial\Omega$ may be used as the boundary condition.

Discretization. A natural discretization of the boundary condition in Eq. (A.1) or (A.4) has the form of a matrix equation connecting the values of ψ on the boundary points of the mesh to the values of ψ on those interior points that border on the boundary. Let $\vec{\psi}_0$ denote the vector of values of ψ on the boundary mesh points, taken in some definite order, and let $\vec{\psi}_1$ denote the vector of values of ψ on the neighbouring interior points. Then the discretization of the boundary conditions, Eq. (A.4), should obtain the form,

$$\vec{\psi}_0 = A\vec{\psi}_1 + \mathbf{b}. \qquad (A.5).$$

In the case of Eq. (A.1) the inhomogeneous term, **b**, will be absent. That this form of a discrete boundary condition is reasonable follows also from the well-posedness of the exterior problem for the equation $\Delta^*\psi = -\mu_m r j_l$ on Ω_e.

In order to obtain an accurate discretization of the above form, the methods discussed in Section 4 are again applicable. These methods are in this case best employed in the defect correction mode, after first obtaining a low order discretization by straightforward analytical procedures.

The boundary condition in the form (A.5) leads to an iterative procedure, and is therefore useful if the equilibrium equation itself is also solved iteratively (as in [**74**], [**75**]; classical iterative methods are not to be recommended for this equation). In case the equilibrium equation is solved by a rapid direct method, the treatment of the boundary conditions as given in [**26**] remains indicated.

We intend to undertake in the near future a more extensive study of the various ways in which to implement numerically the free-field boundary conditions.

Appendix B. The Homogeneous Equilibrium Equation

In Section 2 it has been shown that certain moments of the toroidal current density in the region Ω can be expressed as integrals of linear combinations of the poloidal magnetic field components on the boundary $\partial\Omega$. This theory involves conjugate pairs of solutions to the homogeneous equations, $\mathcal{L}^*\chi = 0$ and $\mathcal{L}(r^{-1}\mu_m^{-1}\xi) = 0$, where functions χ and ξ are called conjugates if Eq. (2.2) holds. Furthermore, in Section 6 it has been shown how the location of the plasma boundary can be found by fitting a solution of $\mathcal{L}^*\chi = 0$ to the external measurements. These two topics demonstrate the need for solutions to the homogeneous equilibrium equation, and in particular for one or more families of solutions that are complete on some domain. Whereas the evaluation of moments of the current density requires only interior solutions, valid throughout the region Ω, for the identification of the plasma boundary also exterior solutions, corresponding to a current density inside Ω are required.

In the important practical case when the magnetic permeability is constant throughout Ω, the equations reduce to $\Delta^*\chi = 0$ and $\Delta(r^{-1}\xi) = 0$, and analytical solutions can be found. This Appendix provides solutions in the form of polynomials and other elementary functions in r and z, and solutions obtained by separation of variables in cylindrical, spherical, and toroidal coordinates. In each case pairs of conjugate functions χ and ξ are given. As $\Delta^*\chi = r\Delta(r^{-1}\chi) - r^{-2}\chi$, it is seen that each solution to $\Delta^*\chi = 0$ corresponds to an "$m = \pm 1$" solution, $r^{-1}\chi$, to the three-dimensional Laplace equation, where m is the toroidal mode number. Of course, $r^{-1}\xi$ is an "$m = 0$" solution to the Laplace equation.

Solutions involving elementary functions. The operators Δ^* and Δ both have the property that they will transform a homogeneous polynomial of degree n in r and z into a homogeneous polynomial of degree $n - 2$. It follows that, if $\Delta^*\chi = 0$ and if χ admits a power series expansion about the origin, then by collecting terms of like order, a family of homogeneous polynomial solutions to the homogeneous equilibrium equation will be generated. Such a family may therefore be expected to exist, and indeed by elementary analysis it is found that the following pairs (χ_n, ξ_n) are all solutions to $\Delta^*\chi = 0$ and to $\Delta(r^{-1}\xi) = 0$, and satisfy the conjugacy relation, Eq. (2.2) or (2.4).

$$
\begin{aligned}
\chi_0 &= 1\,, & \xi_0 &= 0 \\
\chi_1 &= 0\,, & \xi_1 &= r \\
\chi_2 &= \frac{1}{2}r^2\,, & \xi_2 &= rz \\
\chi_3 &= r^2 z\,, & \xi_3 &= rz^2 - \frac{1}{2}r^3 \\
\chi_4 &= \frac{3}{2}r^2z^2 - \frac{3}{8}r^4\,, & \xi_4 &= rz^3 - \frac{3}{2}r^3 z \\
\chi_5 &= 2r^2z^3 - \frac{3}{2}r^4 z\,, & \xi_5 &= rz^4 - 3r^3z^2 + \frac{3}{8}r^5 \\
\chi_6 &= \frac{5}{2}r^2z^4 - \frac{15}{4}r^4z^2 + \frac{5}{16}r^6\,, & \xi_6 &= rz^5 - 5r^3z^3 + \frac{15}{8}r^5 z
\end{aligned}
\tag{B.1}
$$

etc. The general formula, for $n > 0$ is,

$$\begin{cases} \chi_n = \displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor -1} (-4)^{-k} \frac{(n-1)!/2}{k!\,(k+1)!\,(n-2k-2)!} r^{2k+2} z^{n-2k-2}, \\ \xi_n = \displaystyle\sum_{k=0}^{\lceil n/2 \rceil -1} (-4)^{-k} \frac{(n-1)!}{(k!)^2\,(n-2k-1)!} r^{2k+1} z^{n-2k-1}. \end{cases} \tag{B.2}$$

The solution for $n = 0$ does not fit well into this scheme. The moments for $n = 2$, $n = 4$, and $n = 6$ were given by Zakharov and Shafranov [**27**], who employ a different normalization of the functions. The motivation for the present choice of normalization will become clear when we discuss the moments obtained by separation of variables in spherical coordinates. Notice that the relevant Eq. (61) of [**27**] contains two errors, only one of which is obvious.

A related family of elementary solutions can be found by allowing a factor $\ln r$. There results the sequence:

$$\begin{aligned} \chi^l_1 &= -z\,, & \xi^l_1 &= r\ln r, \\ \chi^l_2 &= \chi_2 \ln r - \frac{1}{2}z^2 - \frac{1}{4}r^2, & \xi^l_2 &= \xi_2 \ln r, \\ \chi^l_3 &= \chi_3 \ln r - \frac{1}{3}z^3 - \frac{1}{2}r^2 z\,, & \xi^l_3 &= \xi_3 \ln r + \frac{1}{2}r^3, \\ \chi^l_4 &= \chi_4 \ln r - \frac{1}{4}z^4 - \frac{3}{4}r^2 z^2 + \frac{15}{32}r^4, & \xi^l_4 &= \xi_4 \ln r + \frac{3}{2}r^3 z, \end{aligned} \tag{B.3}$$

etc. The general formula, for $n > 0$ is,

$$\begin{cases} \chi^l_n = \displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor -1} (-4)^{-k} \frac{(n-1)!/2}{k!\,(k+1)!\,(n-2k-2)!} \\ \qquad\qquad \times\, r^{2k+2} z^{n-2k-2} \left(\ln r - \displaystyle\sum_{j=1}^{k} \frac{1}{j} - \frac{1}{2(k+1)}\right) - \frac{1}{n} z^n \\ \xi^l_n = \displaystyle\sum_{k=0}^{\lceil n/2 \rceil -1} (-4)^{-k} \frac{(n-1)!}{(k!)^2\,(n-2k-1)!} \\ \qquad\qquad \times\, r^{2k+1} z^{n-2k-1} \left(\ln r - \displaystyle\sum_{j=1}^{k} \frac{1}{j}\right) \end{cases} \tag{B.4}$$

Other elementary solutions can be found by allowing a power of $\sqrt{r^2+z^2}$. For $n > 1$ the pair

$$\begin{cases} \chi = \dfrac{n}{1-n}(r^2+z^2)^{\frac{1}{2}-n} \chi_n \\ \xi = (r^2+z^2)^{\frac{1}{2}-n} \xi_n \end{cases} \tag{B.5}$$

also provides a solution.

The polynomial solutions can be combined in a form that yields approximations to the lowest order multipole moments about an expansion point $(R_0, 0)$, for arbitrary $R_0 > 0$:

$$f_n^{(even)} = (-R_0)^n 2^{-n}\left[\chi_0 - \sum_{m=1}^{n}\left(\frac{2}{R_0}\right)^{2m}\frac{n!\,m!}{(2m)!\,(n-m)!}\chi_{2m}\right] \tag{B.6}$$

$$f_n^{(odd)} = (-R_0)^{n-1} 2^{-n}\sum_{m=0}^{n}\left(\frac{2}{R_0}\right)^{2m}\frac{n!\,m!}{(2m)!\,(n-m)!}\chi_{2m+1} \tag{B.7}$$

The functions $g_n^{(even)}$ and $g_n^{(odd)}$ are similarly defined with replacement of χ by ξ throughout. In leading order, $f_n^{(even)} + i f_n^{(odd)} \simeq (x+iz)^n$ and $g_n^{(even)} + i g_n^{(odd)} \simeq -i(x+iz)^n$, where $x = r - R_0$ and $i = \sqrt{-1}$. On the symmetry plane, $z = 0$, the exact formulae are

$$\begin{aligned} f_n^{(even)} &= (1 + x/2R_0)^n x^n, \\ g_n^{(odd)} &= -(1 + x/R_0)(1 + x/2R_0)^n x^n, \end{aligned} \tag{B.8}$$

while $g_n^{(even)}$ and $f_n^{(odd)}$ vanish for $z = 0$. We do not have a simple formula for these polynomials that is valid also for $z \neq 0$.

The corresponding approximations to the multipole moments about an arbitrary expansion point (R_0, Z_0) are obtained by substituting $\chi(r, z - Z_0)$ for χ in Eqs. (B.6) and (B.7), together with a similar substitution for ξ, and then employing the representations,

$$\begin{aligned} \chi_n(r, z - Z_0) &= \sum_{k=1}^{n}(-Z_0)^{n-k}\frac{(n-1)!}{(n-k)!\,(k-1)!}\chi_k(r, z), \\ \xi_n(r, z - Z_0) &= \sum_{k=1}^{n}(-Z_0)^{n-k}\frac{(n-1)!}{(n-k)!\,(k-1)!}\xi_k(r, z). \end{aligned} \tag{B.9}$$

As the multipoles $(x + iy)^n$, where $x = r - R_0$ and $y = z - Z_0$, form a complete family of solutions to the Laplace equation on any disk centered at (R_0, Z_0), one may be led to assume that the polynomial solutions given in Eq. (B.2) must also form a complete family on such a disk. This is not the case, however, and the fallacy in the reasoning is that the approximation of Eqs. (B.6) and (B.7) to the real and imaginary parts of $(x + iy)^n$ is not uniform in n. We will see later that the polynomial solutions correspond to a certain family of solutions obtained by separation of variables in spherical coordinates. These polynomial solutions form a complete family on any circular region centered at the origin, which may be of interest for compact toroids or for spheromaks, but hardly for tokamak studies. It has been suggested [**30**] that the polynomial solutions may be combined with solutions involving negative powers of ρ (where $\rho = \sqrt{r^2 + z^2}$), as in Eq. (B.5), to form a complete family. Such a family, however, will only be complete on a region of the form $\rho_a < \rho < \rho_b$, which is not of interest. Useful complete families of analytical solutions are only obtained by separation of variables in toroidal or in cylindrical coordinates.

Solutions from separation in cylindrical coordinates. Separated variables solutions to the Laplace equation in cylindrical coordinates involve Bessel functions and modified Bessel functions of r together with hyperbolic and trigonometric functions of z. To obtain separated solutions to $\Delta^*\chi = 0$ or to $\Delta(r^{-1}\xi) = 0$ the indicated ansatz is $\chi = rf(r)g(z)$ or $\xi = rf(r)g(z)$, which leads to the pair of ordinary differential equations:

$$\begin{cases} r^2 f'' + rf' + (\kappa r^2 - m^2)f = 0 \\ g'' - \kappa g = 0 \end{cases} \tag{B.10}$$

where $m^2 = 1$ for the solution χ, and $m^2 = 0$ for ξ. There are three classes of solutions, depending upon the choice of sign of the separation constant κ.

(a) $\kappa = \lambda^2$, with $\lambda > 0$. Then,

$$\begin{cases} \chi = r(c_1 J_1(\lambda r) + c_2 Y_1(\lambda r))(c_3 \cosh(\lambda z) + c_4 \sinh(\lambda z)) \\ \xi = r(c_1 J_0(\lambda r) + c_2 Y_0(\lambda r))(c_3 \sinh(\lambda z) + c_4 \cosh(\lambda z)) \end{cases} \tag{B.11}$$

J and Y are Bessel functions, in the notation of [**18**], [**19**]. Here and elsewhere c_1, c_2, c_3, and c_4 are arbitrary real constants, and the functions χ and ξ given as a pair are conjugate functions in the sense of Eq. (2.2).

(b) $\kappa = -\lambda^2$, with $\lambda > 0$. Then,

$$\begin{cases} \chi = r(c_1 I_1(\lambda r) - c_2 K_1(\lambda r))(c_3 \cos(\lambda z) - c_4 \sin(\lambda z)) \\ \xi = r(c_1 I_0(\lambda r) + c_2 K_0(\lambda r))(c_3 \sin(\lambda z) + c_4 \cos(\lambda z)) \end{cases} \tag{B.12}$$

I and K are modified Bessel functions. The families (a) and (b), restricted to a discrete set of values of the separation constant, may be used to provide a system that is complete on any desired rectangular region ($r_a < r < r_b$ and $z_a < z < z_b$) in the right half plane.

(c) $\kappa = 0$. This yields again the simplest polynomial solutions,

$$\begin{aligned} \chi &= (c_1 + c_2 r^2)(c_3 + c_4 z) \\ \xi &= (d_1 r + d_2 r \ln r)(d_3 + d_4 z) \end{aligned} \tag{B.13}$$

These are not directly conjugate to each other, but Eq. (B.1) and Eq. (B.3) may be consulted to find the conjugate functions.

All these solutions may be expanded in terms of the solutions (χ_n, ξ_n) and (χ_n^l, ξ_n^l) given earlier, by the use of the following formulae.

$$rJ_0(\lambda r)\exp(\lambda z) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}\xi_n$$

$$rJ_1(\lambda r)\exp(\lambda z) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}\chi_n$$

$$rY_0(\lambda r)\exp(\lambda z) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}\Big(\xi_n^l + (\gamma + \ln(\lambda/2))\xi_n\Big) - \frac{2}{\pi}\lambda^{-1}\xi_0$$

$$rY_1(\lambda r)\exp(\lambda z) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}\Big(\chi_n^l + (\gamma + \ln(\lambda/2))\chi_n\Big) - \frac{2}{\pi}\lambda^{-1}\chi_0$$

$$rI_0(\lambda r)\exp(i\lambda z) = \sum_{n=1}^{\infty}\frac{(i\lambda)^{n-1}}{(n-1)!}\xi_n$$

$$rI_1(\lambda r)\exp(i\lambda z) = -i\sum_{n=1}^{\infty}\frac{(i\lambda)^{n-1}}{(n-1)!}\chi_n$$

$$rK_0(\lambda r)\exp(i\lambda z) = -\sum_{n=1}^{\infty}\frac{(i\lambda)^{n-1}}{(n-1)!}\Big(\xi_n^l + (\gamma + \ln(\lambda/2))\xi_n\Big)$$

$$rK_1(\lambda r)\exp(i\lambda z) = -i\sum_{n=1}^{\infty}\frac{(i\lambda)^{n-1}}{(n-1)!}\Big(\chi_n^l + (\gamma + \ln(\lambda/2))\chi_n\Big)$$

The solutions obtained by separation in cylindrical coordinates may be appropriate for very elongated (belt-shaped) configurations, or for devices with a rectangular cross-section, and may also be convenient for certain modelling studies involving a rectangular grid.

Solutions from separation in spherical coordinates. In spherical coordinates, $r = \rho\sin\vartheta$, $z = \rho\cos\vartheta$, the representation for Δ^* is,

$$\Delta^*\chi = \frac{\partial^2\chi}{\partial\rho^2} + \frac{1}{\rho^2}\sin\vartheta\,\frac{\partial}{\partial\vartheta}\Big(\frac{1}{\sin\vartheta}\,\frac{\partial\chi}{\partial\vartheta}\Big), \tag{B.14}$$

and the relation between the conjugate functions χ and ξ is,

$$\begin{cases} \dfrac{\partial}{\partial\rho}(r^{-1}\xi) = r^{-1}\rho^{-1}\,\dfrac{\partial\chi}{\partial\vartheta} \\ \dfrac{\partial}{\partial\vartheta}(r^{-1}\xi) = -r^{-1}\rho\,\dfrac{\partial\chi}{\partial\rho} \end{cases} \tag{B.15}$$

Starting from the ansatz $\chi = \rho\sin\vartheta f(\rho)g(\vartheta)$, and a similar ansatz for ξ, one obtains for $\Delta^*\chi = 0$ and $\Delta(r^{-1}\xi) = 0$ the systems,

$$\begin{cases} \rho^2 f'' + 2\rho f' - \kappa f = 0 \\ g''(\vartheta) + \dfrac{\cos\vartheta}{\sin\vartheta}g'(\vartheta) + \Big(\kappa - \dfrac{m^2}{(\sin\vartheta)^2}\Big)g(\vartheta) = 0 \end{cases} \tag{B.16}$$

with separation constant $\kappa \in \mathbf{R}$. $m^2 = 1$ corresponds to the equation for χ, and $m^2 = 0$ corresponds to ξ. Three cases must be distinguished: $\kappa > -\frac{1}{4}$, $\kappa < -\frac{1}{4}$, and $\kappa = -\frac{1}{4}$.

In the case $\kappa > -\frac{1}{4}$, there exists a real $\alpha > \frac{1}{2}$ such that $\kappa = \alpha(\alpha - 1)$. The first equation then admits the two independent solutions $f = \rho^{\alpha-1}$ and $f = \rho^{-\alpha}$. The second equation admits for almost every value of α (namely α not an integer in the range $1-m \le \alpha \le m$) the following two independent solutions,

$$\begin{aligned} g(\vartheta) &= \quad P^m_{\alpha-1}(\cos\vartheta) \qquad \text{(notation of [18], [19])} \\ &= (-1)^m P^m_{\alpha-1}(\cos\vartheta) \qquad \text{(notation of [89])} \\ &= (-1)^m \frac{\Gamma(m+\alpha)}{m!\,\Gamma(\alpha-1)}(\tan\frac{1}{2}\vartheta)^{-m} F\Big(1-\alpha,\alpha;1+m;(\sin\frac{1}{2}\vartheta)^2\Big) \\ &= (-1)^m \frac{\Gamma(m+\alpha)}{2^m m!\,\Gamma(\alpha-1)}(\sin\vartheta)^m F\Big(\frac{1-\alpha+m}{2},\frac{\alpha+m}{2};1+m;(\sin\vartheta)^2\Big) \end{aligned} \tag{B.17}$$

and

$$\begin{aligned} g(\vartheta) &= \quad Q^m_{\alpha-1}(\cos\vartheta) \qquad \text{(notation of [18], [19])} \\ &= (-1)^m Q^m_{\alpha-1}(\cos\vartheta) \qquad \text{(notation of [89])} \end{aligned} \tag{B.18}$$

In the sequel we follow the notation of [**18**] and [**19**]. These solutions are invariant under the replacement of α by $1-\alpha$.

As a result one obtains the following two families of conjugate solutions to the equations $\Delta^*\chi = 0$ and $\Delta(r^{-1}\xi) = 0$:

$$\begin{cases} \chi(\rho,\vartheta) = -\dfrac{1}{\alpha}\rho^\alpha \sin\vartheta P^1_{\alpha-1}(\cos\vartheta) \\ \xi(\rho,\vartheta) = \rho^\alpha \sin\vartheta P^0_{\alpha-1}(\cos\vartheta) \end{cases} \tag{B.19}$$

and

$$\begin{cases} \chi(\rho,\vartheta) = -\rho^\alpha \sin\vartheta Q^1_{\alpha-1}(\cos\vartheta) \\ \xi(\rho,\vartheta) = \alpha\rho^\alpha \sin\vartheta Q^0_{\alpha-1}(\cos\vartheta) \end{cases} \tag{B.20}$$

for $\alpha \in \mathbf{R}$, $\alpha \neq \frac{1}{2}$. The limiting values of these solutions as $\alpha \to 0$ exist and are given by

$$\begin{cases} \chi(\rho,\vartheta) = \cos\vartheta - 1 \\ \xi(\rho,\vartheta) = \sin\vartheta \end{cases} \tag{B.21}$$

and

$$\begin{cases} \chi(\rho,\vartheta) = 1 \\ \xi(\rho,\vartheta) = 0 \end{cases} \tag{B.22}$$

For $\alpha = 1$ the solutions are,

$$\begin{cases} \chi(\rho,\vartheta) = 0 \\ \xi(\rho,\vartheta) = \rho\sin\vartheta \end{cases} \tag{B.23}$$

and

$$\begin{cases} \chi(\rho,\vartheta) = \rho \\ \xi(\rho,\vartheta) = -\rho\sin\vartheta\ln(\tan\dfrac{1}{2}\vartheta) \end{cases} \tag{B.24}$$

Notice that for $\alpha = n$ and $n = 1, 2, 3, \ldots$ the solution pair defined by Eq. (B.20) coincides with the polynomial solutions (χ_n, ξ_n) given earlier. This is the motivation for our normalization in Eq. (B.2).

In the case $\kappa < -\frac{1}{4}$ one may set $\kappa = (i\alpha + \frac{1}{2})(i\alpha - \frac{1}{2})$ for some real $\alpha > 0$. The differential equation for f then admits the solutions $f = \sin(\alpha \ln \rho)/\sqrt{\rho}$ and $f = \cos(\alpha \ln \rho)/\sqrt{\rho}$. Solutions to the equation for g are the conical functions, viz. Legendre functions of degree $i\alpha - \frac{1}{2}$. We thus obtain the solutions,

$$\begin{cases} \chi(\rho, \vartheta) = -2\sqrt{\rho}\,(c_1 \sin(\alpha \ln \rho) + c_2 \cos(\alpha \ln \rho)) \sin\vartheta\, P^1_{i\alpha - \frac{1}{2}}(\cos\vartheta) \\ \xi(\rho, \vartheta) = \sqrt{\rho}\,(d_1 \sin(\alpha \ln \rho) + d_2 \cos(\alpha \ln \rho)) \sin\vartheta\, P^0_{i\alpha - \frac{1}{2}}(\cos\vartheta) \end{cases} \tag{B.25}$$

and

$$\begin{cases} \chi(\rho, \vartheta) = -2\sqrt{\rho}(c_1 \sin(\alpha \ln \rho) + c_2 \cos(\alpha \ln \rho)) \sin\vartheta\, Q^1_{i\alpha - \frac{1}{2}}(\cos\vartheta)) \\ \xi(\rho, \vartheta) = \sqrt{\rho}\,(d_1 \sin(\alpha \ln \rho) + d_2 \cos(\alpha \ln \rho)) \sin\vartheta\, Q^0_{i\alpha - \frac{1}{2}}(\cos\vartheta) \end{cases} \tag{B.26}$$

where χ and ξ are conjugate functions provided that $d_1 = c_1 - 2\alpha c_2$ and $d_2 = c_2 + 2\alpha c_1$.

The case $\kappa = -\frac{1}{4}$ can be dealt with by taking the appropriate limits of the solutions found earlier. A complete system of solutions for this case is given by,

$$\begin{cases} \chi(\rho, \vartheta) = -2\sqrt{\rho}\,(c_1 + c_2 \ln \rho)\, \sin\vartheta\, (c_3 P^1_{-\frac{1}{2}}(\cos\vartheta) + c_4 Q^1_{-\frac{1}{2}}(\cos\vartheta)) \\ \xi(\rho, \vartheta) = \sqrt{\rho}\,(c_1 + c_2(\ln \rho + 2))\, \sin\vartheta\, (c_3 P^0_{-\frac{1}{2}}(\cos\vartheta) + c_4 Q^0_{-\frac{1}{2}}(\cos\vartheta)) \end{cases} \tag{B.27}$$

Solutions from separation in toroidal coordinates. For the study of tokamak configurations the most generally useful family of analytical solutions is obtained by separation of variables in a toroidal coordinate system. These coordinates are defined by,

$$\begin{aligned} r &= r_0 \sinh\varsigma/(\cosh\varsigma - \cos\eta) \\ z - z_0 &= r_0 \sin\eta/(\cosh\varsigma - \cos\eta) \end{aligned} \tag{B.28}$$

(r_0, z_0) being the location of the singularity of the coordinate system, at $\varsigma \to \infty$. The operator Δ^* is,

$$\begin{aligned} \Delta^*\chi = \frac{\cosh\varsigma - \cos\eta}{r_0} \Big(\sinh\varsigma \frac{\partial}{\partial\varsigma}\Big(\frac{\cosh\varsigma - \cos\eta}{r_0 \sinh\varsigma} \frac{\partial\chi}{\partial\varsigma}\Big) \\ + \frac{\partial}{\partial\eta}\Big(\frac{\cosh\varsigma - \cos\eta}{r_0} \frac{\partial\chi}{\partial\eta}\Big)\Big) \end{aligned} \tag{B.29}$$

and the connection between the conjugate functions χ and ξ is,

$$\begin{cases} \dfrac{\partial}{\partial\varsigma}(r^{-1}\xi) = r^{-1} \dfrac{\partial\chi}{\partial\eta} \\ \dfrac{\partial}{\partial\eta}(r^{-1}\xi) = -r^{-1} \dfrac{\partial\chi}{\partial\varsigma} \end{cases} \tag{B.30}$$

As $\Delta(r^{-1}\chi) = 0$ separates with the substitution $r^{-1}\chi = (\cosh\varsigma - \cos\eta)^{\frac{1}{2}} f(\varsigma)g(\eta)$, one is motivated to try for $\Delta^*\chi = 0$ the ansatz,

$$\chi = \frac{r_0 \sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}} f(\varsigma)g(\eta).$$

and a similar ansatz for ξ in the equation $\Delta(r^{-1}\xi) = 0$. This leads to the separated equations,

$$\begin{cases} \dfrac{1}{\sinh\varsigma}\dfrac{\partial}{\partial\varsigma}\left(\sinh\varsigma\,\dfrac{\partial f}{\partial\varsigma}\right) - \dfrac{m^2}{(\sinh\varsigma)^2} f - \left(n^2 - \dfrac{1}{4}\right) f = 0 \\ \dfrac{\partial^2 g}{\partial\eta^2} + n^2 g = 0 \end{cases} \tag{B.31}$$

where n^2 is the separation constant, and $m^2 = 1$ for the solution χ, $m^2 = 0$ for the solution ξ. We assume that the point $(r = r_0, z = z_0)$ should lie inside the domain of the solution, so that η is a periodic coordinate, and n must be an integer.

The solutions for $g(\eta)$ are known. Solutions to the equation for $f(\varsigma)$, when $m = 0$ or $m = 1$, are of two forms:

$$\begin{aligned} f(\varsigma) = {} & P^m_{n-\frac{1}{2}}(\cosh\varsigma) \qquad \text{(notation of [18], [19])} \\ = {} & (-i)^m P^m_{n-\frac{1}{2}}(\cosh\varsigma) \qquad \text{(notation of [89])} \\ = {} & \frac{\Gamma(n+m+\frac{1}{2})}{2^m m!\,\Gamma(n-m+\frac{1}{2})}(\tanh\varsigma)^m(\cosh\varsigma)^{-n-\frac{1}{2}} \\ & \times F\Big(\frac{n+m+\frac{3}{2}}{2}, \frac{n+m+\frac{1}{2}}{2}; m+1; (\tanh\varsigma)^2\Big) \\ = {} & \frac{\Gamma(n+m+\frac{1}{2})}{2^{2m} m!\,\Gamma(n-m+\frac{1}{2})}(1-e^{-2\varsigma})^m(\exp\varsigma)^{-n-\frac{1}{2}} \\ & \times F\Big(n+m+\frac{1}{2}, m+\frac{1}{2}; 2m+1; 1-e^{-2\varsigma}\Big) \end{aligned} \tag{B.32}$$

which is singular at $\varsigma \to \infty$ $(r = a,\ z = 0)$, and

$$\begin{aligned} f(\varsigma) = {} & Q^m_{n-\frac{1}{2}}(\cosh\varsigma) \qquad \text{(notation of [18], [19])} \\ = {} & (-1)^m Q^m_{n-\frac{1}{2}}(\cosh\varsigma) \qquad \text{(notation of [89])} \\ = {} & (-1)^m\sqrt{\pi}\frac{\Gamma(n+m+\frac{1}{2})}{2^{n+\frac{1}{2}}\Gamma(n+1)}(\tanh\varsigma)^m(\cosh\varsigma)^{-n-\frac{1}{2}} \\ & \times F\Big(\frac{n+m+\frac{3}{2}}{2}, \frac{n+m+\frac{1}{2}}{2}; n+1; (\cosh\varsigma)^{-2}\Big) \\ = {} & (-1)^m\sqrt{\pi}\,\frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n+1)}(1-e^{-2\varsigma})^m(\exp\varsigma)^{-n-\frac{1}{2}} \\ & \times F\Big(n+m+\frac{1}{2}, m+\frac{1}{2}; n+1; e^{-2\varsigma}\Big) \end{aligned} \tag{B.33}$$

which is regular throughout Ω. In these formulae, F denotes the ${}_2F_1$ hypergeometric function. In the sequel we will stay with the notation of [**18**] and [**19**]. Notice that both $P^m_{n-\frac{1}{2}}$ and $Q^m_{n-\frac{1}{2}}$ are invariant under the replacement of n by $-n$.

It may appear strange that the definitions of P^m_ν in our references should differ by a factor $(-1)^m$ when the argument is $\cos\vartheta$ and by a factor $(-i)^m$ when the argument is $\cosh\varsigma$. This is nevertheless the case, and is related to the presence of branch points at ± 1. Other authors have argued the merits of (re-)defining the special functions in such a way that they are free of unnecessary singularities and branch points.

A complete family of solutions to $\Delta^*\chi = 0$ in any toroidal region of the form $\varsigma \geq \varsigma_0$ is therefore obtained with the functions,

$$\begin{aligned} \chi^c_n &= \frac{r_0 \sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}} Q^1_{n-\frac{1}{2}}(\cosh\varsigma)\cos(n\eta), \qquad n \geq 0 \\ \chi^s_n &= \frac{r_0 \sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}} Q^1_{n-\frac{1}{2}}(\cosh\varsigma)\sin(n\eta), \qquad n \geq 1 \end{aligned} \tag{B.34}$$

The conjugate functions, ξ, must satisfy $\Delta(r^{-1}\xi) = 0$, and a complete family of solutions to this equation is provided by the functions,

$$\begin{aligned} \xi^c_n &= \frac{r_0 \sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}} Q^0_{n-\frac{1}{2}}(\cosh\varsigma)\cos(n\eta), \qquad n \geq 0 \\ \xi^s_n &= \frac{r_0 \sinh\varsigma}{\sqrt{\cosh\varsigma - \cos\eta}} Q^0_{n-\frac{1}{2}}(\cosh\varsigma)\sin(n\eta), \qquad n \geq 1 \end{aligned} \tag{B.35}$$

It turns out, however, that the conjugacy relation, Eq. (B.30), does not allow a simple one-to-one correspondance between the members of Eq. (B.34) and those of Eq. (B.35). The conjugate function to a single member of Eq. (B.34) can be an infinite series from Eq. (B.35) and vice versa. The following definition for the basis functions χ and ξ has been chosen in order to restore the symmetry.

$$\begin{cases} \chi^{(1)}_n(\varsigma,\eta) = \chi^c_n - \chi^c_{n+1} \\ \xi^{(1)}_n(\varsigma,\eta) = -(n+\frac{1}{2})(\xi^s_n - \xi^s_{n+1}) \end{cases} \tag{B.36}$$

and

$$\begin{cases} \chi^{(2)}_n(\varsigma,\eta) = \chi^s_n - \chi^s_{n+1} \\ \xi^{(2)}_n(\varsigma,\eta) = (n+\frac{1}{2})(\xi^c_n - \xi^c_{n+1}) \end{cases} \tag{B.37}$$

for $n \geq 0$, with the understanding that $\chi^s_0 = 0$ and $\xi^s_0 = 0$.

Appendix C. Determination of the Current Distribution From the Flux Surface Structure

An interesting different approach to plasma current profile identification has been proposed by Christiansen and Taylor [**68**], who discuss the possibility of determining the current profile from purely geometric information about the shape of the magnetic surfaces, as may be available from electron temperature measurements for instance. They conclude that for a toroidal configuration this determination is always possible, but the analysis below shows their argument to be incorrect, and clarifies under which circumstances the procedure of [**68**] will be well-conditioned. This analysis will be seen to be of interest for the question of uniqueness of a fit to the magnetic measurements, and is also relevant to MHD equilibrium determination from an extended set of diagnostics.

A differential equation for the flux function. Assume that some function σ, known to be a flux-surface quantity, has been measured over the poloidal cross-section of the plasma. Thus the flux surface structure of the plasma is known as the isocontours of σ, but the functional relationship between ψ and σ is as yet undetermined. Notice that σ must for consistency satisfy a certain differential equation: as

$$\begin{aligned}\Delta^* \sigma &= \frac{d\sigma}{d\psi}\Delta^*\psi + \frac{d^2\sigma}{d\psi^2}|\nabla\psi|^2 \\ &= -\frac{d\sigma}{d\psi}F\frac{dF}{d\psi} - \mu_0\frac{d\sigma}{d\psi}\frac{dp}{d\psi}r^2 + \frac{d^2\sigma}{d\psi^2}|\nabla\sigma|^2\Big/\left(\frac{d\sigma}{d\psi}\right)^2, \end{aligned} \tag{C.1}$$

where the equilibrium relation, Eq. (1.16), has been used, it follows that

$$\Delta^*\sigma = -\alpha - \beta r^2 - \gamma|\nabla\sigma|^2, \tag{C.2}$$

where α, β, and γ are flux surface quantities; specifically, $\alpha = \sigma' F F'$, $\beta = \mu_0 \sigma' p'$, and $\gamma = -\sigma''/(\sigma')^2$, where $'$ denotes $\partial/\partial\psi$. The representation of $\Delta^*\sigma$ in Eq. (C.2) is unique, and the functions α, β, and γ are determinable from knowledge of the function σ, provided that $|\nabla\sigma|^2$ is not, on any flux surface, linearly dependent on 1 and r^2.

One may then consider ψ as a function of σ, and derive from $\gamma = -\sigma''/(\sigma')^2$ the differential equation,

$$\frac{d}{d\sigma}\ln\left(\frac{d\psi}{d\sigma}\right) = \gamma(\sigma). \tag{C.3}$$

After two integrations ψ is obtained as a function of σ, and therefore also as a function of (r, z). Of the two free constants arising from the integrations, one corresponds to the indeterminacy of the total current (indeterminate for this geometric information on its own), and the other is the irrelevant constant term in ψ. It follows that, except in the degenerate case to be discussed below, the current profile in a toroidal configuration can be determined from knowledge of the geometry of the flux surfaces together with a measurement of the total plasma current.

A degenerate configuration. The above procedure is related to the program presented in [**68**], although in that work a different and not unambiguous route is followed to obtain the function γ in Eq. (C.3). Clearly the procedure fails completely if over an extended range of values of σ there is a linear dependence of $|\nabla\sigma|^2$ on 1 and r^2, that is, if $|\nabla\sigma|^2 = c_1(\sigma) + c_2(\sigma)r^2$ over a range of values of σ, for some flux functions c_1 and c_2. If this relation holds only on isolated flux surfaces or only approximately, then the procedure may still be feasible, but will be badly conditioned. These restrictions are also implicit in the derivation employed in [**68**].

The fact that this degenerate case, $|\nabla\psi|^2$ linearly dependent on 1 and r^2 over a range of flux surfaces, can actually occur in a toroidal system, was shown some years ago in a different context by Palumbo [**49**], [**50**] and recently rediscovered in the present context [**51**], [**52**]. An explicit construction of such an equilibrium is given in those references, and it is shown that the associated flux surface configuration is unique up to a scale factor. This construction is rather complicated; a much simpler argument [J.B. Taylor, private communication, August 1984] for the corresponding problem in straight geometry shows that there it is only for concentric circular flux surfaces that $|\nabla\psi|^2$ can be constant on each surface.

Notice that the family of equilibria in [**49**]–[**52**] also provides an example to show that in toroidal geometry the external magnetic measurements need not define uniquely the current profile, even if these measurements are made to arbitrary accuracy. A set of measurements that is consistent with one current profile from this family is automatically consistent with any other current profile that gives the same flux surface structure and the same total current.

Partial profile determination. The procedure of [**68**] becomes badly conditioned when $|\nabla\sigma|^2 \simeq c_1(\sigma)+c_2(\sigma)r^2$, and it is of interest to investigate whether partial information about the current profile can then still be obtained from the geometric information. Indeed, in such a degenerate, or in a nearly degenerate case, it follows from Eq. (C.2) that what can be determined correctly is not the three functions α, β, and γ, but only the combinations $\alpha + \gamma c_1$ and $\beta + \gamma c_2$, where c_1 and c_2 are known. Eliminating γ it is seen that the combination $c_1\beta - c_2\alpha$ is always measurable, or equivalently,

$$\eta = \sigma'(c_1\mu_0 p' - c_2 F F') \tag{C.4}$$

is measurable.

A limited study of the use of geometric information for current profile determination has been reported [**90**], but the authors were unaware of the limitations on this method of analysis, and it is not clear what to make of that work. The same comment applies to the discussion of current profile reconstruction given by Lao et al. [**62**].

Final remarks. We conclude this Appendix with some speculation as to what combination of diagnostic systems, magnetic and other, would be most suitable for an accurate current profile determination. The magnetic diagnostics measure the current I_t, the plasma position (r_c, z_c) as defined in Eqs. (2.10) and (2.11), the parameter

$\beta_I + l_i/2$, the location of the plasma boundary, and in good approximation also the position of the magnetic axis. For sufficiently shaped cross-section or with the aid of a diamagnetic flux measurement it is also possible to obtain separate estimates for β_I and l_i, but beyond this no information on the radial shape of the current profile is obtained.

The radial shape of the current profile does show up to some extent in Faraday rotation measurements [**64**]–[**67**]. Taken in isolation these are difficult to interpret, but it would seem that a limited number of such measurements in conjunction with the usual magnetic diagnostics should suffice for a rough identification of the shape of the current profile in the interior plasma. The Faraday rotation measurements can be shown to provide information on the quantity

$$h = r^2 \nabla \cdot (r^{-2} n_e \nabla \psi), \tag{C.5}$$

via the Radon transform. (n_e is the electron density).

That leaves the problem of obtaining in more detail the separate contributions from the $\mu_0 p'$ and the FF' terms in the current profile. Measurements of the pressure would suffice for that purpose, but only the electron component of the pressure is accurately measured. In principle though, some data on the flux surface geometry (most likely from electron temperature measurements) in conjunction with magnetic measurements and Faraday rotation measurements, could serve to separate these two contributions to the current, with the aid of Eq. (C.4). These diagnostics (magnetic, FIR interferometry and polarimetry, and electron cyclotron emission) are all suitable for near continuous monitoring of the plasma, making a successful effort at a consistent interpretation particularly worthwhile.

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Figure Captions

Fig. 1. Schematic view of the magnetic diagnostics and the field shaping coils on Doublet III.

Fig. 2. Schematic view of the magnetic diagnostics on JET.

Fig. 3. Schematic view of the magnetic diagnostics and the field shaping coils on ISX-B.

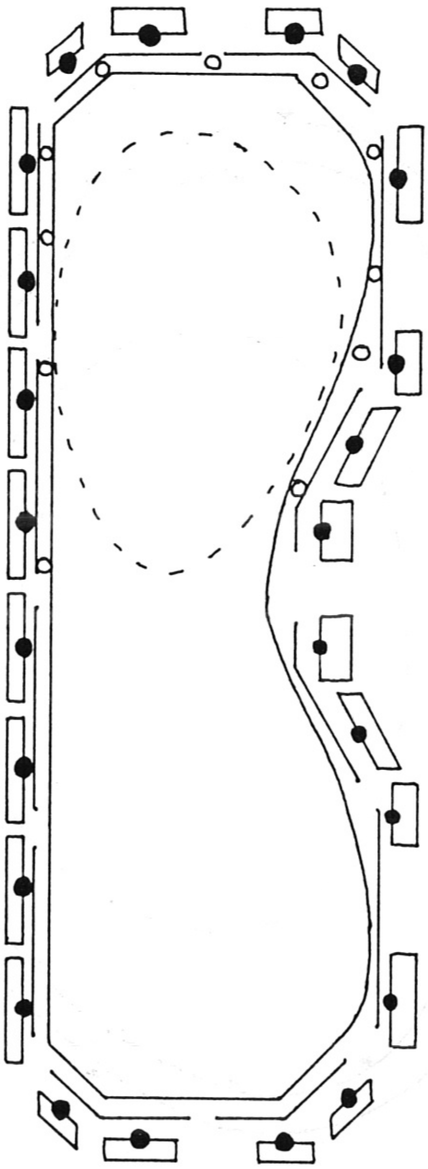

Fig. 1. Doublet III

- ● flux loop
- o magnetic probe
- — partial Rogowski coil
- ▯ field shaping coil

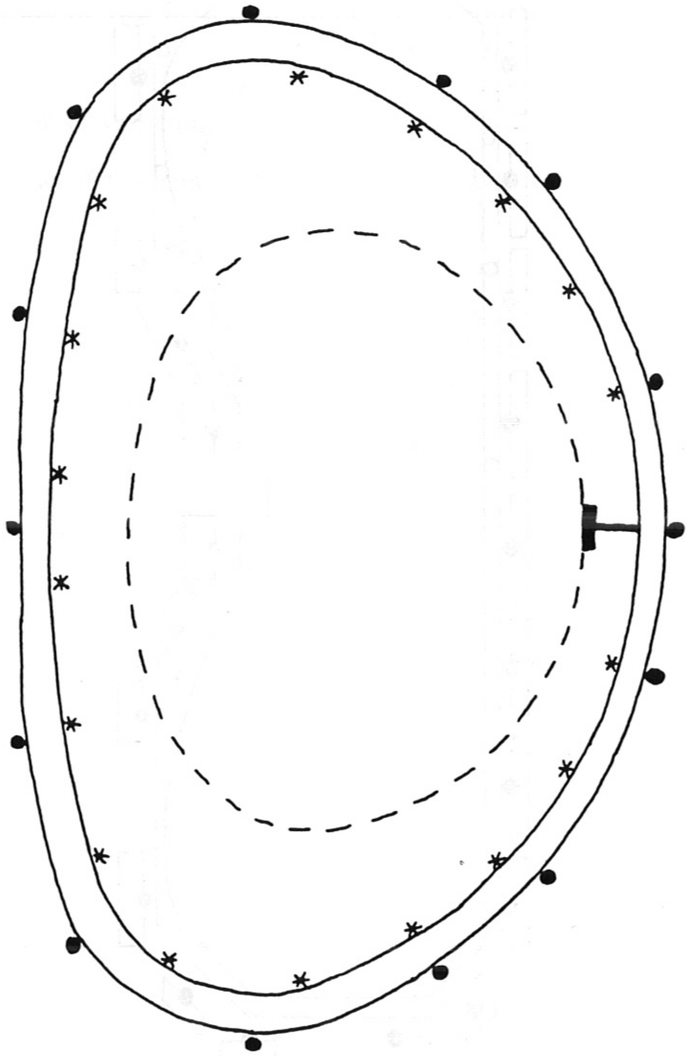

Fig. 2. JET

• flux loop

* magnetic probe

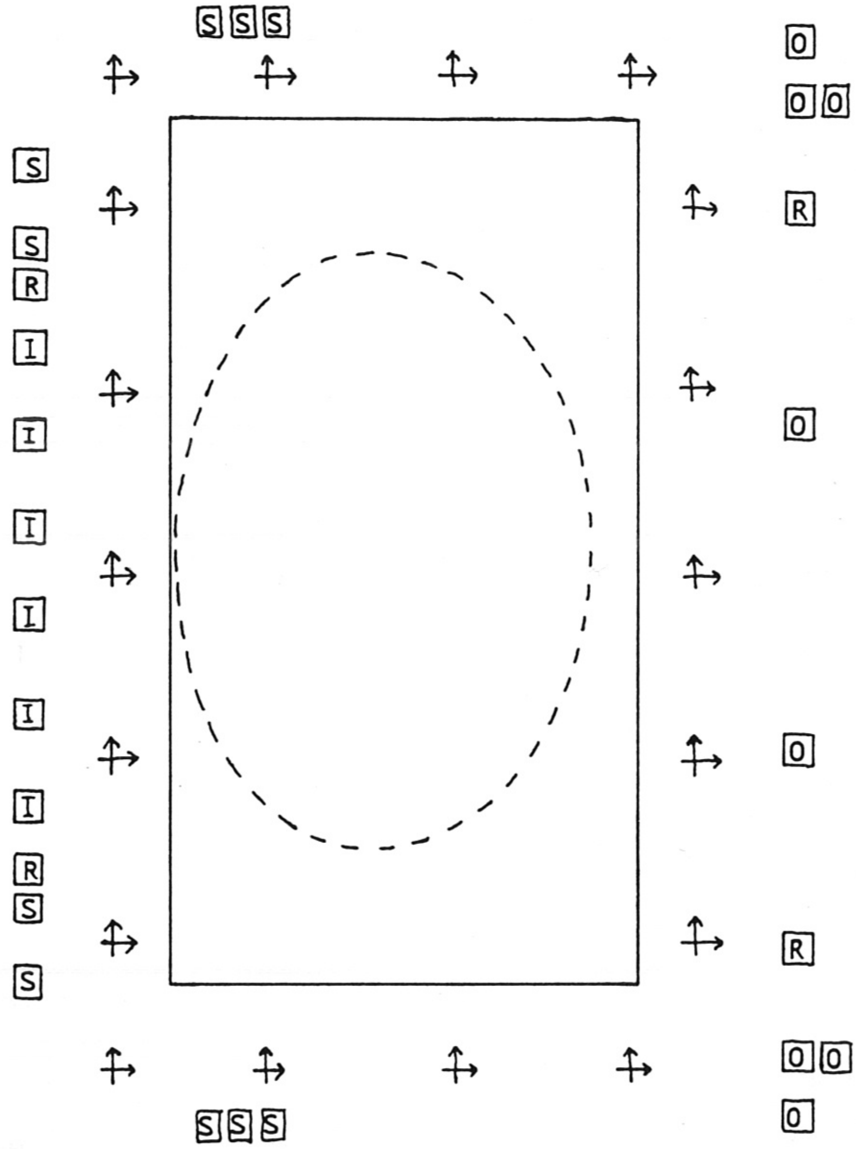

Fig. 3. ISX - B

↑ B_z magnetic probe

→ B_r magnetic probe

□ field shaping coil