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# Generalized Hamiltonians, Functional Integration and Statistics of Continuous Fluids and Plasmas

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Abstract:

Generalized Hamiltonian formalism including generalized Poisson brackets and Lie-Poisson brackets is presented in Sec. II. Gyroviscous magnetohydrodynamics is treated as a relevant example in Euler and Clebsch variables. Section III is devoted to a short review of functional integration containing the definition and a discussion of ambiguities and methods of evaluation. The main part of the contribution is given in Sec. IV, where some of the content of the previous sections is applied to Gibbs statistics of continuous fluids and plasmas. In particular, exact fluctuation spectra are calculated for relevant equations in fluids and plasmas.

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## I. Introduction

Hamiltonian formalism has had a basic role in quantum theory and statistical mechanics and many important applications in stability theory, adiabatic motion, perturbation theory etc. For almost all these purposes, in the case of both particles and vacuum fields, the introduction of a canonical phase space seemed to be appropriate. Non-canonical transformations were known but usually avoided. Continuous fluids and plasmas are the exception to the rule. They can be described by Lagrangian variables and through variational formulation by a canonical phase space, but at the expense of a complicated set of trajectories and their possible breakdown by, for example, shock waves. Euler variables can readily be interpreted and are convenient but do not allow a canonical phase space to be defined. They permit, however, a "generalized" Hamiltonian description in terms of generalized Poisson brackets [1] (GPB). If Euler variables are decomposed in Clebsch-like potentials, a canonical Hamiltonian formulation can be obtained in terms of those potentials but at the expense of introducing "gauge" freedom and ambiguities [2] in phase space.

Integrals over such a phase space of functions are functional integrals [3] similar to those encountered in quantum field theory and solid-state physics. Functional integrals are not proved, in general, to be uniquely defined objects except for some standard cases. In addition to this mathematical problem, there is the problem of removing the freedom of gauge and ambiguities from the integration domain.

Functional integration is essential not only for the formulation [4] of the statistical properties of continuous fluids and plasmas but also for the actual computation of, for example, the partition function or higher moments of the canonical Gibbs-distribution-like correlation functions. This yields a general formula [5] for the k-spectrum of linearized fluids or plasma systems. This formula is useful for small wave numbers but displays "ultraviolet" divergences for large k. For special nonlinear equations such as the Korteweg-de Vries (K-dV) equation an exact k-spectrum [6] is obtained free of divergences.

The paper is arranged as follows: Section II is devoted to generalized Hamiltonian formulation, with the emphasis on the case of gyroviscous magnetohydrodynamics (MHD). In Sec. III the problems related to functional integration are briefly discussed. Section IV uses Hamiltonian formulation and techniques of functional integration to obtain exact k-spectra for relevant fluid and plasma equations. Finally, the remaining problems are discussed in Sec. V.

## II. Generalized Hamiltonian Formulation

### (a) Discrete Case [1]

The most obvious examples of noncanonical Hamiltonian formulations are those obtained from canonical formulations via noncanonical transformations. They lead to the coordinate-free concept of symplectic forms [7] and to geometrization of dynamics. This covers essentially the case of massive particles. The rigid top [1,7], for example, is not included in this class. Its generalized Hamiltonian formulation leads to degeneracy and existence of a Casimir [1] invariant, as will be explained below. It can be shown, however, that for each value of the Casimir a symplectic leaf can be introduced.

GPB appeared for the first time in a text-book by Sudarshan and Mukunda [1], where essentially the discrete case is investigated as follows. Let $z^{\mu}, \mu = 1,2...N$ be the components of a vector z and let

$f(z)$ and $g(z)$ be two arbitrary functions of $z$. The GPB is defined as

$$[f,g] = \eta^{\mu\nu}(z)\frac{\partial f}{\partial z^\mu}\frac{\partial g}{\partial z^\nu} , \tag{1}$$

where $\eta^{\mu\nu}$ is an antisymmetric matrix obeying the condition

$$\eta^{\rho\lambda}\frac{\partial}{\partial z^\rho}\eta^{\mu\nu} + \eta^{\rho\nu}\frac{\partial}{\partial z^\rho}\eta^{\lambda\mu} + \eta^{\rho\mu}\frac{\partial}{\partial z^\rho}\eta^{\nu\lambda} = 0 \tag{2}$$

for all $\lambda,\mu,\nu$. Condition (2) ensures that Jacobi identity is verified by the GPB. For proof see, for example, ref. [8].

If $\eta^{\mu\nu}$ is nonsingular, i.e. $|\eta^{\mu\nu}| \neq 0$, which implies that N is even, then $\eta^{\mu\nu}$ defines a symplectic form. By means of the Darboux theorem a local transformation can be made to bring it to canonical form. The nonsingular case corresponds essentially to a system of massive particles expressed in general noncanonical coordinates.

If, however, the kinematics of a system can be expressed in terms of the elements of some continuous group as in the case of a rigid body spinning around a fixed point, the situation may change. The motion of the rigid body can be described [1] in terms of the motion of a body frame denoted by $S(t)$ with respect to a laboratory frame $S(0)$: $S(t) = TS(0)$, where T is an element of the group of rotations $O_3$. If $S_k$ are the 3 components of the angular momentum $\underline{J}$ with respect to the body frame $S(t)$, then the equation of motion can be written as

$$\dot{S}_k = [S_k, H(S)] \tag{3}$$

with
$$[f(S), g(S)] = -\varepsilon^{mnj} S_j \frac{\partial f}{\partial S_m}\frac{\partial g}{\partial S_n} \tag{4}$$

and
$$H = \frac{1}{2} I^{-1}_{mn} S_m S_n \ , \tag{5}$$

where $I_{mn}$ is the moment of inertia tensor. In this case $\eta^{\mu\nu} = -\varepsilon^{\mu\nu j} S_j$ is obviously singular because it is an odd (3x3) anti-symmetric matrix. It has one null eigenvector $S_\nu$, from which it follows that

$$\left[ f(S) \ , \ \sum_\nu S_\nu^2 \right] = 0 \qquad \text{for all } f(S). \tag{6}$$

A constant of motion with property (6) is called a Casimir invariant. In this case there is only one Casimir. In general, their number equals the dimension of the null space of $\eta^{\mu\nu}$.

Another important point is to verify Jacobi identity for GPB (4) or to check relation (2). If $\varepsilon^{mnj}$ is rewritten as $\varepsilon^{j}_{mn}$, relation (2) becomes [1]

$$\varepsilon^m_{rs}\,\varepsilon^j_{mn} + \varepsilon^m_{sn}\,\varepsilon^j_{mr} + \varepsilon^m_{nr}\,\varepsilon^j_{ns} = 0, \tag{7}$$

which is the relation for the structure constants of the Lie algebra of $O_3$.

(b) Continuum Case

The extension of the GPB to continua has been considered by several authors [9-15]. Instead of vectors $z^\mu$ one considers functions $u^i$, and instead of functions as observables one considers functionals F,G,..., so that the obvious extension of the GPB is

$$\{F, G\} = \sum_{i,j=1}^{N} \int \frac{\delta F}{\delta u^i} A^{ij} \frac{\delta G}{\delta u^j}\, d\tau \ , \tag{8}$$

where $A^{ij}$ are antisymmetric operators and $A^{ij} = A^{ji}$.

The Jacobi identity contains Fréchet derivatives of the $A^{ij}$ and becomes a condition that is far less transparent [8] than condition (2) in the discrete case. The condition becomes simple if the $A^{ij}$ are linear [16] in the dynamic variables. The case of the rotating top is the analogous discrete case, as can be seen from GPB (4), which is also linear in the dynamic variable $S_j$.

A fluid can be described kinematically by the group of diffeomorphism [15] similarly to a rigid top [1], which can be described by $O_3$. If the momentum density is chosen as one of the dynamical variables by analogy with the angular momentum of the top, a GPB can be defined in the following way:

$$\{F, G\} = \int u^i \left[\frac{\delta F}{\delta \underline{u}}, \frac{\delta G}{\delta \underline{u}}\right]_i d\tau \; , \tag{9}$$

$[\;,\;]_i$ is the $i^{th}$ component of the product for a Lie algebra of vector-valued functions, which will be called the inner Lie algebra, while GPB (9) is the outer Lie algebra on functionals. GPB (9) is a special case of GPB (8) and is called [15] Lie-Poisson bracket.

Cases where the $A^{ij}$ do not depend upon the dynamic variables have already been suggested [9-12]. They are appropriate to vacuum fields. A combination of the two types of brackets can occur for such simple equations as, for example, the Korteweg-de Vries equation [17], as can be seen in Sec. IV or generally for interacting fluids and fields [13,18].

Though the Lie-Poisson bracket is rather old, its appearence in fluids and plasmas is recent. In accordance with Morrison and others [13,14], ideal magnetohydrodynamics (MHD), Vlasov-Maxwell equations, Vlasov-Poisson and other equations have been formulated in this way. On the other hand, a mathematical investigation of the Lie group properties underlying these formulations have been intensified since then especially by Marsden and others [15].

More recently, gyroviscous MHD, which contains unfamiliar higher-order derivatives different from those encountered in elasticity theory, also found such a formulation [19] with a correspondingly unfamiliar inner Lie algebra of the Lie-Poisson bracket. Since 2-dimensional gyroviscous MHD may have an interesting impact on statistical mechanics of continua, let us sketch it here. The equations are

$$\dot{M}_s = -\partial_i\left(M_i M_s/\rho\right) - \partial_s\left(\beta|B| + \frac{B^2}{2}\right) - \partial_i \Pi_{is}\,, \tag{10}$$

$$\dot{\rho} = -\partial_s M_s\,, \tag{11}$$

$$\dot{B} = -\partial_s\left(\frac{B M_s}{\rho}\right), \tag{12}$$

$$\dot{\beta} = -\partial_s\left(\frac{\beta M_s}{\rho}\right), \tag{13}$$

where the gyroviscous tensor $\Pi_{is}$ is given by

$$\Pi_{is} = N_{sjik}\,\partial_k\left(\frac{M_j}{\rho}\right) \tag{14}$$

and
$$N_{sjik} = c\left(\delta_{sk}\varepsilon_{ji} - \delta_{ji}\varepsilon_{sk}\right). \tag{15}$$

$M_s$ are the components of the momentum density. For other notations and details see ref. [19]. It turns out that the system (10)-(13) can be written as

$$\dot{M}_s = \{M_s, H\}\,,\ \dot{\rho} = \{\rho, H\}\,,\ \dot{B} = \{B, H\}\,,\ \dot{\beta} = \{\beta, H\} \tag{16}$$

if the GPB is defined by

$$\begin{aligned} \{F, G\} = -\int d\tau \Big[ & M_\ell \left( \frac{\delta F}{\delta M_k} \partial_k \frac{\delta G}{\delta M_\ell} - \frac{\delta G}{\delta M_k} \partial_k \frac{\delta F}{\delta M_\ell} \right) + \\ & B \left( \frac{\delta F}{\delta M_k} \partial_k \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M_k} \partial_k \frac{\delta F}{\delta B} \right) + \\ & \beta \left( \frac{\delta F}{\delta M_k} \partial_k \frac{\delta G}{\delta \beta} - \frac{\delta G}{\delta M_k} \partial_k \frac{\delta F}{\delta \beta} - N_{ijst} \left( \partial_s \frac{\delta F}{\delta M_i} \right) \left( \partial_t \frac{\delta G}{\delta M_j} \right) \right) \Big] \end{aligned} \tag{17}$$

and

$$H = \int \left( \frac{M^2}{2\rho} + \beta |B| + \frac{B^2}{2} \right) d\tau\,. \tag{18}$$

GPB (17) is very similar to GPB (9) up to the term containing $N_{ijst}$. The parentheses are namely the components of the product of the Lie algebra constructed as a semi-direct product extension of the diffeomorphism algebra [15]. The $N_{ijst}$ term contains quadratic spatial derivatives, which should not appear in a semi-direct product extension.

This term can be compensated by an isomorphism

$$i : (f_1, f_2, f_\rho, f_B, f_\beta) \rightarrow (f_1, f_2, f_\rho, f_B, f_\beta + c\,\varepsilon_{sj} \partial_s f_j)\,,$$

where $\underline{f} = (f_1, f_2, f_\rho, f_B, f_\beta)$ is any element of the inner Lie algebra of gyroviscous MHD,
such that

$$\left[ i(\underline{f}), i(\underline{g}) \right] = i \left[ \underline{f}, \underline{g} \right] .$$

This isomorphism motivates a shift in $M_s$ in order to obtain the intensively sought [20] Clebsch decomposition in the form

$$M_s = \rho \partial_s \chi + B \partial_s \psi + \beta \partial_s \alpha - c \, \varepsilon_{sk} \partial_k \beta . \tag{19}$$

This leads to a canonical formulation with eq. (18) as Hamiltonian and

$$\begin{array}{ll} \chi \quad , & \Pi_\chi = -\rho \\ \psi \quad , & \Pi_\psi = -B \\ \alpha \quad , & \Pi_\alpha = -\beta \end{array}$$

as canonically conjugate variables. One more function is needed in this case to achieve canonical formulation. This is a sort of "gauge" freedom.

Note finally that GPB (17) is degenerate and admit Casimirs

$$C = \sum_i \lambda_i \int \rho^{a_i} B^{b_i} \beta^{c_i} \, d\tau \tag{20}$$

with $a_i + b_i + c_i = 1$. This sort of degeneracy seems to be generic [21] for all fluid GPB.

## III. Functional Integration

A functional integral [3] consists of an integrand which, in general, is a functional of some function and of a domain of summation which extends to all possible values of the function. The original definition is based on a limiting process starting with a polygonal approximation to the function at N mesh points separated by $\Delta x$, so that the functional integral becomes a limit of a multiple integral. If for $\Delta x \to 0$ and $N \to \infty$ with $N\Delta x$ fixed the appropriately normalized multiple integral converges, the limit is precisely the value of the functional integral. Let us take as an example

$$\int D(u)\, e^{-\int_a^b [u_x^2 + f(u,x)]\, dx} =$$

$$\lim_{\substack{N \to \infty \\ \Delta x \to 0}} A(\Delta x, N) \int_{-\infty}^{+\infty} du_1 \cdots \int_{-\infty}^{+\infty} du_N \, e^{-\sum_{i=1}^{N} \left[\left(\frac{u_{i+1}-u_i}{\Delta x}\right)^2 + f(u_i, x_i)\right]\Delta x} \tag{21}$$

with $N\Delta x = b-a$ and

$$A(\Delta x, N) = (\pi \Delta x)^{-\frac{N}{2}} . \tag{22}$$

If we choose another way to approximate the function $u(x)$, e.g. by Fourier expanding and then integrating over all possible values of the Fourier coefficients, do we recover the same value for the functional integral? This question is obviously not trivial, in general, especially if the functional contains in a strongly nonlinear way higher-order derivatives of the function. Fortunately, functional integrals of the type (21) happen to be uniquely defined, as proved in ref. [22]. The statistical calculations of Sec. IV reduce to functional integrals of the type (21). The case proposed there for a Monte Carlo numerical evaluation is different, but this does not need to be a serious drawback in view of the fact that the proof given in ref. [22] does not imply that the only well-defined functional integrals are those given by eq. (21).

A more serious question is how to restrict the class of functions on which functional integration is being done. This is particularly difficult if one has several functions u(x) which are not physical and possess, for example, a gauge freedom. The problem is to carry the integration in such a way that the physical values are only counted once. Such ambiguities [23] are known from the quantization of Yang-Mills fields and there is good reason to believe that they also occur if the functional integration is performed through Clebsch potentials.

The practical problem of evaluating functional integrals has not yet been solved in a sufficiently general and satisfactory way. There is the obvious case of a quadratic functional in (21) which can be "diagonalized" in the variables of integration. The functional integral then reduces to the limit of a multiple Gaussian integral which is well known. This means physically that the system is free of interaction. Since this is the only known multidimensional case, it became the starting point [24] of standard perturbation techniques.

The higher-order terms of the perturbation series are moments of the multidimensional Gaussian combined in several ways. These terms can be calculated exactly but display, in general, infrared or ultraviolet divergences which necessitate renormalization techniques [25]. The renormalization only helps to get each term finite but cannot prevent the explosion in the number of terms which causes such perturbations series to have, in general, zero convergence radius [24-26] because they are essentially of the form

$$\approx \sum_n n!\, \varepsilon^n \, , \tag{23}$$

the factorial being due to the explosion in the number of terms or diagrams.

Though these series, when explicitly known, can be occasionally resummed [26], there is no satisfactory general way of using them to all orders. One is obliged to avoid perturbation expansion and solve exactly either analytically or numerically or in a combined way. One-dimensional cases having the form of (21) can be reduced [27,28] to the eigenvalues and eigenfunctions of a nonharmonic oscillator which can then be evaluated numerically. For more general cases Monte Carlo methods based on ref. [29] seem to be the only remaining tool, but their resolution is of course limited.

## IV. Statistics of Continua and Exact k-spectra

Equilibrium statistical mechanics [30] has had a great impact on physics by obtaining thermodynamics out of the Hamiltonian of particles. This is done through the partition function or the zero moment of the canonical Gibbs distribution. Higher-order moments such as correlation functions can deliver fluctuation spectra. In this respect the main interest is in continuous systems instead of particles. Our starting point is then some ideal fluid model such as MHD, Vlasov, hydrodynamics etc... . In order to proceed, we need the Hamiltonian of the system, which in our case is a functional, and so we need to do phase space integrals, which here are functional integrals. Let us begin with linearized equations.

(a) Linearized systems [5]

The linearized equations [31-33] of motion of ideal fluid systems can be written in Lagrange variables as

$$N\ddot{\zeta} + P\dot{\zeta} + Q\zeta = 0 , \tag{24}$$

where $\zeta$ is a vector in some appropriate functional space, N and Q are symmetric operators and P is antisymmetric. If N is positive, it can be transformed away without loss of generality by a congruent transformation. The Lagrangian is then

$$L = \frac{1}{2}\left[(\dot{\zeta},\dot{\zeta}) - (\zeta, P\dot{\zeta}) - (\zeta, Q\zeta)\right] , \tag{25}$$

where ( , ) is the scalar product in the Hilbert space of functions mentioned above. From the Lagrangian we can derive the momentum and the Hamiltonian:

$$\Pi = \frac{\delta L}{\delta \dot{\zeta}} = \dot{\zeta} + \frac{1}{2} P\zeta$$

and

$$H = \frac{1}{2}\left((\Pi - \tfrac{1}{2}P\zeta), (\Pi - \tfrac{1}{2}P\zeta)\right) + \frac{1}{2}(\zeta, Q\zeta) . \tag{26}$$

Let us now assume the system to be in a heat bath and randomized. Then the partition function is the following functional integral:

$$Z = \int D(\Pi)\, D(\zeta)\; e^{-\beta H} .$$

Let Y and M be introduced as follows

$$Y = \begin{pmatrix} \Pi \\ \\ \zeta \end{pmatrix} , \qquad M = \begin{pmatrix} I & -\frac{P}{2} \\ \\ \frac{P}{2} & Q - \frac{P^2}{4} \end{pmatrix} .$$

M is a symmetric operator and Z can be written as

$$Z = \int D(Y)\; e^{-\beta/2\,(Y,\,MY)} .$$

Let Y be expanded in terms of orthonormal eigenfunctions $Y_i$ of M:

$$Y = \sum_{i=1}^{\infty} a_i Y_i \qquad \text{with } \lambda_i \text{ as eigenvalues.}$$

This then gives $(Y, MY) = \sum_{i=1}^{\infty} a_i^2 \lambda_i$ and

$$Z = \int \cdots \int \prod_{i=1}^{\infty} da_i \; e^{-\frac{\beta}{2} \sum_{i=1}^{\infty} \lambda_i a_i^2} . \tag{27}$$

If <...> denotes the canonical average, we have from eq. (27)

$$\langle a_k^2 \rangle = -\frac{2}{\beta Z} \frac{\partial Z}{\partial \lambda_k} = \frac{1}{\beta \lambda_k} \tag{28}$$

and

$$\langle E_k \rangle = \lambda_k \langle a_k^2 \rangle = \frac{1}{\beta} , \tag{29}$$

where $E_k$ is the energy in mode k.
Equation (29) means that there is equipartition of the energy expectation. This is due to the Gaussian, which itself is due to linearization. Equipartition holds for any physics which expresses itself in a redefinition of the energy. In fact, the expectation value for

the amplitudes given by eq. (28) can have strong ultraviolet divergences [34] which cause $<a_k^2> \approx k^2$ for linearized gyroviscous MHD.

This shows that linearization does not give good results for large k. Ultraviolet divergences dominate. The advantages of the linearized problem are in generality and feasibility of the functional integration, whose integrand is a Gaussian (see Sec. III). The nonlinear problem is certainly needed but its solution needs a bounded Hamiltonian and a functional integral with non-Gaussian integrand. This is possible only for special cases [27,28]. We treat here the case of the Korteweg-de Vries (K-dV) equation.

(b) Statistics of the K-dV equation [36,6]

The "soliton gas" approach [35] to drift wave turbulence was a motivation to perform a rigorous treatment [36] of the statistics of the K-dV equation considered as a model for drift wave turbulence. The standard form of K-dV equation is

$$u_t - 6\, u u_x + u_{xxx} = 0\,, \tag{30}$$

where u stands for the electrostatic potential of the drift wave, the steepening term is due [37] to the electron temperature gradient, and the dispersive term should model gyroviscous effects. It is worth noting that the statistical treatment is rigorous for K-dV independently of the underlying physics and is of general interest.

The Hamiltonian formulation of the K-dV equation was given in ref. [9] with a GPB

$$\{F, G\} = \int \frac{\delta F}{\delta u}\, \partial_x \frac{\delta G}{\delta u}\, dx \tag{31}$$

and a Hamiltonian

$$H_1 = \int u^3 dx + \frac{1}{2}\int u_x^2 \, dx , \tag{32}$$

so that eq. (30) can be written

$$u_t = \{u, H_1\} .$$

The Hamiltonian $H_1$ is not interpretable as an energy because of the cubic term and is not useful for a canonical distribution.

Fortunately, there is another Hamiltonian [17]

$$H_2 = \int \frac{u^2}{2} \, dx \tag{33}$$

and another GPB

$$\{F, G\} = \int \frac{\delta F}{\delta u} \left( \frac{\partial^3}{\partial x^3} - 4u \frac{\partial}{\partial x} - 2 u_x \right) \frac{\delta G}{\delta u} \, dx . \tag{34}$$

These brackets are a combination of Lie-Poisson and vacuum-field-type brackets. The Hamiltonian is now bounded at the expense of a GPB which does not allow [8] the Liouville theorem to be verified. This situation calls for a change of phase space. It can be done by the transformation [38]

$$u = v^2 + v_x , \tag{35}$$

which takes us to

$$v_t = \{v, H_2\} = 6 v^2 v_x - v_{xxx} \tag{36}$$

with $$H_2 = \frac{1}{2}\int (v^4 + v_x^2) \, dx \tag{37}$$

and $$\{F, G\} = \int \frac{\delta F}{\delta v} \partial_x \frac{\delta G}{\delta v} \, dx . \tag{38}$$

The bracket operator $\frac{\partial}{\partial x}$ is now independent of dynamic variables and the Hamiltonian is positive, so that Gibbs distributions are all well defined. The partition function is

$$Z = \int D(v) \, e^{-\beta H_2(v)} , \tag{39}$$

$$Z = \lim_{\substack{N \to \infty \\ \Delta x \to 0}} \int \prod_{i=-N}^{N} dv_i \, e^{-\beta \frac{\Delta x}{2} \sum_i \left( v_{i+1}^4 + \left( \frac{v_{i+1} - v_i}{\Delta x} \right)^2 \right)} . \tag{40}$$

The situation is very lucky because Hamiltonian (37) is the 1-d case of a Ginzburg-Landau [39] potential for which Z can be calculated exactly via the transfer integral operator [27]

$$\int dv_{i-1} \, e^{-\beta f(v_i, v_{i-1})} \psi_n(v_{i-1}) = e^{-\beta \varepsilon_n} \psi_n(v_i) . \tag{41}$$

This operator can be approximated by the 1-d Schrödinger operator in the limit of large N and $\Delta x \to 0$. The problem reduces to calculation of the eigenvalues and eigenfunctions of the anharmonic oscillator. One finds

$$Z \approx e^{-N\beta \varepsilon_0} ,$$

where $\varepsilon_0$ is the lowest eigenvalue of eq. (41).

The next step [6] is to calculate the correlation function

$$C(x) = \langle \delta u(x) \delta u(0) \rangle = \langle \delta(v^2 + v_x) \delta(v^2 + v_x)_0 \rangle , \tag{42}$$

where $\delta u = u - \langle u \rangle = v^2 + v_x - \langle v^2 + v_x \rangle .$

C(x) can also be written as a functional integral

$$C(x) = \int D(v) \, \delta(v^2 + v_x) \delta(v^2 + v_x)_0 \, \frac{e^{-\beta H_2(v)}}{Z} . \tag{43}$$

It turns out that this functional integral can be calculated in a similar [6,27] way to Z. One finds

$$C(x) = \sum_{n=1}^{\infty} \{ \ \}_n \, e^{-\frac{|x|}{\xi} \left(\frac{\beta_0}{\beta}\right)^{1/3} (E_n - E_0)} , \tag{44}$$

with $\{ \ \}_n$ and other notations are given in ref. [6]. The Fourier transform of C(x) delivers the spectrum, which is of the form

$$S(k) = 2 \sum_{n=1}^{\infty} \frac{q_n}{p_n} \frac{1}{1 + \frac{k^2}{p_n^2}} . \tag{45}$$

In the case of drift wave turbulence the first Lorentzian dominates and comparison with experiment [40] gives qualitative agreement [6] except for low k values, where "magnetic shear" may cause a strong damping which is not included in the present theory.

(c) How to go beyond special systems?

In some cases the Liouville theorem can be proved directly for noncanonical variables [41-43], so that Hamiltonian formulation can be bypassed. Unfortunately, the known cases do have ultraviolet divergences and are not adequate for plasmas where gyroviscosity is more important than collisional viscosity. In general, it seems unavoidable to obtain first a canonical Hamiltonian formulation. This formulation is only possible for fluids in Clebsch variables and will not be gauge invariant.

At this point I should like to make some connection with the K-dV case treated in Sec. IVb.

1) The Miura transformation (35) is reminiscent of a particular Clebsch ansatz.

2) The dispersive term in K-dV is similar to gyroviscosity.

This suggests to take as next realistic model gyroviscous MHD, for which a Hamiltonian (18) and a Clebsch decomposition were presented in Sec. II.

It is very likely that all conservative systems have a Clebsch representation for which a canonical formalism exists. This may be the unifying way to treat general systems if one can circumvent difficulties due to ambiguities and evaluation of functional integrals.

## V. Remaining Problems

This paper treats the problem of statistics of conservative and continuous fluids and plasmas in the spirit of Gibbs equilibrium distributions. It faces the two main problems of Hamiltonian formulation and functional integration.

Functional integration is certainly the most difficult unsolved problem in general. Exact calculation is only possible for special cases, as already mentioned, and expansion around the linearized case leads, in general, to divergent series even after renormalization. Another problem is the removal of ambiguities, which has not yet been solved. It is worthwhile to learn about functional integration and removal of ambiguities by applying Monte Carlo techniques to the 2-d gyroviscous MHD system presented in Sec. II.

It may be a consolation to know that these difficulties are also encountered in the quantization of nonlinear fields such as Yang-Mills fields. We may even occasionally borrow there some techniques such as summing up divergent series. But in the meantime people in lattice gauge theory [44,45] are using Monte Carlo techniques [29], which are well known among fluid and plasma physicists.

This paper has been restricted to equilibrium statistics for reasons of applicability, feasibility and rigor. Gibbs distributions are known if the Hamiltonian is known. Many phenomena in fluids and plasmas are certainly much more involved and do not display complete chaos in all phase space. There are driving and damping effects and the statistical distributions have themselves to be found, e.g. solutions to the Hopf equation [46]. This is the problem of turbulence which takes place outside statistical equilibrium and will necessitate quite new techniques for its solution. It makes sense, however, to look first at the less difficult problem of statistical equilibrium approached here in order to pave the way to the more common problem of turbulence in fluid dynamics and plasma physics.

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