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Combined Maxwell and Kinetic Guiding-Center Theory with Polarization Drift: Regularized Variational Formulation with Local Charge and Energy Conservation

D. Correa-Restrepo, D. Pfirsch, H.K. Wimmel

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Abstract

A formerly derived regularization method is applied to time-dependent Lagrangian guiding-center mechanics, with the polarization drift included. This approach removes the singularity that occurs for **B**-fields with non-vanishing parallel curl. From the Lagrangian equations of motion, Liouville's theorem and a collisionless kinetic equation for the "regularized guiding centers" are derived. A common Lagrangian density for both the guiding centers and the Maxwell fields is obtained by using a "constrained" Hamiltonian and a formerly derived, new variational principle. From this variational formalism local conservation laws for electric charge and energy are derived, together with the correct charge, current, energy and energy flux densities. These densities combine point-like contributions with electric polarization and magnetization terms.

1. Introduction

Recently, guiding-center (G.C.) mechanics and guiding-center kinetic theory, in combination with Maxwell's theory for the electromagnetic fields, were reformulated and fundamentally improved in three important ways. In the first place, single guiding-center mechanics was provided with exact conservation laws and a Liouville's theorem for arbitrary (but slowly varying) field configurations. This was accomplished by employing Lagrangian and/or non-canonical Hamiltonian formalisms, i.e. a variational representation [1-5]. By using Liouville's theorem, Liouville-Vlasov-type kinetic equations were derived from the Lagrangian equations of motion of the guiding centers [2,3,5]. Together with Maxwell's equations these kinetic equations satisfy local conservation of charge (in the continuum sense) [2,3,5]. However, local conservation of energy seemed only to hold exactly in the absence of the polarization drift [5], i.e. for the non-relativistic and relativistic standard G.C. theories [2,3].

This somewhat unsatisfactory aspect of the new theory was, in the second place, remedied by the work of Pfirsch [6]. He introduced a new variational principle that uses a common Lagrangian density for the Maxwell fields and a wide class of mechanical systems (in the Vlasov representation) that includes charged guiding-center systems and also allows the polarization drift to be taken into account. In this way, correct densities of charge, current, energy, and energy flux were derived that satisfy exact local conservation laws of charge, momentum, and energy. These new densities combine point-like (particle-like) contributions with electric polarization and magnetization terms.

The resulting theories are perfectly satisfactory for field configurations with vanishing parallel curl of the magnetic field. However, for $\mathbf{B} \cdot \mathrm{curl}\, \mathbf{B} \neq 0$ the theories are singular on a hyper-surface of G.C. phase space, e.g. for large values of $|v_{\|}|$ [7]. Because the G.C. drift velocity $\mathbf{v}$ and the "parallel acceleration" $\dot{v}_{\|}$ diverge there, non-causal G.C. orbits occur

and G.C. conservation in phase space is violated. As a consequence, all G.C. distribution functions that involve such unphysical orbits are forbidden. Maxwell distributions and all other distributions involving (arbitrarily) large values of $|v_{||}|$ may not be used. Introduction of diffusion-type, e.g. Fokker-Planck, collision terms is therefore also forbidden. Correa-Restrepo and Wimmel [7]) (in the third place) gave a general regularization method by which this singular behavior is removed while conserving the variational form of the theory. This regularization method was formerly applied to the non-relativistic, standard G.C. theory (i.e. without polarization drift) in ref. 7.

The present paper employs both the regularization method of Correa-Restrepo and Wimmel [7]) and the variational principle of Pfirsch [6]) in order to arrive at a non-singular and fully variational, non-relativistic G.C. theory *including the polarization drift* and self-consistent electromagnetic fields. Correct densities of charge, current, energy, and energy flux will be derived for this case and local conservation laws of charge and energy will be proved. It is thought that this new type of theory is needed in order to avoid spurious sources or sinks of charge and/or energy. This ought to be particularly important in any analysis of plasma stability based on a guiding-center model of the plasma.

Section 2 presents the regularized G.C. Lagrangian with polarization drift, Sec. 3 the equations of motion of the guiding centers and Liouville's theorem in the "energy representation" $\{\mathbf{x},\ W\}$. In Sec. 4 some of the results are given in the $v_{||}$ representation $\{\mathbf{x},\ v_{||}\}$. The collisionless kinetic equation for the guiding centers is derived in Sec. 5. Section 6 gives the "constrained" Hamiltonian that is needed in the variational principle for the combined Maxwell and kinetic G.C. theories (Sec. 7). In Sec. 8 the correct densities of charge, current, energy, and energy flux are derived, together with the local conservation laws of charge and energy. Section 9 presents the conclusions. Finally, in Appendix A a transformation used in Sec. 8 is verified.

2. Regularized guiding-center Lagrangian with the polarization drift included

We start from the *unregularized* G.C. mechanics, with the polarization drift included, as given in ref. 5. The unregularized G.C. Lagrangian is

$$L \equiv \frac{e}{c} \mathbf{A}^* \cdot \mathbf{v} - W \quad , \tag{2.1}$$

with the definitions

$$\mathbf{A}^* \equiv \mathbf{A} + \frac{mc}{e} (v_{\|} \hat{\mathbf{b}} + \mathbf{v}_E) \quad , \tag{2.2}$$

$$W \equiv e\Phi + \mu B + \frac{m}{2} (v_{\|}^2 + v_E^2) \quad . \tag{2.3}$$

Here L depends on the time t, the G.C. *variables* $\mathbf{x}$, $\mathbf{v} \equiv \dot{\mathbf{x}}$, $v_{\|}$, and the parameters μ, m, e, c. Furthermore, $\mathbf{A}\ (t,\ \mathbf{x})$ and $\Phi\ (t,\ \mathbf{x})$ are the electromagnetic potentials, $\mathbf{E}\ (t,\ \mathbf{x})$ and $\mathbf{B}\ (t,\ \mathbf{x})$ are the Maxwell fields, $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ is the unit vector in the direction of $\mathbf{B}$, $\mathbf{v}_E$ is the usual $\mathbf{E} \times \mathbf{B}$ drift, viz.

$$\mathbf{v}_E \equiv \frac{c}{B} \mathbf{E} \times \hat{\mathbf{b}} \quad , \tag{2.4}$$

μ is the magnetic moment and an adiabatic invariant, with $\dot{\mu} = 0$, and $v_{\|}$ turns out to be the "parallel velocity" (i.e. parallel to $\mathbf{B}$) of the unregularized G.C. because one of the unregularized Lagrangian equations leads to $v_{\|} = \mathbf{v} \cdot \hat{\mathbf{b}}$.

As explained in ref. 7, the singularity of the above theory occurs for $B_{\|}^* = 0$, with[5])

$$B_{\|}^{*} \equiv B + \frac{mc}{e}\left\{v_{\|}\,\hat{\mathbf{b}}\cdot\mathrm{curl}\,\hat{\mathbf{b}} + \hat{\mathbf{b}}\cdot\mathrm{curl}\,\mathbf{v}_E\right\} \quad , \tag{2.5}$$

when the polarization drift is included. The condition $B_{\|}^{*} = 0$ can be expressed as $v_{\|} = v_c\,(t,\,\mathbf{x})$, with the critical velocity

$$v_c \approx -\,\Omega/(\hat{\mathbf{b}}\cdot\mathrm{curl}\,\hat{\mathbf{b}}) \gtrless 0 \tag{2.6}$$

and

$$\Omega \equiv \frac{eB}{mc} \quad . \tag{2.7}$$

It is the $v_{\|}$ -dependence of $\mathbf{A}^{*}$ [*eq.* (2.2)] that produces the singularity mentioned. In order to regularize this singularity, $v_{\|}$ in eq. (2.2) is replaced by a function $v_o\,g(v_{\|}/v_o)$ of $v_{\|}$ that approaches $v_{\|}$ in the validity range of G.C. mechanics ($|v_{\|}|$ small), while it approaches constant values outside the validity range, i.e. for large values of $|v_{\|}|$. Thus, eq. (2.2) is replaced by

$$\mathbf{A}^{+} \equiv \mathbf{A} + \frac{mc}{e}\left\{v_o\,g\left(\frac{v_{\|}}{v_o}\right)\hat{\mathbf{b}} + \mathbf{v}_E\right\} \quad , \tag{2.8}$$

with $v_o = \mathrm{const} > 0$, $g(z) \sim z$, $g'(z) \sim 1$ for $|z| << 1$, $g(z)$ being defined in $-\infty < z < +\infty$. The function $g(z)$ has to be monotonically increasing and antisymmetric with respect to $z = 0$, and it is required that $g(z) \sim \pm\, g_\infty = \pm\,\mathrm{const}$ $(g_\infty > 0)$ for $z \to \pm\infty$. We shall also assume that

$$g'(z) \sim |z|^{-\sigma} \qquad \text{for} \qquad z \to \pm\,\infty \quad , \tag{2.9}$$

with $1 < \sigma < \infty$, in order for $\mathbf{v}$ and $\dot{v}_{\|}$ to diverge not faster than a finite power of $|v_{\|}|$ for $v_{\|} \to \pm\infty$. It should be noted that this new divergence is completely harmless because it occurs for infinite $|v_{\|}|$, contrary to the original singularity at finite v_c of eq. (2.6). Finally, v_o and g_∞ must satisfy the conditions $v_{th} << v_o$ and

$$v_o \, g_\infty << |v_c| \equiv |\Omega|/|\hat{\mathbf{b}} \cdot \text{curl}\, \hat{\mathbf{b}}| \, . \tag{2.10}$$

As an example, one may choose $g(z) = \text{arctg}\, z$, i.e., $g_\infty = \frac{\pi}{2}$, $g'(z) = 1/(1 + z^2)$, $\sigma = 2$. When this regularized G.C. mechanics is used in a kinetic equation, the distribution functions should, of course, be vanishingly small outside the validity range of the G.C. approximation and outside the validity range of $v_o \, g \, (v_{\|}/v_o) \sim v_{\|}$. A suitable value of v_o can always be determined in the validity range of G.C. theory because of $|v_c| >> V_\perp$, where $V_\perp$ is the gyration speed of the particle. Of course, v_o may be chosen different for ions and electrons.

The regularized G.C. Lagrangian, with the polarization drift included, is now given by

$$L \equiv \frac{e}{c} \mathbf{A}^+ \cdot \mathbf{v} - W, \tag{2.11}$$

with the "modified vector potential"

$$\mathbf{A}^+ \equiv \mathbf{A} + \frac{mc}{e} \left\{ v_o \, g\left(\frac{v_{\|}}{v_o}\right) \hat{\mathbf{b}} + \mathbf{v}_E \right\} \tag{2.12}$$

and the energy

$$W \equiv e\Phi + \mu B + \frac{m}{2} (v_{\|}^2 + v_E^2). \tag{2.13}$$

It will turn out in Sec. 3 that $v_{\|}$ is now no longer identical with the "parallel velocity", defined as $\mathbf{v}\cdot\hat{\mathbf{b}}$. We have, nevertheless, continued to use the symbol $v_{\|}$ for this new variable in order to facilitate comparison with earlier work. In the limit of small $|v_{\|}|$ one still has, of course, $v_{\|} \approx \mathbf{v}\cdot\hat{\mathbf{b}}$. When the regularization is removed, i.e. for $v_o\ g \rightarrow v_{\|}$, $g' \rightarrow 1$, the above Lagrange function [eqs. (2.11) to (2.13)] agrees with that given in refs. 5 and 6.

3. Equations of G.C. motion in the energy representation

The equations of G.C. motion assume a particularly simple form if the set of variables $\{t,\ \mathbf{x},\ v_{\|}\}$ is replaced by the set $\{t,\ \mathbf{x},\ W\}$, which we call the *energy representation*. The energy W is defined by eq. (2.13). The variable $v_{\|}$ will then be replaced by the function

$$V_{\|} \ = \ V_{\|}\ (t,\ \mathbf{x},\ W), \tag{3.1}$$

which also depends on the parameters μ, e, m, c. Its derivatives will be needed, e.g.

$$\frac{\partial V_{\|}}{\partial t} = \ - \ \frac{1}{mV_{\|}} \left\{ m\mathbf{v}_E \cdot \frac{\partial \mathbf{v}_E}{\partial t} \ + \ e \ \frac{\partial \Phi}{\partial t} \ + \ \mu \ \frac{\partial B}{\partial t} \right\}, \tag{3.2}$$

$$\nabla V_{\|} \ = \ - \ \frac{1}{mV_{\|}} \left\{ \frac{m}{2} \nabla v_E^2 \ + \ e\nabla\Phi \ + \ \mu\nabla B \right\}, \tag{3.3}$$

$$\frac{\partial V_{\|}}{\partial W} \ = \ \frac{1}{mV_{\|}} \ . \tag{3.4}$$

The modified vector potential is now defined as

$$\mathbf{A}^{+}\ (t,\ \mathbf{x},\ W) \ \equiv \ \mathbf{A} \ + \ \frac{mc}{e} \left\{ v_o \ g\left(\frac{V_{\|}}{v_o}\right) \hat{\mathbf{b}} \ + \ \mathbf{v}_E \right\} . \tag{3.5}$$

In what follows the "modified fields" $\mathbf{B}^{+}$ and $\mathbf{E}^{+}$ are needed. They are defined as

$$\mathbf{B}^{+} \ \equiv \ \text{curl}\ \mathbf{A}^{+} \tag{3.6}$$

and

$$\mathbf{E}^+ \equiv -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}^+}{\partial t}, \tag{3.7}$$

so that

$$\text{div}\ \mathbf{B}^+ = 0 \tag{3.8}$$

and

$$\frac{\partial \mathbf{B}^+}{\partial t} + c\ \text{curl}\ \mathbf{E}^+ = 0\ . \tag{3.9}$$

Written explicitly, the modified fields read

$$\mathbf{B}^+ = \mathbf{B} + \frac{mc}{e}\left[v_o\ g\ \text{curl}\ \hat{\mathbf{b}} + g'\ \nabla V_{\|} \times \hat{\mathbf{b}} + \text{curl}\ \mathbf{v}_E\right] \tag{3.10}$$

and

$$\mathbf{E}^+ = \mathbf{E} - \frac{m}{e}\left[v_o\ g\ \frac{\partial \hat{\mathbf{b}}}{\partial t} + g'\frac{\partial V_{\|}}{\partial t}\ \hat{\mathbf{b}} + \frac{\partial \mathbf{v}_E}{\partial t}\right]\ . \tag{3.11}$$

An important quantity is the "parallel component" of $\mathbf{B}^+$, viz.

$$\begin{aligned} B_{\|}^+ &\equiv \mathbf{B}^+ \cdot \hat{\mathbf{b}} \\ &= B + \frac{mc}{e}\left[v_o\ g\ \hat{\mathbf{b}} \cdot \text{curl}\ \hat{\mathbf{b}} + \hat{\mathbf{b}} \cdot \text{curl}\ \mathbf{v}_E\right]\ . \end{aligned} \tag{3.12}$$

In order to obtain the equations of G.C. motion from the Lagrangian in the W-representation

$$L^+ (t, \mathbf{x}, W, \mathbf{v} \equiv \dot{\mathbf{x}}) \equiv \frac{e}{c} A^+ \cdot \mathbf{v} - W, \tag{3.13}$$

one observes that the total time derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \dot{W} \frac{\partial}{\partial W} + \dot{\mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} + \ddot{W} \frac{\partial}{\partial \dot{W}} + \dot{\mu} \frac{\partial}{\partial \mu} \quad , \tag{3.14}$$

where the last three terms make no contribution. From

$$\frac{\partial L^+}{\partial W} = 0 \tag{3.15}$$

one obtains

$$\mathbf{v} \cdot \hat{\mathbf{b}} = \frac{V_\parallel}{g'} \quad . \tag{3.16}$$

The other equations of motion are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L^+}{\partial \mathbf{v}} \right) = \nabla L^+ \quad . \tag{3.17}$$

This is equivalent to

$$\frac{c\, g'}{e\, V_\parallel} \dot{W}\, \hat{\mathbf{b}} = \mathbf{v} \times \text{ curl } A^+ - \frac{\partial A^+}{\partial t} \quad . \tag{3.18}$$

Scalar multiplication by $\mathbf{v}$ and use of eq. (3.16) yields the energy equation:

$$\dot{W} = - \frac{e}{c} \mathbf{v} \cdot \frac{\partial \mathbf{A}^+}{\partial t} \quad . \tag{3.19}$$

It is seen that the energy W is conserved when $\mathbf{A}^+$ is time-independent. One may define the kinetic energy by

$$W_k \equiv W - e\Phi = \mu B + \frac{m}{2}\left(V_{\|}^2 + v_E^2\right) \quad . \tag{3.20}$$

Its (total) time derivative can be written as

$$\dot{W}_k = \frac{\partial W_k}{\partial t} + e\,\mathbf{E}^+ \cdot \mathbf{v} \tag{3.21}$$

or, more explicitly, as

$$\dot{W}_k = \mathbf{v} \cdot \left\{ e\,\mathbf{E} - m v_o\, g\, \frac{\partial \hat{\mathbf{b}}}{\partial t} - m\, \frac{\partial \mathbf{v}_E}{\partial t} \right\} + \mu\, \frac{\partial B}{\partial t} + \frac{m}{2}\, \frac{\partial v_E^2}{\partial t} \quad , \tag{3.22}$$

where it must be remembered that all time and space derivatives must be taken with W kept constant.

Forming the vector product of eq. (3.18) and $\hat{\mathbf{b}}$ and observing eq. (3.16) yields the G.C. velocity:

$$\mathbf{v} = \frac{V_{\|}}{g'\, B_{\|}^+}\, \mathbf{B}^+ + \frac{1}{B_{\|}^+}\, \hat{\mathbf{b}} \times \frac{\partial \mathbf{A}^+}{\partial t} \quad . \tag{3.23}$$

It is one of the advantages of the energy representation that $\mathbf{v}$ can be expressed in such a concise form. The explicit expression for $\mathbf{v}$ is then

$$\mathbf{v} = \frac{V_{\|}}{g'}\,\hat{\mathbf{b}} + \frac{c}{B_{\|}^{+}}\,\mathbf{E}\times\hat{\mathbf{b}} + \frac{\mu}{m\Omega^{+}}\,\hat{\mathbf{b}}\times\nabla B + \frac{V_{\|}v_o\,g}{g'\Omega^{+}}\,\hat{\mathbf{b}}\times\frac{\partial\hat{\mathbf{b}}}{\partial s} + \frac{v_o\,g}{\Omega^{+}}\,\hat{\mathbf{b}}\times\frac{\partial\hat{\mathbf{b}}}{\partial t}$$

$$+ \frac{1}{2\Omega^{+}}\,\hat{\mathbf{b}}\times\nabla v_E^2 + \frac{1}{\Omega^{+}}\,\hat{\mathbf{b}}\times\frac{\partial\mathbf{v}_E}{\partial t} + \frac{V_{\|}}{g'\Omega^{+}}\left\{\text{curl}\,\mathbf{v}_E - (\hat{\mathbf{b}}\cdot\text{curl}\,\mathbf{v}_E)\hat{\mathbf{b}}\right\}, \quad (3.24)$$

with the definitions

$$\Omega^{+} = \frac{e\,B_{\|}^{+}}{mc} \quad (3.25)$$

and

$$\frac{\partial}{\partial s} \equiv \hat{\mathbf{b}}\cdot\nabla. \quad (3.26)$$

Liouville's theorem [8]) in the energy representation can be derived by the methods indicated in refs. 7 and 8. It reads

$$\frac{\partial}{\partial t}\left(\frac{g'\,B_{\|}^{+}}{V_{\|}}\right) + \nabla\cdot\left(\frac{g'\,B_{\|}^{+}}{V_{\|}}\,\mathbf{v}\right) + \frac{\partial}{\partial W}\left(\frac{g'\,B_{\|}^{+}}{V_{\|}}\dot{W}\right) = 0. \quad (3.27)$$

Because of $\dot{\mu} = 0$ this results [8]) in the phase space volume element

$$\mathrm{d}\tau_W \equiv \frac{g'\,B_{\|}^{+}}{V_{\|}}\,\mathrm{d}^3x\,\mathrm{d}W\,\mathrm{d}\mu \quad (3.28)$$

being conserved when it moves according to the above equations of motion [e.g. eqs. (3.19) and (3.23)].

4. Results in the $\{\mathbf{x},\ v_{\|}\}$ representation

Even though some aspects of the energy representation exhibit a particular simplicity, the set of variables $\{\mathbf{x},\ v_{\|}\}$ is more adapted to the treatment of the G.C. kinetic equations (Secs. 5 to 7). Hence we shall use the $\{\mathbf{x},\ v_{\|}\}$ representation in this section and throughout the rest of the paper. Henceforth, all time and space derivatives are to be taken with $v_{\|}$ (rather than W) kept constant. It will suffice simply to list the main results in the $\{\mathbf{x},\ v_{\|}\}$ representation.

The Lagrangian is given by

$$L^*(t,\ \mathbf{x},\ v_{\|}\ ,\mathbf{v} \equiv \dot{\mathbf{x}}) \equiv \frac{e}{c}\,\mathbf{v}\cdot\mathbf{A}^* - e\Phi - W_k, \tag{4.1}$$

with the definitions

$$\mathbf{A}^* \equiv \mathbf{A} + \frac{mc}{e}\left\{v_o\, g\left(\frac{v_{\|}}{v_o}\right)\hat{\mathbf{b}} + \mathbf{v}_E\right\}, \tag{4.2}$$

$$W_k \equiv \mu B + \frac{m}{2}(v_{\|}^2 + v_E^2)\,, \tag{4.3}$$

$$\mathbf{v}_E \equiv \frac{c}{B}\mathbf{E}\times\hat{\mathbf{b}}\,, \tag{4.4}$$

and the modified fields are

$$\mathbf{B}^* \equiv \text{curl}\ \mathbf{A}^*\,, \tag{4.5}$$

$$\mathbf{E}^* \equiv -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}^*}{\partial t} \tag{4.6}$$

or, explicitly,

$$\mathbf{B}^* = \mathbf{B} + \frac{mc}{e}\left\{v_o\, g\, \mathrm{curl}\,\hat{\mathbf{b}} + \mathrm{curl}\,\mathbf{v}_E\right\}, \tag{4.7}$$

$$\mathbf{E}^* = \mathbf{E} - \frac{m}{e}\left\{v_o\, g\, \frac{\partial \hat{\mathbf{b}}}{\partial t} + \frac{\partial \mathbf{v}_E}{\partial t}\right\}, \tag{4.8}$$

with

$$B_{\|}^* \equiv \mathbf{B}^* \cdot \hat{\mathbf{b}} = B + \frac{mc}{e}\left[v_o\, g\, \hat{\mathbf{b}} \cdot \mathrm{curl}\,\hat{\mathbf{b}} + \hat{\mathbf{b}} \cdot \mathrm{curl}\,\mathbf{v}_E\right]. \tag{4.9}$$

Comparison with Sec. 3 yields

$$B_{\|}^* = B_{\|}^+, \tag{4.10}$$

$$\mathbf{E}^* \times \hat{\mathbf{b}} = \mathbf{E}^+ \times \hat{\mathbf{b}}. \tag{4.11}$$

The Lagrangian equations of G.C. motion are equivalent to

$$\mathbf{v} \cdot \hat{\mathbf{b}} = v_{\|}/g' \tag{4.12}$$

and

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \equiv \frac{e}{c}\frac{\mathrm{d}\mathbf{A}^*}{\mathrm{d}t} = \frac{e}{c}\nabla(\mathbf{A}^*\cdot\mathbf{v}) - e\nabla\Phi - \nabla W_k \,. \tag{4.13}$$

If $\mathbf{A}^*$, Φ, and W_k are independent of $\mathbf{x}$, then the canonical momentum $\mathbf{p}$ is conserved. By expanding the total time derivative, viz.

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \dot{v}_{\|}\frac{\partial}{\partial v_{\|}} + \dot{\mu}\frac{\partial}{\partial \mu} + \ddot{v}_{\|}\frac{\partial}{\partial \dot{v}_{\|}} + \dot{\mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}} \,, \tag{4.14}$$

where the last three terms make no contribution, expressions for $\mathbf{v}$ and $\dot{v}_{\|}$ can be obtained:

$$\mathbf{v} = \frac{v_{\|}}{g'}\frac{\mathbf{B}^*}{B^*_{\|}} + \frac{c}{eB^*_{\|}}(e\mathbf{E}^* - \nabla W_k)\times\hat{\mathbf{b}} \tag{4.15}$$

and

$$\dot{v}_{\|} = \frac{1}{mg'B^*_{\|}}\mathbf{B}^*\cdot(e\mathbf{E}^* - \nabla W_k) \tag{4.16}$$

$$= \frac{1}{mv_{\|}}\mathbf{v}\cdot(e\mathbf{E}^* - \nabla W_k) \,. \tag{4.16a}$$

More explicit expressions are

$$\dot{v}_{\|} = \frac{1}{g'}\left(\frac{e}{m}E_{\|} - \frac{\mu}{m}\frac{\partial B}{\partial s} - \frac{1}{2}\frac{\partial v_E^2}{\partial s} - \hat{\mathbf{b}}\cdot\frac{\partial \mathbf{v}_E}{\partial t}\right)$$

$$+ \frac{1}{g'}\left(v_o\, g\,\frac{\partial \hat{\mathbf{b}}}{\partial s} - \hat{\mathbf{b}}\times \mathrm{curl}\,\mathbf{v}_E\right)\cdot\mathbf{v} \tag{4.17}$$

and eq. (3.24) for $\mathbf{v}$, with

$$\Omega^* = \Omega^+ = \frac{e\, B_{\|}^*}{mc} \tag{4.18}$$

and, again, $\partial/\partial s \equiv \hat{\mathbf{b}} \cdot \nabla$.

Liouville's theorem in the $\{\mathbf{x},\ v_{\|}\}$ representation can be derived by the methods indicated in refs. 7, 8 or by transforming eq. (3.27). It reads

$$\frac{\partial}{\partial t}\left(g'\ B_{\|}^*\right) + \nabla \cdot \left(g'\ B_{\|}^*\ \mathbf{v}\right) + \frac{\partial}{\partial v_{\|}}\left(g'\ B_{\|}^*\ \dot{v}_{\|}\right) = 0. \tag{4.19}$$

Because of $\dot{\mu} = 0$ this results in

$$\mathrm{d}\tau \equiv \frac{2\pi}{m}\,|g'\ B_{\|}^*|\ \mathrm{d}^3x\ \mathrm{d}v_{\|}\ \mathrm{d}\mu \tag{4.20}$$

being conserved when moving according to the G.C. equations of motion. When the regularization is removed, i.e. for $v_o\ g \rightarrow v_{\|}$, $g' \rightarrow 1$, the equations of motion and the other results of Secs. 3 and 4 agree with those given in refs. 5 and 6.

5. Collisionless kinetic equation for the guiding centers

Using the regularized equations of motion of Sec. 4, one can readily derive a collisionless G.C. kinetic equation. The phase space is 5-dimensional, with coordinates $\{x_i\} \equiv \{\mathbf{x},\ v_{\|}\ ,\mu\}$, $i = 1$ to 5. The volume element $d\tau$ in phase space, as given by eq. (4.20), is Liouvillian (conserved) [8]), i.e. $d\dot{\tau} = 0$ for a $d\tau$ that moves according to the G.C. equations of motion, where the dot in $d\dot{\tau}$ designates the total time derivative.

The guiding-center distribution function f is defined by

$$dN \equiv f\,(t,\ \mathbf{x},\ v_{\|},\ \mu)\,d\tau\ , \tag{5.1}$$

dN being the number of guiding centers in $d\tau$. The collisionless kinetic equation expresses conservation of dN in a volume element $d\tau$ that moves with the guiding centers, i.e.

$$d\dot{N} \equiv \frac{d}{dt}\,(f\,d\tau) = 0. \tag{5.2}$$

This equation can be reformulated [8]) to read

$$\frac{\partial}{\partial t}\,(g'\,B_{\|}^{*}f) + \mathrm{div}\,(g'\,B_{\|}^{*}f\,\mathbf{v}) + \frac{\partial}{\partial v_{\|}}\,(g'\,B_{\|}^{*}f\,\dot{v}_{\|}) = 0, \tag{5.3}$$

where $\dot{\mu} = 0$ has been used. In this form the kinetic equation holds independently of whether $d\tau$ is conserved or not. In the case of a conserved $d\tau$ [eq. (4.20)] eq. (5.3) simplifies to read

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \dot{v}_{\|}\,\frac{\partial f}{\partial v_{\|}} = 0, \tag{5.4}$$

where, again, $\dot{\mu} = 0$ has been used. In eqs. (5.3) and (5.4) $\mathbf{v}$ must be taken from eq. (3.24) and $\dot{v}_{\|}$ from eq. (4.17). When eq. (5.4) is applied to a G.C. species of a plasma, it describes a G.C. Vlasov fluid (in phase space). It will be shown in Sec. 8 that the combination of G.C. Vlasov fluids and Maxwell fields satisfies local conservation laws of charge and energy. Local conservation of momentum will also hold, as indicated in ref. 6, but will not be considered in this paper.

6. The "constrained " Hamiltonian of the guiding centers

In Secs. 7 and 8 the method of ref. 6 will be employed in order to derive local conservation theorems of the combined system of Maxwell fields and G.C. Vlasov fluids. For this purpose a *Hamiltonian* of the guiding centers is required. Because the G.C. Lagrangian of eq. (4.1) is of the non-standard type [8-10]), Dirac's method [9,8,10,11]) of constructing a "constrained" Hamiltonian must be used. As will be explained below, we use a modified and improved version of Dirac's method [8]).

Let us write L^* of eq. (4.1) in the general form

$$L(t,\ z_\nu,\ \dot{z}_\nu)\ \equiv\ \sum_n\ \gamma_n\ (t,\ z_\nu)\ \dot{z}_n\ -\ \varphi(t,\ z_\nu), \tag{6.1}$$

with n and ν running from 1 to 4. Because one has

$$\omega\ \equiv\ \det\ (\omega_{nm})\ \neq\ 0, \tag{6.2}$$

with the definition

$$\omega_{nm}\ \equiv\ \frac{\partial \gamma_n}{\partial z_m}\ -\ \frac{\partial \gamma_m}{\partial z_n}\quad , \tag{6.3}$$

the Lagrangian equations yield unique functions for the $\dot{z}_n$:

$$\dot{z}_n\ =\ V_n\ (t,\ z_\nu)\quad , \tag{6.4}$$

where the phase space is spanned by the set of coordinates $\{z_\nu\}$. The modified method of Dirac [8]) yields the Hamiltonian

$$H\ (t,\ z_\nu,\ p_\nu) \quad \equiv \quad \varphi\ (t,\ z_\nu) \ + \ \sum_m V_m\ (t,\ z_\nu)\ [p_m \ - \ \gamma_m\ (t,\ z_\nu)], \qquad (6.5)$$

defined in the "super phase space" $\{z_\nu,\ p_\nu\}$. At this point, the p_ν are independent variables not yet subject to the usual *constraint*

$$p_n \ = \ \gamma_n \quad . \qquad (6.6)$$

The canonical equations read

$$\dot{z}_n \ = \ \frac{\partial H}{\partial p_n} \ = \ V_n\ (t,\ z_\nu) \qquad (6.7)$$

and

$$\dot{p}_n \quad = \quad -\frac{\partial H}{\partial z_n} \quad = \quad -\frac{\partial \varphi}{\partial z_n} \ - \ \sum_m \frac{\partial V_m}{\partial z_n}\ p_m \ + \ \sum_m \frac{\partial}{\partial z_n}(V_m \gamma_m) \quad . \qquad (6.8)$$

It follows that

$$\dot{p}_n \ - \ \dot{\gamma}_n \ = \ -\sum_m \frac{\partial V_m}{\partial z_n}\ (p_m - \gamma_m). \qquad (6.9)$$

Hence, when the initial condition $p_m - \gamma_m \ = \ 0$, all m, is applied, it follows that $p_m - \gamma_m \ = 0$ holds for all later times. One may therefore return to the original phase space $\{z_\nu\}$ by using the *constraint* $p_m \ = \ \gamma_m$, which agrees with the usual definition of the canonical momenta. This is a modified and, in fact, improved version of Dirac's formalism in that the original equations of motion [eqs. (2.4)] do not result from the time-independence of the constraints, but hold in fact even when the constraints are not applied at all [8]). If one wants Liouville's theorem to hold in super phase space, i.e. $d\dot{\tau}_s = 0$ for

$$\mathrm{d}\tau_s \equiv \prod_m \mathrm{d}z_m \, \mathrm{d}p_m \quad , \tag{6.10}$$

then the constraints [eq. (6.6)] may not be used.

If the above general scheme is compared with eqs. (4.1) to (4.6), it follows that

$$z_i = x_i \quad , \quad i = 1 \text{ to } 3 \quad , \tag{6.11}$$

$$z_4 = v_{||} \quad , \tag{6.11a}$$

$$\gamma_i \quad = \quad \frac{e}{c} A_i^* \quad , \quad i = 1 \quad \text{to} \quad 3 \quad , \tag{6.12}$$

$$\gamma_4 = 0, \tag{6.12a}$$

$$\varphi \quad = \quad W \quad \equiv \quad W_k \, + e \, \Phi \, . \tag{6.13}$$

Hence the G.C. Hamiltonian reads

$$H(t, \mathbf{x}, v_{||}, p_\nu) \equiv W_k(t, \mathbf{x}, v_{||}) + e\Phi(t, \mathbf{x}) + \mathbf{v} \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A}^*) + \dot{v}_{||} p_4 \, , \tag{6.14}$$

with $\mathbf{A}^*$ and W_k given in Sec. 4. In eq. (6.14) the quantities $\mathbf{v}$ and $\dot{v}_{||}$ are the functions $\mathbf{v}(t, \mathbf{x}, v_{||})$ and $\dot{v}_{||}(t, \mathbf{x}, v_{||})$ given by eqs. (3.24) and (4.17).

The t and $\mathbf{x}$ dependence of H is defined via the potentials and the fields. Hence one may express H as

$$H \;=\; H\,(\,W_k + e\Phi,\; \mathbf{p} - \frac{e}{c}\mathbf{A},\; \mathbf{E},\; \mathbf{B},\; v_{\|},\; p_4\,) \quad . \tag{6.15}$$

Then the derivatives $\partial H/\partial \mathbf{B}$ and $\partial H/\partial \mathbf{E}$, which are needed in Secs. 7 and 8, can be determined. The evaluation uses the constraints of eq. (6.6), i.e. the results hold, and are only needed, for $\mathbf{p} = \frac{e}{c}\mathbf{A}^*$ and $p_4 = 0$, viz.

$$\begin{aligned} \frac{\partial H}{\partial \mathbf{B}} \;&=\; \mu\hat{\mathbf{b}} \;-\; \frac{m}{B}\left\{v_0 g\,\mathbf{v}_\perp - \frac{v_{\|}}{g'}\,\mathbf{v}_E\right\} \\ &-\; \frac{mc}{B^2}\,(\,\mathbf{v}_\perp \,-\, \mathbf{v}_E\,)\times\mathbf{E} \;+\; \frac{2m}{B}\left\{(\,\mathbf{v}_\perp - \mathbf{v}_E\,)\cdot\mathbf{v}_E\right\}\hat{\mathbf{b}} \end{aligned} \tag{6.16}$$

and

$$\frac{\partial H}{\partial \mathbf{E}} \;=\; \frac{mc}{B}\,(\,\mathbf{v}_\perp - \mathbf{v}_E\,)\times\hat{\mathbf{b}} \quad . \tag{6.17}$$

In ref. 6 the quantity $\hat{H} \equiv H - e\Phi$ and its derivatives with respect to the fields were introduced, but the results are, of course, the same. The quantity $\mathbf{v}_\perp$ is defined as

$$\mathbf{v}_\perp \;\equiv\; \mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{b}})\,\hat{\mathbf{b}} \;=\; \mathbf{v} - \frac{v_{\|}}{g'}\,\hat{\mathbf{b}} \quad . \tag{6.18}$$

7. Variational principle for combined Maxwell and kinetic G. C. theories

We switch to an Eulerian picture of the Hamiltonian G.C. dynamics of Sec. 6 by employing the Hamilton-Jacobi formalism. A complete solution

$$S = S(t, \mathbf{z}, \mathbf{a}) \tag{7.1}$$

of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(t, \mathbf{z}, \frac{\partial S}{\partial \mathbf{z}}) = 0 \tag{7.2}$$

is assumed to exist for each G.C. species. Then the transformation $\{\mathbf{z}, \mathbf{p}\} \rightarrow \{\mathbf{z}, \mathbf{a}\}$ in the super phase space is effected by evaluating

$$\mathbf{p} \equiv \frac{\partial S}{\partial \mathbf{z}} \quad , \tag{7.3}$$

where $\mathbf{a}$ is a set of constants of the motion, $\dot{\mathbf{a}} = 0$ [12]). Then the Lagrangian $L(t)$ for the combined Maxwell and kinetic G.C. system is given by [6])

$$\begin{aligned} L(t) &= \int \mathrm{d}^3\mathrm{x} \, \frac{1}{8\pi} \, (\mathbf{E}^2 - \mathbf{B}^2) \\ &- \sum_{\nu} \int \mathrm{d}^n\mathrm{a} \int \mathrm{d}^n\mathrm{z} \left(\frac{\partial S_\nu}{\partial t} + H_\nu(t, \mathbf{z}, \frac{\partial S_\nu}{\partial \mathbf{z}}) \right) \phi_\nu(t, \mathbf{z}, \mathbf{a}) \quad , \end{aligned} \tag{7.4}$$

where the index ν counts the G.C. species and $\mathbf{E}$ and $\mathbf{B}$ are defined by the potentials $\mathbf{A}$ and Φ:

$$\mathbf{B} = \text{curl } \mathbf{A} \quad , \tag{7.4a}$$

$$\mathbf{E} \;=\; -\nabla\Phi \;-\; \frac{1}{c}\frac{\partial \mathbf{A}}{\partial \mathbf{t}} \quad . \tag{7.4b}$$

Using Hamilton's principle

$$\delta \int_{t_1}^{t_2} L(t)\ \mathrm{dt} = 0 \tag{7.5}$$

(where the variation is taken by varying ϕ_ν, S_ν, Φ, and $\mathbf{A}$) together with the boundary conditions at the times t_1, t_2

$$\delta\phi_\nu \;=\; \delta S_\nu \;=\; \delta\Phi \;=\; \delta\mathbf{A} \;=\; 0 \quad , \tag{7.6}$$

and assuming that certain partial integrations over $\mathbf{z}$ do not give any boundary contributions, one arrives at the following results. Variation of the $\{\phi_\nu\}$ yields the Hamilton-Jacobi equations

$$\frac{\partial S_\nu}{\partial t} \;+\; H_\nu(t,\ \mathbf{z},\ \frac{\partial S_\nu}{\partial \mathbf{z}}) \;=\; 0 \quad ; \tag{7.7}$$

variation of the S_ν yields the collisionless kinetic equations, in the form

$$\frac{\partial \phi_\nu}{\partial t} \;+\; \frac{\partial}{\partial \mathbf{z}} \cdot (\mathbf{V}\phi_\nu) \;=\; 0 \quad ; \tag{7.8}$$

variation of Φ and $\mathbf{A}$ yields the inhomogeneous Maxwell's equations [6]) and, hence, the definitions of the charge and current densities ρ and $\mathbf{j}$ (see Sec. 8). Note that the ho-

mogeneous Maxwell's equations are implied to hold. In eq. (7.8), the $\mathbf{V}$ are the sets of generalized velocities V_n given by [12])

$$\dot{\mathbf{z}} = \mathbf{V}(t,\ \mathbf{z},\ \mathbf{a}) \ = \ \frac{\partial H_\nu}{\partial(\partial S_\nu/\partial \mathbf{z})} \quad , \tag{7.9}$$

where $\mathbf{V}$ has been written without the species index ν, as in the case of the coordinates $\mathbf{z}$ and $\mathbf{a}$. Furthermore, by using the Van Vleck determinant [13,8,14])

$$w_\nu(t,\ \mathbf{z},\ \mathbf{a}) \ \equiv \ \det\left(\frac{\partial^2 S_\nu}{\partial a_i\ \partial z_k}\right) \quad , \tag{7.10}$$

with the property

$$\frac{\partial w_\nu}{\partial t} \ + \ \frac{\partial}{\partial \mathbf{z}} \cdot (\mathbf{V}\ w_\nu) = 0 \quad , \tag{7.11}$$

a Vlasov-type distribution function $f_\nu(t, \mathbf{z}, \mathbf{a})$ can be constructed:

$$f_\nu \ \equiv \ \phi_\nu / w_\nu \quad , \tag{7.12}$$

with the property

$$\frac{\mathrm{d}f_\nu}{\mathrm{d}t} \quad \equiv \quad \frac{\partial f_\nu}{\partial t} \ + \ \mathbf{V} \cdot \frac{\partial f_\nu}{\partial \mathbf{z}} \ = \ 0 \quad , \tag{7.13}$$

i.e. f_ν is a constant of the motion. Equation (7.13) is the Liouville-Vlasov-type kinetic equation in the $\{\mathbf{z},\ \mathbf{a}\}$ representation, and eq. (7.11) states [8]) that the volume element in super phase space

$$\mathrm{d}\tau_s \quad \equiv \quad w_\nu \prod_{i=1}^{n} \mathrm{d}z_i\ \ \mathrm{d}a_i \tag{7.14}$$

and the one in the original phase space $\{z_i\}$, viz.

$$d\tau \equiv w_\nu \prod_{i=1}^{n} dz_i \quad , \tag{7.15}$$

are conserved, i.e. $d\dot{\tau}_s = d\dot{\tau} = 0$. Hence, eq. (7.11) represents Liouville's theorem.

8. Densities and local conservation laws

It was shown in ref. 6 that the new variational principle sketched in Sec. 7 generally leads to the following densities and local conservation laws. The charge density is generally given by

$$\begin{aligned} \rho &= \frac{1}{4\pi}\,\mathrm{div}\,\mathbf{E} \\ &= \sum_{\nu} e_\nu \int \mathrm{d}^n a\;\mathrm{d}^{n_2} z_2\; w_\nu\; f_\nu \\ &+ \sum_{\nu} \mathrm{div} \int \mathrm{d}^n a\;\mathrm{d}^{n_2} z_2\; w_\nu\; f_\nu\; \frac{\partial H_\nu}{\partial \mathbf{E}} \quad . \end{aligned} \tag{8.1}$$

Here, again, ν counts the G.C. species. The vector $\mathbf{z}_2$ (of dimension $n_2 = n - 3$) contains the coordinates z_m other than the spatial coordinates x, y, z. The current density generally reads [6])

$$\begin{aligned} \mathbf{j} &= \frac{1}{4\pi}\left(c\;\mathrm{curl}\,\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\right) \\ &= \sum_{\nu} e_\nu \int \mathrm{d}^n a\;\mathrm{d}^{n_2} z_2\; w_\nu\; f_\nu\; \mathbf{v}_\nu \\ &- \sum_{\nu} \frac{\partial}{\partial t} \int \mathrm{d}^n a\;\mathrm{d}^{n_2} z_2\; w_\nu\; f_\nu\; \frac{\partial H_\nu}{\partial \mathbf{E}} \\ &- \sum_{\nu} c\;\mathrm{curl} \int \mathrm{d}^n a\;\mathrm{d}^{n_2} z_2\; w_\nu\; f_\nu\; \frac{\partial H_\nu}{\partial \mathbf{B}} \quad . \end{aligned} \tag{8.2}$$

These densities combine point-like contributions with electric polarization and magnetization terms. Clearly, charge is locally conserved, i.e. [6])

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0 \quad , \tag{8.3}$$

where the cancellation of the point-like contributions follows from the G.C. kinetic equation [eq. (7.13) or, alternatively, eq. (7.8)].

Furthermore, the energy density is generally given by [6])

$$\epsilon = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) + \sum_\nu \int \mathrm{d}^n a \, \mathrm{d}^{n_2} z_2 \, w_\nu \, f_\nu \left(\hat{H}_\nu - \mathbf{E} \cdot \frac{\partial H_\nu}{\partial \mathbf{E}} \right) \quad , \tag{8.4}$$

with $\hat{H}_\nu \equiv H_\nu - e_\nu \Phi$. Likewise, the energy flux density generally reads [6])

$$\mathbf{h} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} + \sum_\nu \int \mathrm{d}^n a \, \mathrm{d}^{n_2} z_2 \, w_\nu \, f_\nu \left(\hat{H}_\nu \mathbf{v}_\nu + c \, \mathbf{E} \times \frac{\partial H_\nu}{\partial \mathbf{B}} \right) \quad . \tag{8.5}$$

Local conservation of energy holds [6]) and is expressed by

$$\frac{\partial \epsilon}{\partial t} + \operatorname{div} \mathbf{h} = 0. \tag{8.6}$$

Local conservation laws for momentum and angular momentum also hold, but are derived elsewhere [15]).

The above equations for ρ, $\mathbf{j}$, ϵ, and $\mathbf{h}$ will be evaluated for the regularized G.C. mechanics with polarization drift of Secs. 2 to 6. In order to do this, one must return from the super phase space $\{\mathbf{z}, \mathbf{a}\}$ to the physical G.C. phase space $\{\mathbf{z}\}$ and use eqs. (6.16) and (6.17) for $\partial H_\nu / \partial \mathbf{B}$ and $\partial H_\nu / \partial \mathbf{E}$. It is shown in Appendix A that any integral of the form

$$I_\nu \equiv \int \mathrm{d}^n a \, \mathrm{d}^{n_2} z_2 \, w_\nu \, f_\nu \, F(t, \mathbf{z}) \quad , \tag{8.7}$$

with

$$\phi_\nu \equiv w_\nu \, f_\nu = \delta(\mathbf{a} - \mathbf{a}_0) \, \phi_{\nu g}(t, \mathbf{z}) \quad , \tag{8.8}$$

can be given the form

$$I_\nu = \int \mathrm{d}^{n_2} z_2 \, \phi_{\nu g}(t, \mathbf{z}) \; F(t, \mathbf{z}) \quad , \tag{8.9}$$

where the **a**-integration has been performed. Here $\mathbf{a}_0$ is the "constrained "value of **a**, in the sense of Dirac's theory (Sec. 6) and $\phi_{\nu g}$ is the distribution function in **z**-space. It satisfies the kinetic equation in the form

$$\frac{\partial \phi_{\nu g}}{\partial t} + \frac{\partial}{\partial \mathbf{z}} \cdot (\phi_{\nu g} \mathbf{V}) = 0. \tag{8.10}$$

By introducing the Liouvillian phase space volume element $\mathrm{d}\tau_\nu$ of eq.(4.20) in the abbreviated form

$$\mathrm{d}\tau_\nu \equiv \lambda_\nu(t, \mathbf{z}) \prod_n \mathrm{d}z_n \tag{8.11}$$

a Liouville-Vlasov-type G.C. distribution $f_{\nu g}$ is constructed:

$$f_{\nu g} = \phi_{\nu g} \, / \, \lambda_\nu \quad , \tag{8.12}$$

with the kinetic equation taking the form

$$\frac{\mathrm{d}f_{\nu g}}{\mathrm{d}t} \equiv \frac{\partial f_{\nu g}}{\partial t} + \mathbf{V}_\nu \cdot \frac{\partial f_{\nu g}}{\partial \mathbf{z}} = 0 \quad . \tag{8.13}$$

The resulting expressions for ρ, $\mathbf{j}$, ϵ, $\mathbf{h}$ are supplemented by allowing for a μ-dependence of $\phi_{\nu g}$, λ_ν, $f_{\nu \bar{g}}$ and also integrating over μ, to yield

$$\rho = \sum_\nu e_\nu \int \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \lambda_\nu \, f_{\nu g}$$

$$+ \sum_\nu \mathrm{div} \int \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \lambda_\nu \, f_{\nu g} \, \frac{m_\nu c}{B}(\mathbf{v}_\nu - \mathbf{v}_E) \times \hat{\mathbf{b}} \quad , \tag{8.14}$$

$$\mathbf{j} = \sum_\nu e_\nu \int \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \lambda_\nu \, f_{\nu g} \, \mathbf{v}_\nu$$

$$- \sum_\nu \frac{\partial}{\partial t} \int \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \lambda_\nu \, f_{\nu g} \, \frac{m_\nu c}{B}(\mathbf{v}_\nu - \mathbf{v}_E) \times \hat{\mathbf{b}}$$

$$- \sum_\nu c \, \mathrm{curl} \int \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \lambda_\nu \, f_{\nu g} \left[\mu \, \hat{\mathbf{b}} - \frac{m_\nu}{B} \left(v_{0\nu} \, g_\nu \, \mathbf{v}_{\nu\perp} - \frac{v_\parallel}{g'_\nu} \mathbf{v}_E \right) \right.$$

$$\left. - \frac{m_\nu c}{B^2}(\mathbf{v}_{\nu\perp} - \mathbf{v}_E) \times \mathbf{E} + \frac{2m_\nu}{B} \left\{ (\mathbf{v}_{\nu\perp} - \mathbf{v}_E) \cdot \mathbf{v}_E \right\} \hat{\mathbf{b}} \right] \quad , \tag{8.15}$$

$$\epsilon = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2)$$

$$+ \sum_\nu \int \mathrm{d}v_\parallel \, \mathrm{d}\mu \, \lambda_\nu \, f_{\nu g} \left[\mu B + \frac{m_\nu}{2} \left(v_\parallel^2 + v_E^2 \right) + m_\nu \left(\mathbf{v}_\nu \cdot \mathbf{v}_E - v_E^2 \right) \right] , \tag{8.16}$$

$$
\begin{aligned}
\mathbf{h} \;=\; & \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \\
& + \sum_{\nu}\int \mathrm{d}v_{\|}\,\mathrm{d}\mu\;\lambda_{\nu}\,f_{\nu g}\Big[\Big\{\mu B + \frac{m_{\nu}}{2}\Big(v_{\|}^{2}+v_{E}^{2}\Big)\Big\}\mathbf{v}_{\nu} \\
& + \Big\{\mu B + 2m_{\nu}\,\mathbf{v}_{E}\cdot(\mathbf{v}_{\nu\perp}-\mathbf{v}_{E})\Big\}\mathbf{v}_{E} \\
& - \frac{m_{\nu}c^{2}}{B^{2}}\,\mathbf{E}\times\Big\{(\mathbf{v}_{\nu\perp}-\mathbf{v}_{E})\times\mathbf{E}\Big\} \\
& - \frac{m_{\nu}c}{B}\,\mathbf{E}\times\Big\{v_{0\nu}\,g_{\nu}\,\mathbf{v}_{\nu\perp} - \frac{v_{\|}}{g'_{\nu}}\,\mathbf{v}_{E}\Big\}\Big] \quad , \qquad (8.17)
\end{aligned}
$$

with λ_{ν} given explicitly by [eqs. (4.20), (8.11)]

$$
\lambda_{\nu} \;\equiv\; \frac{2\pi}{m_{\nu}}\,\mid g'_{\nu}\,B^{*}_{\nu\|}\mid \quad . \qquad (8.18)
$$

The integration ranges are $-\infty < v_{\|} < +\infty$ and $0 \le \mu < \infty$. When the regularization is removed, i.e. for $v_{0}g \;\to\; v_{\|}$, $g' \;\to\; 1$, the above densities agree with those given in eqs. (8.1) to (8.4) of Ref. 6. It has thus been established that the G.C. kinetic equations of Sec. 5 together with Maxwell's equations, with ρ, $\mathbf{j}$, ϵ, $\mathbf{h}$ given by the above equations, form a selfconsistent Lagrangian system with local conservation of charge and energy. As in the unregularized theory [6]), the current density $\mathbf{j}$ of eq. (8.15) has the property of vanishing in any time-independent $\mathbf{B}$-field (with $\mathbf{E} = 0$ assumed), for distributions $f_{\nu g}$ depending only on energy, i.e. $f_{\nu g} = f_{\nu g}\,(\mu B + \frac{m_{\nu}}{2}\,v_{\|}^{2})$. This means that drift and magnetization currents exactly cancel each other as in the exact Vlasov theory of charged particles.

9. Conclusion

Regularized [7]) non-relativistic guiding-center equations of motion that include the polarization drift [1,4,5]) were derived from a regularized G.C. Lagrangian .The regularization procedure removes a singularity present if $\mathbf{B} \cdot \text{curl}\, \mathbf{B} \neq 0$. The equations of motion conserve single-guiding-center energy in time-independent fields and obey a Liouville's theorem [8]) for an appropriate phase space volume element, which is constructed. From this G.C. mechanics, collisionless kinetic guiding-center equations were derived. In the second part of the paper, the "constrained" Hamiltonian in the sense of Dirac [9–11]) was given for the guiding centers. It was used in the new variational principle of one of the authors [6]) in order to construct correct densities of charge, current, energy, and energy flux and thus obtain local conservation laws of charge and energy for the combined system of Maxwell fields and guiding-center Vlasov fluids. The variational principle mentioned [6]) yields the Hamilton-Jacobi equation for the single guiding centers, the collisionless kinetic equations for the different species of G.C. Vlasov fluids, and the inhomogeneous Maxwell's equations (with the definitions of the correct charge and current densities) together with the local conservation laws for the combined system. The local conservation laws of momentum and angular momentum have already been derived elsewhere [15]). This new theory should be particularly appropriate for stability calculations in view of its exact local conservation laws, the absence of any high-energy singularities, and its inclusion of the polarization drift. Spurious sources or sinks of charge and energy are absent, and the regularization allows an unrestricted use of distribution functions in arbitrary field configurations.

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[+]) The University of Texas at Austin, Institute for Fusion Studies, Austin, Texas 78712, U.S.A.

Appendix A. Transformation of density integrals

The transformation used in Sec. 8 in order to reduce the density integrals to become integrals over $\{\mathbf{z}_2\} \equiv \{v_{\|}\}$ space is now deduced. We shall drop the index ν that counts the G.C. species. We introduce the distribution function ϕ in super phase space by

$$\mathrm{d}N = \phi \, \mathrm{d}^n a \, \mathrm{d}^n z \quad , \tag{A.1}$$

where $\mathrm{d}N$ is the number of guiding centers in $\mathrm{d}\tau_0 \equiv \mathrm{d}^n a \, \mathrm{d}^n z$. For a $\mathrm{d}\tau_0$ moving with the guiding centers $\mathrm{d}N$ is conserved ($\mathrm{d}\dot{N} = 0$) and hence [8]) eq. (7.8) holds for ϕ. To return to the original phase space $\{z_m\}$, one introduces the constraint $\mathbf{a} = \mathbf{a}_0$ and the G.C. distribution function $\phi_g(t, \mathbf{z})$ by

$$\phi(t, \mathbf{z}, \mathbf{a}) = \delta(\mathbf{a} - \mathbf{a}_0) \, \phi_g(t, \mathbf{z}) \tag{A.2}$$

[cf. eq. (8.8)] . This yields

$$\mathrm{d}N_g \equiv \int_{\mathbf{a}} \mathrm{d}N = \left(\int_{\mathbf{a}} \phi \, \mathrm{d}^n a \right) \mathrm{d}^n z = \phi_g(t, \mathbf{z}) \, \mathrm{d}^n z \quad . \tag{A.3}$$

Here $\mathrm{d}N_g$ is also conserved because the a_m are constants of the motion. One therefore has [8])

$$\frac{\partial \phi_g}{\partial t} + \frac{\partial}{\partial \mathbf{z}} \cdot \left(\mathbf{V} \phi_g \right) = 0 \quad , \tag{A.4}$$

which is eq. (8.10). The equivalence of eqs. (8.7) and (8.9) now follows because I_ν of eq. (8.7) can be written as

$$I \, \mathrm{d}^3x = \int_a \int_{\mathbf{z}_2} \mathrm{d}N \; F(t, \mathbf{z}) = \int_{\mathbf{z}_2} \mathrm{d}N_g \; F(t, \mathbf{z}) \quad , \tag{A.5}$$

where the index ν has again been dropped.

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