Liouville's Theorem for Time-Dependent, Non-Standard and Standard Lagrangians

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Abstract

Various forms of Liouville's theorem are considered which are appropriate for non-canonical representations of mechanical systems and for applications. Special attention is given to time-dependent, non-standard Lagrangians (without constraints) for which the usual transition to an ordinary Hamiltonian is impossible. In addition, applications to integrals of the motion and kinetic equations are listed and a basic equivalence relation is proved. The results are important for modern Lagrangian guiding-center theories.

1. Introduction

In classical mechanics, statistical mechanics, and kinetic theory Liouville's theorem [1] is of great importance. Where valid, it makes the motion in the phase space of a mechanical system incompressible. Here, the terms "phase space" or "evolution space" designate any state space such that the equations of motion for any complete set of coordinates are of first order. Points in phase space will be described by arbitrary, i.e. generally non-canonical, coordinates z_ν.

A phase space volume element

$$d\tau \equiv \lambda\,(t, z_\nu) \prod_n dz_n \tag{1.1}$$

that moves according to the equations of motion is conserved, i.e.

$$\frac{d}{dt}\,(d\tau) \equiv d\dot{\tau} = 0 \tag{1.2}$$

holds, if λ satisfies the condition

$$\frac{\partial \lambda}{\partial t} + \sum_n \frac{\partial}{\partial z_n}\,(\lambda \dot{z}_n) = 0, \tag{1.3}$$

with the generalized velocities $\dot{z}_n$ determined from the equations of motion:

$$\dot{z}_n = V_n\,(t, z_\nu). \tag{1.4}$$

The equivalence of eqs. (1.1), (1.2) with eq. (1.3) is proved in Appendix A. When eqs. (1.2) and (1.3) are satisfied, $d\tau$ will be called a Liouvillian, or conserved, volume element, and λ a Liouville weight function.

Conventionally, Liouville's theorem says that

$$d\tau \equiv \prod_i dq_i \, dp_i \tag{1.5}$$

is conserved, the q_i, p_i being canonical coordinates of a Hamiltonian system. For systems given by standard Lagrangians, i.e. $L\ (t, q_k, \dot{q}_k)$ with [2]

$$\det\left(\frac{\partial p_i}{\partial \dot{q}_j}\right) \equiv \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j}\right) \neq 0, \tag{1.6}$$

eq. (1.5) may be employed in order to derive a conserved phase space volume element $d\tau$ in arbitrary coordinates. However, in the case of non-standard Lagrangians [2], with eq. (1.6) violated, the velocities $\dot{q}_i$ cannot be expressed by t, q_j, p_j, and it is therefore impossible to employ eq. (1.5) by passing from the Lagrangian to the Hamiltonian (expressed by t and canonical coordinates). An alternative method, for constructing a conserved $d\tau$ direct from the non-standard Lagrangian, is then desired. Non-standard Lagrangians are of practical importance, e.g. in modern Lagrangian guiding-center theories [3 – 9] and in other fields [10, 11]. In the case of standard Lagrangians the use of a Hamilton-Jacobi representation [12, 13] has been of practical importance [7]. This paper lists and derives various forms of Liouville's theorem which are appropriate for non-canonical system representations and for applications [7 – 9].

It should be noted that, ordinarily, a conserved phase space volume element $d\tau$ is only of practical use if $\lambda\ (t, z_\nu)$ is a known function of state that does not require integration along orbits for its construction. For instance, if a complete set of constants of the motion c_n is considered, viz.

$$c_n \equiv c_n\,(t,\; z_\nu) \;=\; const. \tag{1.7}$$

the number of c_n equalling the number of coordinates z_n, it is true that

$$d\tau \;\equiv\; \prod_n dc_n \tag{1.8}$$

is conserved; but the Liouville weight function

$$\lambda\,(t,\; z_\nu) \;=\; \det\left(\frac{\partial c_n}{\partial z_m}\right) \tag{1.9}$$

is only a known function of its arguments if the constants of the motion c_n are as well, i.e. after the equations of motion have been solved and the solution has been inverted to obtain eq. (1.7). However, special circumstances can exist [7, 9] where a Liouville weight function that is only formally given proves to be useful. This point deserves special attention in applications.

In Secs. 2 and 3 conserved phase space volume elements are derived for time-dependent first-order and second-order Lagrangians without constraints, respectively. The results of Littlejohn [3], Dirac-Pfirsch [9], Lutzky-Wimmel [14], and Van Vleck-Pfirsch [12, 13] are covered. Section 4 presents applications and Sec. 5 gives the conclusions. In Appendix A the equivalence of eqs. (1.1), (1.2) with eq. (1.3) is proved.

2. First-order Lagrangian systems without constraints

First-order equations of motion without constraints are obtained from (time-dependent) non-standard Lagrangians of the form

$$L\,(t,\, z_\nu,\, \dot{z}_\nu) \;\equiv\; \sum_n \gamma_n\,(t,\, z_\nu)\,\dot{z}_n \;-\; \phi\,(t,\, z_\nu)\;, \tag{2.1}$$

with n and ν ranging from 1 to 2 N (see below). The Lagrangian equations are of the form

$$\sum_m \omega_{nm}\,(t,\, z_\nu)\,\dot{z}_m \;=\; -\left(\frac{\partial \gamma_n}{\partial t} \;+\; \frac{\partial \phi}{\partial z_n}\right)\;, \tag{2.2}$$

with the definition

$$\omega_{nm} \;\equiv\; \frac{\partial \gamma_n}{\partial z_m} \;-\; \frac{\partial \gamma_m}{\partial z_n}\;\;. \tag{2.3}$$

Hence the matrix (ω_{nm}) is antisymmetric. In order for a unique solution

$$\dot{z}_n \;=\; V_n\,(t,\, z_\nu) \tag{2.4}$$

for the generalized velocities $\dot{z}_n$ to follow from eq. (2.2), it is necessary that

$$\omega \;\equiv\; \det\,(\omega_{nm}) \;\neq\; 0 \tag{2.5}$$

be valid, which is only possible when the number of coordinates is even. According to Littlejohn [3] the following phase space volume elements $\mathrm{d}\tau$ are conserved ($\mathrm{d}\dot{\tau} = 0$):

$$\mathrm{d}\tau \equiv C(t,\ z_\nu)\ \lambda\ (t,\ z_\nu) \prod_n \mathrm{d}z_n\ , \tag{2.6}$$

where C is any integral of the motion (see Sec. 1), and the Liouville weight function λ is given by

$$\lambda = |\omega|^{\frac{1}{2}} \equiv |\ \det\ (\omega_{nm})|^{\frac{1}{2}}\quad . \tag{2.7}$$

To prove this theorem [3], it is sufficient, because of the equivalence proved in Appendix A, to show that λ satisfies eq. (1.3).

The proof goes as follows. Consider the total time derivative of $\omega \equiv \det\ (\omega_{nm})$:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \omega \sum_n \sum_m \dot{\omega}_{nm}\ J_{mn}\quad , \tag{2.8}$$

where the dot again designates the total time derivative and (J_{nm}) is the inverse of (ω_{nm}), i.e.

$$\sum_m \omega_{nm}\ J_{ml} = \delta_{nl}\quad . \tag{2.9}$$

Equation (2.8) follows from the Laplacian development [15] of the determinant ω, together with the relation

$$\omega^{nm} \;=\; \omega\, J_{mn} \quad , \tag{2.10}$$

ω^{nm} being the co-factor of ω_{nm} . On expanding in eq. (2.8):

$$\dot{\omega}_{nm} \;=\; \frac{\partial \omega_{nm}}{\partial t} + \sum_i \dot{z}_i \, \frac{\partial \omega_{nm}}{\partial z_i} \tag{2.11}$$

and substituting

$$\frac{\partial \omega_{nm}}{\partial t} \;=\; \frac{\partial}{\partial z_n} \sum_k \omega_{mk}\, \dot{z}_k \;-\; \frac{\partial}{\partial z_m} \sum_k \omega_{nk}\, \dot{z}_k \;, \tag{2.12}$$

a relation that follows from eq. (2.2) by differentiations, one obtains

$$\dot{\omega}_{nm} \;=\; \sum_i \left\{ \dot{z}_i \left(\frac{\partial \omega_{mi}}{\partial z_n} + \frac{\partial \omega_{in}}{\partial z_m} + \frac{\partial \omega_{nm}}{\partial z_i} \right) + \omega_{mi}\, \frac{\partial \dot{z}_i}{\partial z_n} + \omega_{in}\, \frac{\partial \dot{z}_i}{\partial z_m} \right\} \tag{2.13}$$

and hence

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} \;=\; -2\,\omega \sum_m \frac{\partial \dot{z}_m}{\partial z_m} + \omega \sum_i \sum_k \sum_m \left\{ \dot{z}_m\, J_{ki} \left(\frac{\partial \omega_{km}}{\partial z_i} + \frac{\partial \omega_{mi}}{\partial z_k} + \frac{\partial \omega_{ik}}{\partial z_m} \right) \right\} \tag{2.14}$$

The second term on the r.h.s. of eq. (2.14) vanishes owing to eq. (2.3) so that one ends up with

$$\frac{d\omega}{dt} = -2\omega \sum_m \frac{\partial \dot{z}_m}{\partial z_m} \tag{2.15}$$

or

$$\frac{d\lambda}{dt} = -\lambda \sum_m \frac{\partial \dot{z}_m}{\partial z_m} , \tag{2.16}$$

which is the same as eq. (1.3) owing to

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \sum_m \dot{z}_m \frac{\partial}{\partial z_m} . \tag{2.17}$$

This completes the proof showing that $d\tau$ of eq. (2.6) is Liouvillian and λ of eq. (2.7) is a Liouville weight function. Littlejohn [3] also mentions that the ω_{nm} and J_{nm} are Lagrange and Poisson brackets, respectively, when a set of canonical coordinates $\{q_i,\ p_i\}$ of phase space exists. This provides another route to proving Liouville's theorem.

We consider the relation between Dirac's constrained Hamiltonian [10, 11] and Liouville's theorem. Given the non-standard Lagrangian of eq. (2.1), one may construct the Hamiltonian [10, 11, 9, 20]

$$H\ (t,\ z_\nu,\ p_\nu) \equiv \phi\ (t,\ z_\nu) + \sum_m V_m\ (t,\ z_\nu) \Big(p_m - \gamma_m\ (t,\ z_\nu)\Big), \tag{2.18}$$

which is defined in the "super phase space" spanned by the canonical coordinates $\{z_\nu,\ p_\nu\}$. This super phase space has twice the dimension of the original phase space $\{z_\nu\}$. The functions $V_m\ (t,\ z_\nu)$ are the solutions for the generalized velocities $\dot{z}_m$ $\Big($see eq.(1.4)$\Big)$ as found by inverting the Lagrangian equations, eq. (2.2). The canonical equations read

$$\dot{z}_n = \frac{\partial H}{\partial p_n} = V_n\,(t,\, z_\nu) \tag{2.19}$$

and

$$\dot{p}_n = -\frac{\partial H}{\partial z_n} = -\frac{\partial \phi}{\partial z_n} - \sum_m \frac{\partial V_m}{\partial z_n}\, p_m + \sum_m \frac{\partial}{\partial z_n}\,(V_m\, \gamma_m). \tag{2.20}$$

It follows that

$$\dot{p}_n - \dot{\gamma}_n = -\sum_m \frac{\partial V_m}{\partial z_n}\,(p_m - \gamma_m). \tag{2.21}$$

Hence, when the initial condition $p_m - \gamma_m = 0$, all m, is applied, it follows that $p_m - \gamma_m = 0$ for all (later) times. One may therefore return to the original phase space $\{z_\nu\}$ by using the constraint $p_m = \gamma_m$, which agrees with the usual definition of the canonical momenta. This is a modified and, in fact, improved version of Dirac's formalism in that the original equations of motion (eqs. (2.19)) do not result from the time-independence of the constraints, but hold in fact even when the constraints are not applied at all [9, 20].

It is obvious that Liouville's theorem holds in super phase space, i.e.

$$d\tau_s \equiv \prod_m dz_m\, dp_m \tag{2.22}$$

is conserved ($d\dot{\tau}_s = 0$), as follows from the relation (see Appendix A)

$$\sum_m \left(\frac{\partial \dot{z}_m}{\partial z_m} + \frac{\partial \dot{p}_m}{\partial p_m} \right) = 0 \quad . \tag{2.23}$$

However, what one needs is Liouville's theorem in the original phase space $\{z_\nu\}$, and the question arises whether the conserved $d\tau$ of eq. (2.6) can be reconstructed from the modified Dirac formalism [9, 20] in a straight-forward manner. The answer to this is that this cannot be done. One can only write down a formal expression for a conserved $d\tau$ by transforming to non-canonical coordinates $\{z_\nu, \; c_\nu\}$ in super phase space, where the c_ν are constants of the motion ($\dot{c}_\nu = 0$) [9]. Then from

$$d\tau_s \equiv \prod_m dz_m \, dp_m = \lambda \, (t, \; z_\nu, c_\nu) \prod_m dz_m \, dc_m \quad , \tag{2.24}$$

which is conserved in super phase space, one derives

$$d\tau \equiv \lambda \, (t, \; z_\nu, \; c_\nu) \prod_m dz_m \quad , \tag{2.25}$$

which is conserved in the phase space $\{z_\nu\}$, where

$$\lambda = \det \left(\frac{\partial p_m}{\partial c_n} \right) \quad , \tag{2.26}$$

and λ satisfies eq. (1.3).

3. Second-order Lagrangian systems without constraints

Second-order equations of motion without constraints are obtained from (time-dependent) standard Lagrangians $L\ (t,\ q_i,\ \dot{q}_i)$, i.e. ones that satisfy eq. (1.6) [2]. Then the Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \ , \tag{3.1}$$

or, explicitly,

$$\sum_n \ddot{q}_n \frac{\partial^2 L}{\partial \dot{q}_m\, \partial \dot{q}_n} = \frac{\partial L}{\partial q_m} - \frac{\partial^2 L}{\partial \dot{q}_m\, \partial t} - \sum_n \dot{q}_n \frac{\partial^2 L}{\partial \dot{q}_m\, \partial q_n} \ , \tag{3.2}$$

yield a unique solution for the generalized accelerations:

$$\ddot{q}_n = a_n\ (t,\ q_i,\ \dot{q}_i). \tag{3.3}$$

The following abbreviations are used:

$$D_{nm} \equiv \frac{\partial^2 L}{\partial \dot{q}_n\, \partial \dot{q}_m} \equiv \frac{\partial p_n}{\partial \dot{q}_m} \equiv \frac{\partial p_m}{\partial \dot{q}_n} \tag{3.4}$$

and

$$D \equiv \det\ (D_{nm}). \tag{3.5}$$

The matrix (D_{nm}) is symmetric. It will be shown that

$$d\tau \equiv C\, D \prod_i dq_i\, d\dot{q}_i \tag{3.6}$$

is a Liouvillian phase space volume element, i.e. $d\dot{\tau} = 0$, where C is again any integral of the motion. Equivalently (see Appendix A), it suffices to prove that D is a Liouville weight function, i.e. satisfies the equation

$$\frac{\partial D}{\partial t} + \sum_i \left\{ \frac{\partial}{\partial q_i} (D\dot{q}_i) + \frac{\partial}{\partial \dot{q}_i} (D\ddot{q}_i) \right\} = 0 \quad , \tag{3.7}$$

with $\ddot{q}_i$ given by eq. (3.3). Instead of starting with eq. (1.5), a proof that uses the equations of motion [*eq.*(3.2)], is given.

To prove eq. (3.7), one expands the total time derivative of D :

$$\frac{dD}{dt} = D \sum_i \sum_k \dot{D}_{ik}\, E_{ki} \tag{3.8}$$

with E_{ki} given by

$$\sum_k D_{ik}\, E_{kj} = \delta_{ij} \tag{3.9}$$

[compare eqs. (2.8), (2.9)]. By also expanding the total time derivative of D_{ik} and using eq. (3.2) one obtains

$$\dot{D}_{ik} = \frac{\partial^2 L}{\partial q_k\, \partial \dot{q}_i} - \frac{\partial^2 L}{\partial q_i\, \partial \dot{q}_k} - \sum_n \frac{\partial a_n}{\partial \dot{q}_i} D_{nk} \tag{3.10}$$

$$= \frac{\partial^2 L}{\partial q_i\, \partial \dot{q}_k} - \frac{\partial^2 L}{\partial q_k\, \partial \dot{q}_i} - \sum_n \frac{\partial a_n}{\partial \dot{q}_k} D_{ni} \tag{3.10a}$$

or, by combining the two expressions,

$$\dot{D}_{ik} = -\frac{1}{2} \sum_n \left(\frac{\partial a_n}{\partial \dot{q}_i} D_{nk} + \frac{\partial a_n}{\partial \dot{q}_k} D_{ni} \right) . \tag{3.11}$$

Substituting eq. (3.11) in eq. (3.8) and using eq. (3.9) yields Lutzky's [14] result:

$$\frac{dD}{dt} = -D \sum_n \frac{\partial a_n}{\partial \dot{q}_n} \tag{3.12}$$

Owing to $\partial \dot{q}_i / \partial q_k = 0$ this can be written as

$$\frac{dD}{dt} = -D \sum_i \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial a_i}{\partial \dot{q}_i} \right) , \tag{3.13}$$

which is equivalent to eq. (3.7) because

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_i \left(\dot{q}_i \frac{\partial}{\partial q_i} + a_i \frac{\partial}{\partial \dot{q}_i} \right) . \tag{3.14}$$

This completes the proof of Liouville's theorem [*eqs.* (3.6) *and* (3.7)] by showing that D is a Liouville weight function.

When Lagrangian densities are to be constructed from Lagrangians, the use of Hamilton-Jacobi theory enables one to pass over to the desired Eulerian picture [7]. It is then desirable to construct a Liouvillian phase space volume element $d\tau$ from

Hamilton's principal function S rather than from the Lagrangian. This was done by Van Vleck [12] and later, independently, by Pfirsch [13]. A brief presentation of this construction will be given. Let

$$S = S_o(t, Q_i, \alpha_i) + s_o \tag{3.15}$$

be a complete integral of the Hamilton-Jacobi partial differential equation

$$\frac{\partial S}{\partial t} + H\left(t, Q_i, \frac{\partial S}{\partial Q_i}\right) = 0 \quad , \tag{3.16}$$

with H being the Hamiltonian [1, 16]. Let us pass from a set of canonical coordinates $\{q_i, p_i\}$ to the set $\{Q_i, \alpha_i\}$ by means of the transformation equations [1, 16]

$$q_i = Q_i \quad ; \quad p_i = \frac{\partial S}{\partial Q_i} \quad . \tag{3.17}$$

Using eq. (1.5), a conserved $\mathrm{d}\tau$ is given by

$$\mathrm{d}\tau \equiv \prod_i \mathrm{d}q_i\, \mathrm{d}p_i = \lambda(t, Q_k, \alpha_k) \prod_i \mathrm{d}Q_i\, \mathrm{d}\alpha_i \quad , \tag{3.18}$$

where the Liouville weight function λ is the Van Vleck determinant [12, 13]:

$$\lambda(t, Q_k, \alpha_k) = \frac{\partial(q_i, p_i)}{\partial(Q_k, \alpha_k)} = \det\left(\frac{\partial^2 S}{\partial Q_i\, \partial \alpha_k}\right) . \tag{3.19}$$

Conservation of $\mathrm{d}\tau$ ($\mathrm{d}\dot{\tau} = 0$) is equivalent to (see Appendix A)

$$\frac{\partial \lambda}{\partial t} + \sum_i \left\{ \frac{\partial}{\partial Q_i} (\lambda \dot{Q}_i) + \frac{\partial}{\partial \alpha_i} (\lambda \dot{\alpha}_i) \right\} = 0, \tag{3.20}$$

with $\dot{\alpha}_i = 0$, the α_i being constants of the motion [1, 16]. Hence eq. (3.20) is equivalent to

$$\frac{\partial \lambda}{\partial t} + \sum_i \frac{\partial}{\partial Q_i} (\lambda \dot{Q}_i) = 0 \tag{3.21}$$

where the partial derivatives $\partial/\partial Q_i$ are to be performed with the α_k kept constant. The generalized velocities $\dot{Q}_i$ are given by [1, 16]

$$\dot{Q}_i = \dot{q}_i = \frac{\partial H}{\partial p_i} . \tag{3.22}$$

Equation (3.21) states that

$$d\tau_Q \equiv \lambda(t, Q_k, \alpha_k) \prod_i dQ_i \tag{3.23}$$

is a conserved volume element in coordinate space. The same result is, of course, trivially obtained from the conservation of the $d\tau$ of eq. (3.18) and $\dot{\alpha}_k = 0$. Equation (3.21) can also be derived direct, as was done by Pfirsch [13]. This completes the construction of Liouville's theorem from the Hamilton-Jacobi theory.

4. Applications of Liouville's Theorem

A simple, but useful relation between Liouville weight functions and integrals of the motion exists. Suppose two different Liouville weight functions λ_1 , λ_2 are given so that both volume elements

$$d\tau_i \equiv \lambda_i \prod_n dz_n \quad , \quad i = 1,\, 2 \tag{4.1}$$

are conserved. Then

$$C \equiv d\tau_2/d\tau_1 = \lambda_2/\lambda_1 \tag{4.2}$$

is an integral of the motion because the λ_i are assumed to be explicit functions of state as explained in Sec. 1. Two different λ_i may exist if a system can be described by two non-trivially different Lagrangians [14]. The converse relation, namely that the product of a Liouville weight function and an integral of the motion is again a Liouville weight function, was already made use of in Secs. 2 and 3.

Liouville's theorem allows kinetic equations to be greatly simplified. Let the number of representative points, or particles, in phase space be defined as

$$dN \equiv f\, d\tau \quad , \tag{4.3}$$

where f is the distribution function relative to the volume element $d\tau$, which need not be Liouvillian at this point. A collisionless kinetic equation is then just an equation of continuity in phase space, reading

$$\frac{\partial}{\partial t}(\lambda f) + \sum_n \frac{\partial}{\partial z_n}(\lambda f\ \dot{z}_n) = 0. \tag{4.4}$$

This equation is equivalent to the conservation of representative points (e.g. particles) contained in a co-moving phase space volume element $d\tau$, i.e. to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}N) \equiv \frac{\mathrm{d}}{\mathrm{d}t}(f\ \mathrm{d}\tau) = 0 \quad , \tag{4.5}$$

as can be seen from Appendix A. When $\mathrm{d}\tau$ is now chosen to be Liouvillian, i.e.

$$\mathrm{d}\tau \equiv \lambda\,(t,\ z_\nu) \prod_n \mathrm{d}z_n \quad , \tag{4.6}$$

with $\mathrm{d}\dot{\tau} = 0$ and λ satisfying eq.(1.3), the kinetic equation simplifies to read

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \sum_n \dot{z}_n \frac{\partial f}{\partial z_n} = 0. \tag{4.7}$$

That is, f is now a constant of the motion. Hence any

$$f = f_o\,(C_\nu)\ , \tag{4.8}$$

where the

$$C_\nu = C_\nu\,(t,\ z_\mu) \tag{4.9}$$

are integrals of the motion ($\dot{C}_\nu = 0$) , is an explicit solution of eq. (4.7). For C_ν such that

$$\frac{\partial C_\nu}{\partial t} = 0 \;, \tag{4.10}$$

f_o is also time-independent, i.e. $\partial f_o / \partial t = 0$.

5. Conclusion

Usually Liouville's theorem is derived from canonical Hamiltonian systems [1]. In this paper alternative forms of Liouville's theorem have been derived for time-dependent, non-standard and standard Lagrangians. These results are valuable in modern Lagrangian guiding-center theories [3 – 9], particularly when coupling of guiding-center fluids in phase space with electromagnetic fields is to be described by a common Lagrangian density [7, 9].

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Appendix A: An equivalence theorem

For a $d\tau$ defined by

$$d\tau \equiv \lambda\,(t,\ z_\nu) \prod_n dz_n \equiv \lambda\, d\tau_o \quad , \tag{A.1}$$

the equivalence of the conservation of $d\tau$, i.e. of

$$d\dot{\tau} = 0, \tag{A.2}$$

and of the partial differential equation

$$\frac{\partial \lambda}{\partial t} + \sum_n \frac{\partial}{\partial z_n}\,(\lambda \dot{z}_n) = 0 \tag{A.3}$$

will be proved. It is seen, first of all, that

$$d\dot{\tau} = \frac{d\lambda}{dt}\, d\tau_o + \lambda\, d\dot{\tau}_o \tag{A.4}$$

or

$$\frac{d\dot{\tau}}{d\tau} = \frac{1}{\lambda}\frac{d\lambda}{dt} + \frac{d\dot{\tau}_o}{d\tau_o} \quad . \tag{A.5}$$

Hence it suffices to calculate $d\dot{\tau}_o/d\tau_o$.

A first brief derivation makes use of the theory of differential forms [17, 18]. The volume element $d\tau_o$ is written as an external product:

$$d\tau_o \equiv \bigwedge_n dz_n \equiv dz_1 \wedge dz_2 \wedge .... dz_N \quad . \qquad (A.6)$$

The total time derivative is then given by

$$d\dot{\tau}_o = \sum_n (-1)^{n+1} \, (dz_n)^{\bullet} \wedge \bigwedge_{m'(n)} dz_{m'} \quad , \qquad (A.7)$$

with $m'(n)$ running from 1 to N, leaving out n. On expanding

$$(dz_n)^{\bullet} = \sum_k \frac{\partial \dot{z}_n}{\partial z_k} \, dz_k \quad , \qquad (A.8)$$

substituting this in eq. (A.7), and using the antisymmetry of the external product, one obtains

$$\begin{aligned} d\dot{\tau}_o &= \sum_n (-1)^{n+1} \frac{\partial \dot{z}_n}{\partial z_n} \, dz_n \wedge \bigwedge_{m'(n)} dz_{m'} \\ &= \sum_n \frac{\partial \dot{z}_n}{\partial z_n} \bigwedge_m dz_m \\ &= d\tau_o \sum_n \frac{\partial \dot{z}_n}{\partial z_n} \quad . \end{aligned} \qquad (A.9)$$

Because of eqs. (A.5) and (A.9) the condition of incompressibility, $d\dot{\tau} = 0$, thus turns out to be equivalent to

$$\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \sum_n \frac{\partial \dot{z}_n}{\partial z_n} = 0 \quad , \tag{A.10}$$

which, after multiplying by λ and owing to

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \sum_n \dot{z}_n \frac{\partial}{\partial z_n} \quad , \tag{A.10a}$$

is in turn equivalent to eq. (A.3). This completes this first proof of equivalence. It should be noted that the expansion given in eq. (A.8) implies that $\mathrm{d}t = 0$, i.e. it is the linear part of a Taylor expansion of $[\dot{z}_n\,(2) - \dot{z}_n\,(1)]$, with the $\dot{z}_n$ taken at different points $\{z_\nu\}$, but at equal times t.

A second, more pedestrian proof of the equivalence of eqs. (A.2) and (A.3) goes as follows: Consider the solutions of the equations of motion, i.e. the orbits

$$z_n = z_n\,(t,\, C_\nu) \quad , \tag{A.11}$$

the C_ν being a complete set of constants of the motion (e.g. the initial values of the z_ν). If the co-moving volume element

$$\mathrm{d}\tau_o\,(t) \equiv \prod_n \mathrm{d}\,[z_n\,(t)] \tag{A.12}$$

of an infinitesimal neighborhood of an orbit is considered at two slightly different times, $\mathrm{d}\dot{\tau}_o$ can be determined as

$$\mathrm{d}\dot{\tau}_o = \lim_{\Delta t \to 0} \frac{\mathrm{d}\tau_o\,(t + \Delta t) - \mathrm{d}\tau_o\,(t)}{\Delta t} \quad . \tag{A.13}$$

On using the expansion

$$z_n\ (t\ +\ \Delta t)\ =\ z_n\ (t)\ +\ \dot{z}_n\ [t,\ z_\nu\ (t)]\ \Delta t +\ 0\ [(\Delta t^2)] \qquad (A.14)$$

one can also expand the Jacobian

$$\begin{aligned} J\ &\equiv\ \det\left\{\frac{\partial[z_n\ (t\ +\ \Delta t)]}{\partial[z_m\ (t)]}\right\} \\ &= \det\left\{\delta_{nm}\ +\ \frac{\partial \dot{z}_n}{\partial z_m}\ \Delta t\ +\ 0\ [(\Delta t)^2]\right\} \\ &= 1\ +\ \Delta t\ \cdot\ \sum_n \frac{\partial \dot{z}_n}{\partial z_n}\ +\ 0[(\Delta t)^2]\quad . \end{aligned} \qquad (A.15)$$

Because of

$$d\tau_o\ (t\ +\ \Delta t)\ =\ J\ d\tau_o\ (t) \qquad (A.16)$$

one obtains from eq. (A.13)

$$d\dot{\tau}_o\ =\ \lim_{\Delta t\to 0}\ \frac{J\ -\ 1}{\Delta t}\ d\tau_o \qquad (A.17)$$

or

$$d\dot{\tau}_o\ = d\tau_o \sum_n \frac{\partial \dot{z}_n}{\partial z_n}\quad . \qquad (A.18)$$

This agrees with eq. (A.9). The remaining arguments of the equivalence proof are the same as above, after eq. (A.9). This second method of proof was given before, e.g. in Ref. [19], where it was applied to the special case of a Hamiltonian system.

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