Regularization of Hamilton-Lagrangian Guiding Center Theories

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Abstract

The Hamilton-Langrangian guiding-center (G.C.) theories of Littlejohn, Wimmel, and Pfirsch show a singularity for $\underline{B}$-fields with non-vanishing parallel curl at a critical value of $v_\parallel$, which complicates applications. The singularity is related to a sudden breakdown, at a critical $v_\parallel$, of gyration in the exact particle mechanics. While the latter is a real effect, the G.C. singularity can be removed. To this end a regularization method is defined that preserves the Hamilton-Lagrangian structure and the conservation theorems. For demonstration this method is applied to the standard G.C. theory (without polarization drift). Liouville's theorem and G.C. kinetic equations are also derived in regularized form. The method could equally well be applied to the case with polarization drift and to relativistic G.C. theory.

1. Introduction

In recent years, guiding-center mechanics and guiding-center kinetic theory have been reformulated in an important way. By using Langrangian and/or Hamilton theory in non-canonical coordinates, guiding-center theory was endowed with exact conservation theorems for the guiding-center particles and for the guiding center Vlasov fluids; in particular, exact Liouville's theorems were constructed and used to obtain Liouville-Vlasov-type kinetic equations [1 - 6].

While these new theories are superior by virtue of their symmetries and their exact conservation laws, they are nevertheless subject to the following problem. In magnetic fields with non-vanishing parallel curl, i.e. $\underline{B}\cdot\text{curl } \underline{B} \neq 0$, the guiding-center drift velocity $\underline{v}$ and the acceleration $\dot{v}_{\parallel}$ will diverge on a hyper-surface in phase space, e.g. for large values of $|v_{\parallel}|$. Non-causal guiding-center orbits then occur, and particle conservation is violated. This is seen by observing [1 - 6] that $\underline{v}$ and $\dot{v}_{\parallel}$ are always given by the following general expressions:

$$\underline{v} = v_{\parallel}\frac{\underline{B}^*}{B^*_{\parallel}} + \frac{c}{e B^*_{\parallel}}(e\underline{E}^* - \nabla W_K)\times\hat{\underline{b}}\,, \tag{1.1}$$

$$\dot{v}_{\parallel} = \frac{1}{m B^*_{\parallel}}\underline{B}^*\cdot(e\underline{E}^* - \nabla W_K)\,, \tag{1.2}$$

where the definition of the quantity $B_\parallel^*$ in the denominators depends on the particular G.C. theory under consideration [1 - 6], but in any case critical values v_c of $v_\parallel$ exist such that $B_\parallel^*(t, \mathbf{x}, v_\parallel = v_c) = 0$ if $\underline{B}\cdot \mathrm{curl}\, \underline{B} \neq 0$. As an example, in the non-relativistic G.C. mechanics without polarization drift $B_\parallel^*$ is given by

$$B_\parallel^* \equiv \underline{B}^* \cdot \hat{\underline{b}} = B + \frac{mc}{e} v_\parallel \hat{\underline{b}} \cdot \mathrm{curl}\, \hat{\underline{b}} , \qquad (1.3)$$

whence the critical $v_\parallel$ reads

$$v_c = - \frac{eB}{mc\, \hat{\underline{b}} \cdot \mathrm{curl}\, \hat{\underline{b}}} \equiv - \frac{\Omega}{\hat{\underline{b}} \cdot \mathrm{curl}\, \hat{\underline{b}}} \gtrless 0 . \qquad (1.4)$$

Here $\hat{\underline{b}} = \underline{B}/B$, and the remaining notation can be looked up in Refs. [1 - 6] and in Sec.2 of this paper.

We may introduce the "twist length" L_t by the definition

$$L_t = | (\hat{\underline{b}} \cdot \mathrm{curl}\, \hat{\underline{b}}) |^{-1} . \qquad (1.4a)$$

It is seen that $|v_c|$ is large compared with the gyration speed $V_\perp$ of a particle because

$$\left| \frac{v_c}{V_\perp} \right| = \frac{|\Omega| L_t}{V_\perp} = L_t / R_g \qquad (1.4b)$$

and the validity of G.C. theory requires that $R_g \ll L_t$. Numerically, $|v_c|$ can be large, as is shown in Sec. 5 below. If this singularity is not removed by a regularization procedure it is necessary to exclude all orbits that intersect, or are tangential to, the singular hyper-surface mentioned. For instance,

the use of Maxwell distributions and all other distributions with arbitrarily large values of $v_\parallel$ is forbidden. Introduction of diffusion-type collision terms, e.g. Fokker-Planck terms, is also impossible. In order to improve on this situation, a <u>regularized version</u> of Lagrangian G.C. mechanics is presented in Secs. 2 and 3 of this paper. This is followed by a derivation of Liouville's theorem and kinetic equations in Sec. 4. It is noteworthy that a related phenomenon exists in exact particle mechanics, namely a sudden breakdown of gyro-motion at a critical value of $v_\parallel$ that occurs if $\underline{B}\cdot \mathrm{curl}\ \underline{B} \neq 0$. This point is presented in Sec. 5. Section 6 contains the conclusions.

2. Regularized Lagrange equations for guiding-center motion

In order to demonstrate our method of regularization, we start from unregularized G.C. mechanics with the Lagrangian

$$L \equiv \frac{e}{c}\underline{A}^{*}\cdot\underline{v} - e\phi - W_K , \qquad (2.1)$$

with L depending on time t, the G.C. variables $\mathbf{x}$, $\mathbf{v} \equiv \dot{\mathbf{x}}$, $v_\parallel$, and the parameters μ (magnetic moment), m, e, c. Here $\underline{A}^{*}$ is a modified electromagnetic vector potential [1 - 6], viz.

$$\underline{A}^{*}(t,\underline{x},v_\parallel) \equiv \underline{A}(t,\underline{x}) + \frac{mc}{e} v_\parallel \hat{\underline{b}} , \qquad (2.2)$$

$\phi(t,\mathbf{x})$ is the scalar electric potential, and W_K is the G.C. kinetic energy in the form

$$W_K(t,\underline{x},v_\parallel,\mu) \equiv \mu B(t,\underline{x}) + \frac{m}{2} v_\parallel^2 . \qquad (2.3)$$

The other quantities are: $\mathbf{A}(t,\mathbf{x})$ the usual vector potential, $\mathbf{B}(t,\mathbf{x})$ the magnetic field, $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ the unit vector in the magnetic field direction; the scalar magnetic moment μ is an adiabatic invariant, i.e. $\dot{\mu} = 0$, and $v_\parallel$ is the "parallel velocity" of the G.C., i.e. parallel to $\mathbf{B}$; but the relation $v_\parallel = \mathbf{v}\cdot\hat{\mathbf{b}}$

is not implied here; it only follows as one of the (unregularized) Lagrangian equations. Equations (2.1) to (2.3) follow from the corresponding equations of Ref. [5] by putting there $v_E = 0$. Hence, the above equations represent non-relativistic G.C. mechanics without polarization drift. The theory then contains the three standard drifts, slightly modified, and an additional drift related to the time-derivative of $\hat{b}$. We shall refer to it as the "standard G.C. theory".

It is the $v_\parallel$-dependence of A^* [eq. (2.2)] that produces the singularity as indicated by eqs. (1.3) and (1.4). In order to regularize this singularity, $v_\parallel$ in eq. (2.2) is replaced by a function $v_0\, g(v_\parallel/v_0)$ of $v_\parallel$ that approaches $v_\parallel$ in the validity range of G.C. mechanics ($|v_\parallel|$ small), while it approaches constant values outside the validity range, i.e. for large values of $|v_\parallel|$. To be more specific, eq. (2.2) is replaced by

$$\underline{A}^* \equiv \underline{A} + \frac{mc}{e} v_0 g\left(\frac{v_\parallel}{v_0}\right) \hat{\underline{b}} , \tag{2.4}$$

with $v_0 = \text{const} > 0$, $g(z) \sim z$, $g'(z) \sim 1$ for $z \ll 1$, $g(z)$ being defined in $-\infty < z < +\infty$. The function $g(z)$ has to be monotonically increasing and antisymmetric with respect to $z = 0$, and it is required that $g(z) \sim \pm g_\infty = \pm\text{const}$ ($g_\infty > 0$) for $z \to \pm\infty$. We shall also assume that

$$g'(z) \sim |z|^{-\alpha} \quad \text{for} \quad z \longrightarrow \pm\infty , \tag{2.5}$$

with $1 < \alpha < \infty$, in order for $\underline{v}$ and $\dot{v}_\parallel$ to diverge not faster than a finite power of $|v_\parallel|$ for $v_\parallel \rightarrow \pm\infty$. It should be noted that this new divergence is completely harmless because it occurs for infinite $|v_\parallel|$, contrary to the original singularity at finite v_c of eq. (1.4). Finally, v_o and g_∞ must satisfy the conditions $v_{th} \ll v_o$ and

$$v_o g_\infty \ll |v_c| \equiv |\Omega| / |\hat{\underline{b}} \cdot \mathrm{curl}\, \hat{\underline{b}}| . \tag{2.6}$$

As an example, one may choose $g(z) = \mathrm{arctg}\ z$, i.e., $g_\infty = \pi/2$, $g'(z) = 1/(1+z^2)$, $\alpha = 2$. It should be noted that v_o may be chosen different for ions and electrons. When this regularized mechanics is used in a kinetic equation, the distribution functions should, of course, be vanishingly small outside the validity range of the G.C. approximation and outside the validity range of $v_o\, g(v_\parallel / v_o) \sim v_\parallel$. A suitable value of v_o can always be determined in the validity range of G.C. theory because of $|v_c| \gg V_\perp$ [see eq. (1.4b)].

Before deriving the regularized Lagrangian equations of G.C. motion let us define the "modified fields" $\underline{E}^*$ and $\underline{B}^*$ [1 - 6] as

$$\underline{B}^* \equiv \mathrm{curl}\ \underline{A}^* , \tag{2.7}$$

$$\underline{E}^* \equiv -\nabla \phi - \frac{1}{c} \frac{\partial \underline{A}^*}{\partial t} , \tag{2.8}$$

where the partial time and space derivatives are performed with $v_\parallel$ and μ kept constant. It follows that

$$\operatorname{div} \underline{B}^* = 0 , \tag{2.9}$$

$$\frac{\partial \underline{B}^*}{\partial t} = - c \operatorname{curl} \underline{E}^* . \tag{2.10}$$

Explicit regularized expressions of $\underline{B}^*$ and $\underline{E}^*$ are

$$\underline{B}^* = \underline{B} + \frac{mc}{e} v_0 g \operatorname{curl} \underline{\hat{b}} , \tag{2.11}$$

$$\underline{E}^* = \underline{E} - \frac{m}{e} v_0 g \frac{\partial \underline{\hat{b}}}{\partial t} . \tag{2.12}$$

An important quantity is the "parallel" component of $\underline{B}^*$; in its regularized form it now reads

$$B_\parallel^* \equiv \underline{B}^* \cdot \underline{\hat{b}} = B + \frac{mc}{e} v_0 g \, \underline{\hat{b}} \cdot \operatorname{curl} \underline{\hat{b}} , \tag{2.13}$$

which is positive-definite, in contrast to eq. (1.3).

The regularized Lagrange equations of G.C. motion are now given as they follow from eqs. (2.1), (2.3), (2.4) and (2.7) to (2.13). Firstly,

$$\frac{\partial L}{\partial v_{\parallel}} = 0 \tag{2.14}$$

yields

$$\underline{v} \cdot \hat{\underline{b}} = v_{\parallel} / g' . \tag{2.15}$$

The other equations of motion are obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}}\right) = \nabla L , \tag{2.16}$$

with the definition

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \nabla + \dot{v}_{\parallel} \frac{\partial}{\partial v_{\parallel}} + \dot{\underline{v}} \cdot \frac{\partial}{\partial \underline{v}} . \tag{2.17}$$

Because $\partial L / \partial \underline{v}$ is independent of $\underline{v}$, the last term on the r.h.s. of eq. (2.17) does not contribute. The phase space, or state space, of the guiding centers is spanned by the variables $\underline{x}$ and $v_{\parallel}$, and by the parameter μ. Equation (2.16) reads

$$\frac{e}{c} \frac{d\underline{A}^*}{dt} = \frac{e}{c} \nabla (\underline{A}^* \cdot \underline{v}) - \nabla W , \tag{2.18}$$

with the variable $\underline{v} \equiv \dot{\underline{x}}$ to be taken independent of $\underline{x}$ (and of t, in the sense of partial derivatives). The quantity W is defined as

$$W \equiv W_k + e\phi . \tag{2.19}$$

Equation (2.18) can be transformed to read

$$m g' \dot{v}_{\parallel} \hat{\underline{b}} = \frac{e}{c} \underline{v} \times \underline{B}^* + (e\underline{E}^* - \nabla W_k) . \tag{2.20}$$

Equations (2.15) and (2.20) can be combined and solved for $\underline{v}$ and $\dot{v}_{\parallel}$:

$$\underline{v} = \frac{v_{\parallel}}{g'} \frac{\underline{B}^*}{B^*_{\parallel}} + \frac{c}{e B^*_{\parallel}} (e\underline{E}^* - \nabla W_K) \times \hat{\underline{b}} \tag{2.21}$$

and

$$\dot{v}_{\parallel} = \frac{1}{m g' B^*_{\parallel}} \underline{B}^* \cdot (e\underline{E}^* - \nabla W_K) \tag{2.22}$$

$$= \frac{1}{m v_{\parallel}} \underline{v} \cdot (e\underline{E}^* - \nabla W_K) . \tag{2.23}$$

By substituting the modified fields and ∇W_k more explicit expressions are obtained:

$$\underline{v} = \frac{v_\parallel}{g'}\hat{\underline{b}} + \frac{c}{B_\parallel^*}\underline{E}\times\hat{\underline{b}} + \frac{\mu}{m\Omega^*}\hat{\underline{b}}\times\nabla B$$

$$+ \frac{v_\parallel v_0 g}{g'\Omega^*}\hat{\underline{b}}\times\frac{\partial\hat{\underline{b}}}{\partial s} + \frac{v_0 g}{\Omega^*}\hat{\underline{b}}\times\frac{\partial\hat{\underline{b}}}{\partial t}, \qquad (2.24)$$

with

$$\Omega^* \equiv \frac{e B_\parallel^*}{mc} = \Omega + v_0 g\,\hat{\underline{b}}\cdot\mathrm{curl}\,\hat{\underline{b}}, \qquad (2.25)$$

and

$$\dot{v}_\parallel = \frac{1}{mg'}\left(eE_\parallel - \mu\frac{\partial B}{\partial s}\right) + \frac{v_0 g}{g'}\underline{v}\cdot\frac{\partial\hat{\underline{b}}}{\partial s}, \qquad (2.26)$$

with the usual definition $\partial/\partial s \equiv \hat{\underline{b}}\cdot\nabla$. It is seen from eqs. (2.24) to (2.26) and the above definition of g(z) that the divergence of $\underline{v}$ and $\dot{v}_\parallel$ at $v_\parallel = v_c$ [eq. (1.4)] has been eliminated by the regularization procedure.

The energy equation following from the above equations of motion reads

$$\frac{dW_k}{dt} = \mu \frac{\partial B}{\partial t} + e\,\underline{E}^{*} \cdot \underline{v} \tag{2.27}$$

or, more explicitly,

$$\frac{dW_k}{dt} = e\,\underline{E} \cdot \underline{v} + \mu \frac{\partial B}{\partial t} - m v_o g\, \underline{v} \cdot \frac{\partial \hat{\underline{b}}}{\partial t} . \tag{2.28}$$

This can be given the form

$$\frac{dW_k}{dt} = e\,\underline{E} \cdot \underline{v} - \underline{\mu} \cdot \frac{\partial \underline{B}}{\partial t} , \tag{2.29}$$

where the vectorial magnetic moment $\underline{\mu}$ is defined by

$$\underline{\mu} \equiv -\mu \hat{\underline{b}} + \frac{m v_o g}{B} \underline{v}_{\perp} . \tag{2.30}$$

Equations (2.29) and (2.30) will be useful in G.C. kinetic theory, as given in Sec. 4. For time-independent fields conservation of single-guiding-center energy follows in the form

$$\frac{d}{dt}\left(W_k + e\phi\right) \equiv \frac{dW}{dt} = 0 . \tag{2.31}$$

Liouville's theorem will be considered in Sec. 4.

3. Energy representation

In some cases, it is useful to consider $v_\parallel$ as a dependent variable which is determined by the relation

$$W = \frac{m}{2} V_\parallel^2 + e\phi + \mu B . \tag{3.1}$$

It should be noted that we have now written $V_\parallel$ (capitalized) to stress the fact that it is considered to be a function, depending on time t, position $\mathbf{x}$, total energy W and, of course, also on the parameters m, e and μ. We thus have

$$V_\parallel = V_\parallel (t, \underline{x}, W) . \tag{3.2}$$

From eqs. (3.1) and (3.2) we can derive the useful relations

$$\frac{\partial V_\parallel}{\partial t} = - \frac{1}{m V_\parallel} \left(e \frac{\partial \phi}{\partial t} + \mu \frac{\partial B}{\partial t} \right) , \tag{3.3}$$

$$\nabla V_\parallel = - \frac{1}{m V_\parallel} \left(e \nabla \phi + \mu \nabla B \right) , \tag{3.4}$$

$$\frac{\partial V_\parallel}{\partial W} = \frac{1}{m V_\parallel} , \tag{3.5}$$

where, of course, the time and space derivatives are now formed with W kept constant. As in the calculations of Sec. 2, we introduce the modified vector potential $\underline{A}^+$:

$$\underline{A}^+ \equiv \underline{A} + \frac{mc}{e} v_0 g\left(\frac{V_\parallel}{v_0}\right)\hat{\underline{b}} , \tag{3.6}$$

which now also depends on W through the dependent variable $V_\parallel$. The corresponding modified fields $\underline{B}^+$ and $\underline{E}^+$ are then

$$\underline{B}^+ \equiv \operatorname{curl} \underline{A}^+ \tag{3.7}$$

and

$$\underline{E}^+ \equiv -\nabla\phi - \frac{1}{c}\frac{\partial \underline{A}^+}{\partial t} . \tag{3.8}$$

Written explicitly, these expressions read

$$\underline{B}^+ = \underline{B} + \frac{mc}{e} v_0 g \operatorname{curl} \hat{\underline{b}} + \frac{mc}{e} g' \nabla V_\parallel \times \hat{\underline{b}} \tag{3.9}$$

and

$$\underline{E}^{+} = \underline{E} - \frac{m}{e} v_0 g \frac{\partial \hat{\underline{b}}}{\partial t}$$

$$- \frac{m}{e} g' \frac{\partial V_{\parallel}}{\partial t} \hat{\underline{b}} \ . \qquad (3.10)$$

Though $\underline{B}^{+}$ and $\underline{E}^{+}$ obviously differ from $\underline{B}^{*}$ and $\underline{E}^{*}$ [eqs. (2.11) and (2.12)], the components of $\underline{B}^{+}$ and $\underline{B}^{*}$ parallel to $\underline{B}$ and of $\underline{E}^{+}$ and $\underline{E}^{*}$ perpendicular to $\underline{B}$ are the same, viz.

$$\underline{B}^{+} \cdot \hat{\underline{b}} = \underline{B}^{*} \cdot \hat{\underline{b}} \ , \qquad (3.11)$$

$$\underline{E}^{+} \times \hat{\underline{b}} = \underline{E}^{*} \times \hat{\underline{b}} \ . \qquad (3.12)$$

Expressed in the new variables, the regularized Lagrangian is now

$$L^{+} \equiv \frac{e}{c} \underline{v} \cdot \underline{A}^{+} - W \ . \qquad (3.13)$$

It thus follows that

$$L^{+} = L^{+}(t, \underline{x}, W, \underline{v}) \qquad (3.14)$$

and the total time derivative is correspondingly given by

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{v}\cdot\nabla + \dot{W}\frac{\partial}{\partial W} + \dot{\underline{v}}\cdot\frac{\partial}{\partial \underline{v}} . \tag{3.15}$$

Again, the term $\dot{\underline{v}}\cdot\partial/\partial\underline{v}$ does not contribute to the Lagrangian equations of motion.

From the equations above we can now derive the regularized equations of G.C. motion in the W-representation. Firstly, since L^+ does not depend on $\dot{W}$, we have

$$\frac{\partial L^+}{\partial W} = 0 . \tag{3.16}$$

Taking into account the relation

$$\frac{\partial \underline{A}^+}{\partial W} = \frac{c}{e}\frac{g'}{V_\parallel}\hat{\underline{b}} , \tag{3.17}$$

which follows from eqs. (3.5) and (3.6), eq. (3.16) yields the result

$$\underline{v}\cdot\hat{\underline{b}} = V_\parallel / g' , \tag{3.18}$$

which agrees with eq. (2.15). Furthermore, from eqs. (3.6) and (3.13) we obtain

$$\frac{\partial L^+}{\partial \underline{v}} = \frac{e}{c}\underline{A}^+ \tag{3.19}$$

and

$$\nabla L^{+} = \frac{e}{c}\left\{ (\underline{v}\cdot\nabla)\,\underline{A}^{+} + \underline{v}\times \operatorname{curl}\underline{A}^{+}\right\} . \tag{3.20}$$

Then, Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L^{+}}{\partial \underline{v}}\right) = \nabla L^{+} \tag{3.21}$$

lead to the result

$$\frac{\partial \underline{A}^{+}}{\partial t} + \frac{c}{e}\frac{g'}{V_{\parallel}}\dot{W}\hat{\underline{b}} = \underline{v}\times \operatorname{curl}\underline{A}^{+} . \tag{3.22}$$

Recalling eq. (3.18), we take the scalar product of eq. (3.22) with $\underline{v}$ and obtain the energy equation in the form

$$\dot{W} = -\frac{e}{c}\,\underline{v}\cdot\frac{\partial \underline{A}^{+}}{\partial t} . \tag{3.23}$$

This, of course, is the same as eq. (2.28), as can be seen by making the appropriate substitutions.

On the other hand, the cross-product of eq. (3.22) with $\hat{\underline{b}}$ leads to a very simple expression for the guiding-center velocity, viz.

$$\underline{v} = \frac{V_{\parallel}}{B_{\parallel}^{+} g'} \operatorname{curl} \underline{A}^{+} + \frac{\hat{\underline{b}}}{B_{\parallel}^{+}} \times \frac{\partial \underline{A}^{+}}{\partial t} . \qquad (3.24)$$

One of the advantages of the energy representation is that it makes it possible to express $\underline{v}$ through such a concise equation. If it is assumed that the fields are time-independent, and that there is no parallel component of the current, then (for $g' = 1$) eq. (3.24) reduces to the expression for the guiding-center velocity with was derived in Ref. [7].

We can also write eq. (3.24) in a different form which proves useful when deriving kinetic equations. Replacing $\hat{\underline{b}}$ from eq. (3.17), we obtain

$$\underline{v} = \frac{V_{\parallel}}{B_{\parallel}^{+} g'} \left(\operatorname{curl} \underline{A}^{+} + \frac{e}{c} \frac{\partial \underline{A}^{+}}{\partial W} \times \frac{\partial \underline{A}^{+}}{\partial t} \right) . \qquad (3.25)$$

Scalar multiplication of this equation by $\partial \underline{A}^{+} / \partial t$ and comparison with eq. (3.23) yields the relation

$$\frac{\partial \underline{A}^{+}}{\partial t} \cdot (\operatorname{curl} \underline{A}^{+}) = - \frac{c}{e} \frac{B_{\parallel}^{+} g'}{V_{\parallel}} \dot{W} . \qquad (3.26)$$

Taking into account the identity

$$\nabla \cdot \left(\frac{\partial \underline{A}^{+}}{\partial W} \times \frac{\partial \underline{A}^{+}}{\partial t} \right) = \frac{\partial}{\partial W} \left(\frac{\partial \underline{A}^{+}}{\partial t} \cdot \mathrm{curl}\, \underline{A}^{+} \right)$$

$$- \frac{\partial}{\partial t} \left(\frac{\partial \underline{A}^{+}}{\partial W} \cdot \mathrm{curl}\, \underline{A}^{+} \right) \qquad (3.27)$$

and eqs. (3.7), (3.17) and (3.26) then leads [through eq.(3.25)] to the result

$$\frac{\partial}{\partial t}\left(\frac{g' B_{\parallel}^{+}}{V_{\parallel}}\right) + \nabla \cdot \left(\frac{g' B_{\parallel}^{+}}{V_{\parallel}} \underline{v}\right) + \frac{\partial}{\partial W}\left(\frac{g' B_{\parallel}^{+}}{V_{\parallel}} \dot{W}\right) = 0. \qquad (3.28)$$

This is Liouville's theorem [cf. Sec. 4] in the energy representation. However, in Sec. 4 we shall rederive Liouville's theorem in the $(t, \mathbf{x}, v_{\parallel})$ representation since this seems more practical in the context of kinetic equations.

4. Liouville's theorem and kinetic equations

In this section we shall again use the phase space coordinates $(\mathbf{x}, v_\parallel)$, rather than $(\mathbf{x}, W)$, and first of all, derive Liouville's theorem. This theorem is virtually indispensable if one wants to construct a practical kinetic equation from the equations of motion, e.g. those given in Sec. 2 .

Liouville's theorem can be derived direct from the G.C. Lagrangian of Sec. 2 by the following theorem [4, 8]. Let $(\mathbf{x}, v_\parallel)$ be replaced by (z_ν), ν = 1 to 4 and let L of eq. (2.1) be written as

$$L(z_\nu, \dot{z}_\nu, t) \equiv \sum_{m=1}^{4} \gamma_m(t, z_\nu)\dot{z}_m - \varphi(t, z_\nu) \quad . \tag{4.1}$$

The Lagrangian equations assume the form

$$\sum_{m=1}^{4} \omega_{nm}(t, z_\nu)\dot{z}_m = -\left(\frac{\partial \gamma_n}{\partial t} + \frac{\partial \varphi}{\partial z_n}\right) , \tag{4.2}$$

with the definition

$$\omega_{nm} \equiv \frac{\partial \gamma_n}{\partial z_m} - \frac{\partial \gamma_m}{\partial z_n} \quad . \tag{4.3}$$

Then the following phase space volume element $d\tau$ is Liouvillian, i.e. $d\dot{\tau} = 0$ along orbits:

$$d\tau \equiv \lambda(z_\nu, t)\, dz_1 dz_2\, dz_3 dz_4 \equiv \lambda\, d^3x\, dv_{\parallel} , \tag{4.4}$$

with

$$\lambda \equiv |\Delta|^{1/2} , \tag{4.5}$$

$$\Delta \equiv \det(\omega_{mn}) . \tag{4.6}$$

A proof of this theorem is given elsewhere [8]. Upon comparing eq. (4.1) with eqs. (2.1), (2.3), (2.4) it turns out that

$$\omega_{nn} = 0 , \quad n = 1 \text{ to } 4 , \tag{4.7}$$

$$\omega_{0i} = -\omega_{i0} = -mg'b_i , \quad i = 1, 2, 3 , \tag{4.8}$$

$$\left.\begin{aligned} \omega_{12} &= -\omega_{21} = -B_3^* \\ \omega_{23} &= -\omega_{32} = -B_1^* \\ \omega_{31} &= -\omega_{13} = -B_2^* \end{aligned}\right\} , \tag{4.9}$$

where b_i are the Cartesian components of $\hat{\underline{b}}$, and $\underline{B}^*$ is defined in eq. (2.11). It follows that

$$\lambda = \left| mg' \left(\frac{e}{c} B + m v_0 g \, \hat{\underline{b}} \cdot \mathrm{curl} \, \hat{\underline{b}} \right) \right|$$

$$= \left| mg' \frac{e}{c} B^*_{\parallel} \right| \quad . \qquad (4.10)$$

By adjusting the constant factor we obtain the Liouville volume element as

$$d\tau \equiv \frac{2\pi}{m} \left| g' B^*_{\parallel} \right| d^3x \, dv_{\parallel} \, d\mu \, , \qquad (4.11)$$

where we have extended the phase space by the parameter μ . To check that $d\dot{\tau} = 0$ does in fact hold along phase space orbits, one shows that the following partial differential equation is satisfied [1 - 6, 8, 9]:

$$\frac{\partial}{\partial t} (g' B^*_{\parallel}) + \nabla \cdot (g' B^*_{\parallel} \underline{v}) + \frac{\partial}{\partial v_{\parallel}} (g' B^*_{\parallel} \dot{v}_{\parallel})$$

$$+ \frac{\partial}{\partial \mu} (g' B^*_{\parallel} \dot{\mu}) = 0 \quad . \qquad (4.12)$$

This proof is analogous to those given in Refs. [2, 3, 5] and requires using eqs. (2.21) and (2.22) for $\underline{v}$ and $\dot{v}_\parallel$ as well as the modified homogeneous Maxwell equations, eqs. (2.9), (2.10), and the relation $\dot{\mu} = 0$.

We shall now derive a collisionless kinetic theory for the above regularized G.C. mechanics [Secs. 2, 3]. Our phase space is now 5-dimensional, with the coordinates $\{\alpha_i\} \equiv \{\underline{x}, v_\parallel, \mu\}$, i = 1 to 5. The guiding-center distribution f is defined by

$$dN \equiv f \, d\tau , \tag{4.13}$$

with $f = f(t, \underline{x}, v_\parallel, \mu)$, dN being the number of guiding centers in a phase space volume element $d\tau$. The collisionless kinetic equation expresses conservation of dN in a volume element $d\tau$ that moves with the guiding centers, i.e.

$$\frac{d}{dt}(dN) \equiv \frac{d}{dt}(f \, d\tau) = 0 , \tag{4.14}$$

or, equivalently [2, 8, 9],

$$\frac{\partial}{\partial t}(g' B_\parallel^* f) + \nabla\cdot(g' B_\parallel^* \underline{v} f)$$

$$+ \frac{\partial}{\partial v_\parallel}(g' B_\parallel^* \dot{v}_\parallel f) + \frac{\partial}{\partial \mu}(g' B_\parallel^* \dot{\mu} f) = 0 , \tag{4.15}$$

with $\dot{\mu} = 0$. Equation (4.15) holds independently of whether $d\tau$ is Liouvillian or not. By employing the Liouvillian $d\tau$ of eq. (4.11), eq. (4.12) can be used in order to obtain the kinetic equation in Liouville-Vlasov form, viz.

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \dot{v}_{\parallel} \frac{\partial f}{\partial v_{\parallel}} + \dot{\mu} \frac{\partial f}{\partial \mu} = 0 \,, \qquad (4.16)$$

with $\dot{\mu} = 0$. When moment equations are to be constructed from the G.C. kinetic equation, the use of eq. (4.15) is more advantageous. On the other hand, eq. (4.16) can more easily be manipulated and solved.

The moment equations for G.C. number and energy can be derived from the above equations in the same way as in earlier papers [2, 3]. Defining the G.C. density

$$n \equiv \int f \, d\tau_v \,, \qquad (4.17)$$

with $d\tau_v$ being the volume element in G.C. velocity space, viz. [compare with eq. (4.11)]

$$d\tau_v \equiv \frac{2\pi}{m} \left| g' B_{\parallel}^* \right| dv_{\parallel} \, d\mu \,, \qquad (4.18)$$

and defining, furthermore, the G.C. flux density

$$\underline{\Gamma} \equiv \int \underline{v}\, f\, d\tau_v \qquad (4.19)$$

one obtains the equation of continuity

$$\frac{\partial n}{\partial t} + \nabla\cdot\underline{\Gamma} = 0 . \qquad (4.20)$$

As to energy, one defines [2, 3] the G.C. energy density as

$$D \equiv \int W_K\, f\, d\tau_v \quad , \qquad (4.21)$$

the kinetic energy flux density as

$$\underline{F}_1 \equiv \int W_K\, \underline{v}\, f\, d\tau_v \quad , \qquad (4.22)$$

and the vectorial magnetic moment density as

$$\underline{M} \equiv \int \underline{\mu}\, f\, d\tau_v \quad , \qquad (4.22a)$$

with W_K and $\underline{\mu}$ given by eqs. (2.3) and (2.30). A preliminary form of the energy equation is then

$$\frac{\partial D}{\partial t} + \nabla\cdot\underline{F}_1 = \int \frac{dW_K}{dt}\, f\, d\tau_v . \qquad (4.23)$$

On using eq. (2.29) and one of Maxwell's equations one obtains instead

$$\frac{\partial D}{\partial t} + \nabla \cdot \underline{F} = \underline{E} \cdot (e \underline{\Gamma} + c \operatorname{curl} \underline{M}) \quad , \tag{4.24}$$

with the total G.C. energy flux density

$$\underline{F} \equiv \underline{F}_1 + c \, \underline{M} \times \underline{E} \tag{4.25}$$

and the effective G.C. current density

$$\underline{j}_{eff} \equiv e \underline{\Gamma} + c \operatorname{curl} \underline{M} \tag{4.26}$$

while the effective G.C. charge density is simply

$$\rho_{eff} \equiv en \; . \tag{4.27}$$

If the sum is taken over charge and current densities of the G.C. plasma components $\alpha = i, e$:

$$\rho_{total} \equiv \sum_{\alpha} \rho_{eff} \; ; \quad \underline{j}_{total} \equiv \sum_{\alpha} \underline{j}_{eff} \; , \tag{4.28}$$

and if total charge and current densities are introduced into Maxwell's equations, then one finds conservation of total energy:

$$\frac{\partial}{\partial t}\left\{ D_{total} + \frac{E^2 + B^2}{8\pi} \right\} + \nabla\cdot\left\{ \underline{F}_{total} + \frac{c}{4\pi}\, \underline{E}\times\underline{B} \right\} = 0\,, \qquad (4.29)$$

with

$$D_{total} = \sum_{\alpha} D_{\alpha}\;; \qquad \underline{F}_{total} = \sum_{\alpha} \underline{F}_{\alpha}\;. \qquad (4.30)$$

The definitions of G.C. charge, current, energy, and energy flux densities could have been derived in a more systematic way by Pfirsch's variational method [6]; for brevity, this aspect of the G.C. fluid conservation theorems is not covered in the present paper.

5. Exact equations of particle motion

5.a Sudden breakdown of gyration (axially symmetric case)

In the exact particle picture, i.e. without the G.C. approximation, there is a phenomenon that is related to the singularity in G.C. mechanics at $v_\parallel = v_c$ [eq. (1.4)]. Namely, in (static) magnetic fields with $\underline{B}\cdot\mathrm{curl}\,\underline{B} \neq 0$ a sudden breakdown of gyro-motion occurs when $v_\parallel$ is continuously varied. This gyro-breakdown is related to a critical $v_\parallel = \hat{v}_c$, where

$$\hat{v}_c = -\frac{\Omega}{2\,\hat{\underline{b}}\cdot\mathrm{curl}\,\hat{\underline{b}}} = \frac{v_c}{2} \gtrless 0 \qquad (5a.1)$$

is valid in a $\underline{B}$-field of axial symmetry. The gyro-motion is then absent for $v_\parallel/\hat{v}_c > 1$ even though the usual validity conditions of G.C. theory are satisfied, i.e. the gyro-radius R_g is much smaller than all inhomogeneity lengths, including the "twist length" $L_t \equiv |\hat{\underline{b}}\cdot\mathrm{curl}\,\hat{\underline{b}}|^{-1}$, with $R_g \equiv v_\perp/|\Omega|$. Because this gyro-breakdown effect seems to be largely unknown, we think it worthwhile to present it here for the case of simple model fields.

We consider a charged particle in a magnetic field of axial symmetry, viz. with the vector potential (in cylindrical coordinates)

$$A_r = 0 \;;\quad A_\varphi = \frac{B_0}{2}r \;;\quad A_z = -\frac{\alpha B_0}{2}r^2 , \qquad (5a.2)$$

and the field components

$$B_r = 0 \quad ; \quad B_\varphi = \alpha B_0 r \quad ; \quad B_z = B_0 \quad , \tag{5a.3}$$

B_0 and α being constants. This field is "twisted", i.e. has parallel curl $\neq 0$, viz. (on the axis)

$$\underline{\hat{b}} \cdot \mathrm{curl}\, \underline{\hat{b}} = 2\alpha \quad . \tag{5a.4}$$

Of course, the parallel curl of $\underline{B}$ requires a parallel current density. The Lagrange function of the particle is given by

$$L \equiv \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2 \right)$$

$$+ \frac{e}{c} \left(A_r \dot{r} + A_\varphi r \dot{\varphi} + A_z \dot{z} \right) , \tag{5a.5}$$

with the canonical momenta

$$p_r = m \dot{r} \quad , \tag{5a.6}$$

$$p_\varphi = m r^2 \left(\dot{\varphi} + \frac{\Omega_0}{2} \right) , \tag{5a.7}$$

$$p_z = m \left(\dot{z} - \frac{\Omega_0 \alpha}{2} r^2 \right) , \tag{5a.8}$$

with $\Omega_o = eB_o/mc$. Owing to the symmetry, p_φ and p_z are constants of the motion as also, of course, is the kinetic energy. The equations of motion are given by

$$\dot{v}_r = \frac{v_\varphi^2}{r} + \Omega_o v_\varphi - \alpha \Omega_o r v_z , \tag{5a.9}$$

$$\dot{v}_\varphi = -\frac{v_r v_\varphi}{r} - \Omega_o v_r \tag{5a.10}$$

$$\dot{v}_z = \alpha \Omega_o r v_r , \tag{5a.11}$$

with the definitions $v_r \equiv \dot{r}$, $v_\varphi \equiv r\dot{\varphi}$, $v_z \equiv \dot{z}$. In order to determine the gyration frequency ω and the gyro-radius $r = r_g$, we consider a particle with its <u>guiding center</u> at $r = 0$, so that $r \equiv r_g = \text{const}$, $\dot{\varphi} = \text{const}$, $v_\varphi = \text{const}$, $v_z = \text{const}$ must hold for motion periodic in the (r,φ)-plane. Then ω and $r = r_g$ are determined by quadratic equations:

$$\omega^2 + \Omega_o \omega - \alpha \Omega_o v_z = 0 \tag{5a.12}$$

and

$$\left(\frac{v_\varphi}{r}\right)^2 + \Omega_o\left(\frac{v_\varphi}{r}\right) - \alpha \Omega_o v_z = 0 , \tag{5a.13}$$

with v_φ given; sign v_φ = sign ω is necessary in order for r to be positive whenever it is real valued.

It is seen from eqs. (5a.12) and (5a.13) that the solution for ω and $r = r_g$ are only real-valued provided that $v_z/\hat{v}_c \leq 1$, with

$$\hat{v}_c = -\frac{\Omega_o}{4\alpha} \gtrless 0 \; . \tag{5a.14}$$

In the opposite case, e.g. $v_z/\hat{v}_c > 1$, the particle orbits are qualitatively different (see below), and there is nothing resembling gyration any more. It is important to note that this sudden transition occurs for values of ω and $r = r_g$ that would seem to be in agreement with the usual validity conditions of G.C. mechanics, i.e. the critical values of ω and $r = r_g$ are

$$\omega_c = -\Omega_o / 2 \tag{5a.15}$$

and

$$r_c = 2 v_\varphi / \Omega_o \equiv 2 r_{go} \ll 2 L_t \tag{5a.16}$$

where r_{go} is the gyro-radius obtained for $v_z = 0$. More general considerations and calculation of exact orbits are better done in a planar geometry (below).

5.b Exact orbits of unbounded non-gyro motion in planar geometry

In order to further illustrate the breakdown of gyro-motion in a magnetic field with a parallel component of the current ($\underline{B}\cdot \mathrm{curl}\, \underline{B} \neq 0$), let us consider a particle of charge e and mass m moving in a magnetic field $\underline{B}$ with a vector potential $\underline{A}$ given by

$$\underline{A} = \frac{B_0}{a}\left\{ (\cos(ay) - 1)\, \hat{\underline{e}}_x + \sin(ay)\, \hat{\underline{e}}_z \right\}. \qquad (5b.1)$$

Here x, y, z are Cartesian coordinates and B_0 and a are constants. Then, for the magnetic field $\underline{B}$ and the electric current density $\underline{j}$ we obtain

$$\underline{B} = B_0 \left\{ \cos(ay)\, \hat{\underline{e}}_x + \sin(ay)\, \hat{\underline{e}}_z \right\} \qquad (5b.2)$$

and

$$\underline{j} = \frac{c}{4\pi}\, a\, \underline{B} \,, \qquad (5b.3)$$

i.e.

$$a = \frac{\underline{B}\cdot \mathrm{curl}\, \underline{B}}{B^2} \,. \qquad (5b.4)$$

Equation (5b.2) thus describes a magnetic field of constant magnitude ($|\underline{B}| = B_0$), with straight lines which twist as one proceeds in the y-direction. Then, according to the standard guiding center theory, there should be no drifts whatsoever. We shall see that this is not the case: there is a class of particles which "drift" perpendicular to $\underline{B}$ in the $\pm y$-directions, in spite of the fact that they satisfy the usual conditions for the applicability of the guiding-center theory.

We consider the Lagrangian of a particle in a magnetic potential given by eq. (5b.1):

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{m\Omega}{a}\left\{ \dot{x}(\cos(ay) - 1) + \dot{z}\sin(ay) \right\} , \tag{5b.5}$$

with

$$\Omega \equiv \frac{e B_0}{mc} . \tag{5b.6}$$

From eq. (5b.5) we derive the canonical momenta

$$P_x = m\dot{x} + \frac{m\Omega}{a}\{ \cos(ay) - 1 \} , \tag{5b.7}$$

$$P_y = m\dot{y} , \tag{5b.8}$$

$$P_z = m\dot{z} + \frac{m\Omega}{a}\sin(ay) \tag{5b.9}$$

and the Hamiltonian

$$H = \frac{1}{2m}\left[\left\{ p_x - \frac{m\Omega}{a}(\cos(ay) - 1) \right\}^2 \right.$$

$$\left. + p_y^2 + \left\{ p_z - \frac{m\Omega}{a}\sin(ay) \right\}^2 \right] . \tag{5b.10}$$

Since the Hamiltonian does not depend on either x, z or t, the canonical momenta p_x and p_z and the total energy W are constants of the motion. From eqs. (5b.7) and (5b.9) it can easily be seen that p_x/m and p_z/m are, respectively, the x and z-components of the velocity at the plane y = 0, viz. $\dot{x}_0$ and $\dot{z}_0$. The quantity $\dot{x}_0$ can also be identified as the parallel component of the velocity $\underline{v}$ at the plane y = 0:

$$\dot{x}_0 = v_{\parallel 0} = \underline{v} \cdot \frac{\underline{B}(y=0)}{B_0} . \tag{5b.11}$$

Owing to the fact that p_x and p_z are constants of the motion, the particle can be considered as effectively being in a one-dimensional potential $\Psi(y)$ and the Hamiltonian can be written as

$$H = \frac{p_y^2}{2m} + \Psi(y) , \tag{5b.12}$$

where

$$\Psi(y) = \frac{m}{2}\left[\left\{ v_{\parallel 0} - \frac{\Omega}{a}(\cos(ay)-1)\right\}^2 + \left\{ \dot{z}_0 - \frac{\Omega}{a}\sin(ay)\right\}^2\right] . \tag{5b.13}$$

The particles moving in this periodic potential can be divided into two classes: Those whose energy is large enough to overcome the potential barriers determined by the maximum value of $\Psi(y)$, and those with energy so low that they are trapped, their motion being confined to a certain y-region. The maximum value of $\Psi(y)$ can easily be determined from eq. (5b.13):

$$\Psi_{max} = \frac{m}{2}\frac{\Omega^2}{a^2}\left\{ 1 + \sqrt{\left(\frac{a v_{\parallel 0}}{\Omega} + 1\right)^2 + \frac{a^2 \dot{z}_0^2}{\Omega^2}}\right\}^2 . \tag{5b.14}$$

The condition for a particle to move freely in the y-direction is then

$$W > \Psi_{max} . \tag{5b.15}$$

The constant W can be determined from eqs. (5b.8) and (5b.12-13):

$$W = H(y=0) = \frac{m}{2}\left\{ v_{\parallel 0}^2 + \dot{y}_0^2 + \dot{z}_0^2\right\} . \tag{5b.16}$$

The condition (5b.15) can then be written as

$$\frac{1}{2}\frac{a^2}{\Omega^2}\dot{y}_0^2 > 1 + \frac{a}{\Omega}v_{\parallel 0} + \sqrt{\left(\frac{a v_{\parallel 0}}{\Omega} + 1\right)^2 + \frac{a^2 \dot{z}_0^2}{\Omega^2}} , \tag{5b.17}$$

or, equivalently,

$$- \frac{a}{\Omega} v_{\parallel 0} > 1 - \frac{1}{4} \frac{a^2}{\Omega^2} \dot{y}_0^2 + \frac{\dot{z}_0^2}{\dot{y}_0^2} . \tag{5b.18}$$

This equation can also be written as

$$\frac{v_{\parallel 0}}{v_c^+} > 1 - \frac{1}{4} \frac{\dot{y}_0^2}{v_c^{+2}} + \frac{\dot{z}_0^2}{\dot{y}_0^2} , \tag{5b.18'}$$

where the critical $v_{\parallel} = v_c^+$ is twice the critical velocity $\hat{v}_c$ of the axially symmetric case:

$$v_c^+ \equiv - \frac{\Omega}{a} = - \frac{\Omega}{\underline{\hat{b}} \cdot \text{curl } \underline{\hat{b}}} . \tag{5b.18"}$$

On the other hand, the assumptions of the guiding-center theory require that the condition

$$R_g^2 a^2 \ll 1 \tag{5b.19}$$

be satisfied. Here, R_g is the gyro-radius

$$R_g^2 = \frac{\dot{y}_0^2 + \dot{z}_0^2}{\Omega^2} . \tag{5b.20}$$

Combining the inequalities (5b.17) and (5b.19) yields the requirement

$$-\frac{a^2}{\Omega^2}v_{110}^2 + \left\{ 1 + \sqrt{\left(\frac{a v_{110}}{\Omega} + 1\right)^2 + \frac{a^2 \dot{z}_0^2}{\Omega^2}} \right\}^2$$

$$< \frac{a^2(\dot{y}_0^2 + \dot{z}_0^2)}{\Omega^2} \ll 1 \ . \qquad (5b.21)$$

All the particles which satisfy the condition (5b.21) thus "drift" in the $\pm$y-directions and, hence, do not move according to the predictions of the guiding-center theory, in spite of the fact that they satisfy the usual conditions for the applicability of this theory.

The trajectories of the particles are found by integrating eqs. (5b.7-9), together with

$$m\ddot{y} = -\frac{\partial H}{\partial y} \ . \qquad (5b.22)$$

Equation (5b.22) can be integrated explicitly, yielding

$$\dot{y}^2 = \dot{y}_0^2 + \frac{2\Omega}{a}\left\{ \dot{z}_0 \sin(ay) + \left(v_{110} + \frac{\Omega}{a}\right)(\cos(ay) - 1) \right\} . \qquad (5b.23)$$

The computation of a closed analytical expression for y(t) from eq. (5b.23) is only possible in simple cases. As an illustration we study the case with $\dot{z}_0 = 0$ and $v_{\parallel 0} = -\Omega/a$. We can then immediately integrate eq. (5b.23) and obtain

$$y(t) = \dot{y}_0 t \,, \tag{5b.24}$$

where we have set y(t = 0) = 0. With the help of eq. (5b.24) we can integrate eqs. (5b.7) and (5b.9). This yields

$$x(t) = -\frac{\Omega}{a^2 \dot{y}_0} \sin(a\dot{y}_0 t) \tag{5b.25}$$

and

$$z(t) = \frac{\Omega}{a^2 \dot{y}_0} \cos(a\dot{y}_0 t) \,, \tag{5b.26}$$

where we have chosen the integration constants in such a way that x(t = 0) = 0, z(t = 0) = $\Omega/(a^2\dot{y}_0)$. Combining eqs. (5b.25) and (5b.26) leads to

$$x^2(t) + z^2(t) = \frac{\Omega^2}{a^4 \dot{y}_0^2} . \tag{5b.27}$$

It is easy to see that the trajectories given by eqs. (5b.24) and (5b.27) are helices which have their axis of symmetry <u>perpendicular</u> to <u>B</u> and cannot be understood from the point of view of the usual guiding-center mechanics, in spite of the fact that the condition (5b.19), namely $(\dot{y}_0^2 a^2)/\Omega^2 \ll 1$, can always be satisfied by a suitable choice of $\dot{y}_0$.

The sudden breakdown effect of gyro-motion at $v_\parallel = \hat{v}_c$ was mainly presented for the sake of comparison with the singularity occurring in Hamilton-Lagrangian G.C. theories (without regularization applied). In the two examples the curvature drift was identically zero since the "guiding center" of the particle moved on a straight field line (or field lines). Gyro-breakdown therefore occurred well <u>within</u> the "usual", or naive, range of validity of G.C. mechanics. However, the magnetic field lines can be curved and the radius of curvature may be of the order of the "twist length" L_t associated with $\underline{\hat{b}}\cdot\mathrm{curl}\,\underline{\hat{b}} \neq 0$. In these cases the curvature drift calculated for $v_\parallel = \hat{v}_c$ will be well in excess of the gyration velocity $V_\perp$ of the particle and, hence, the critical value $v_\parallel = \hat{v}_c$ then lies <u>outside</u> the validity range of G.C. theory.

It should also be noted that $|\hat{v}_c|$ can be rather large, and in fact be much larger than the velocity of light. For instance, for laboratory values, viz. $B = 3 \times 10^4$ G, $L_t \equiv |\underline{\hat{b}}\cdot\mathrm{curl}\,\underline{\hat{b}}|^{-1} = 10^3$ cm, and for deuterons, one finds $|\hat{v}_c| \approx 7.2 \times 10^{10}$ cm/s, while for electrons $|\hat{v}_c| \approx 2.7 \times 10^{14}$ cm/s. Lower values of $|\hat{v}_c|$ obtain for lower B-fields and smaller "twist lengths". When the non-relativistic equations of particle motion yield $|\hat{v}_c| \gtrsim c$, relativistic equations of motion should be used instead. It can be shown that the gyro-breakdown effect persists when the relativistic equations of particle motion are used. For brevity, these calculations are not presented here.

6. Conclusion

We have given a regularization method for singular Hamilton-Lagrangian guiding-center theories and applied it to the standard theory without polarization drift. The same method can be used in order to regularize the relativistic standard G.C. theory [3] and non-relativistic G.C. mechanics including the polarization drift [5, 6]. In the latter case, one must employ Pfirsch's method [6] of obtaining a total Lagrangian density, and charge and current densities as well as continuum-type conservation theorems, in order to make the G.C. theory compatible with Maxwell's equations. In both cases Liouville's theorem (which is needed in order to obtain useful G.C. kinetic equations) can be derived from either the G.C. Lagrangian [4, 8] or by applying Pfirsch's variational method (in its original version) [6]. It is perhaps not superfluous to mention that all Hamilton-Lagrangian G.C. theories are non-invariant with respect to Galileo transformations (or Lorentz transformations, respectively). It has been shown elsewhere [10] that this is necessarily so (because of exact G.C. energy conservation in time-independent fields and the use of the particle drift approximation).

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6. Conclusion

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References

1. Littlejohn, R.G., Phys. Fluids **24**, 1730 (1981).

2. Wimmel, H.K., Z. Naturforschung **37a**, 985 (1982).

3. Wimmel, H.K., Z. Naturforschung **38a**, 601 (1983).

4. Littlejohn, R.G., J. Plasma Physics **29**, 111 (1983).

5. Wimmel, H.K., Physica Scripta **29**, 141 (1984).

6. Pfirsch, D., Z. Naturforschung **39a**, 1 (1984).

7. Morozov, A.I. and Solov'ev, L.S., in: *Reviews of Plasma Physics* (Leontovich, M.A., Ed.), Vol. **2**, 201 (1966).

8. Wimmel, H.K., to be published.

9. Chapman, S. and Cowling, T.G., *The Mathematical Theory of Non-uniform Gases*, Cambridge Univ. Press 1958, p. 371.

10. Wimmel, H.K., Z. Naturforschung **38a**, 841 (1983).