NEW FORMULATION OF DIRAC'S CONSTRAINT THEORY

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Abstract

Dirac's method of obtaining a Hamiltonian $H(q_1 ... q_N, p_1 ... p_N, t)$ corresponding to a Lagrangian $L(q_1 .... q_N, \dot{q}_1 .... \dot{q}_N, t)$ for which the usual expression $\sum_i p_i\dot{q}_i - L$ does not allow one to find the solutions of the Euler-Lagrange equations via Hamiltons canonical equations is formulated in a more explicit way by making extensive use of the eigenvectors to the matrix $\partial^2 L/\partial\dot{q}_i\partial\dot{q}_k$. The question of secondary and so on and first and second-class constraints is well separated from the basic problem of finding a Hamiltonian and is also discussed in terms of certain eigenvectors. It is also shown that different but equivalent forms of the Hamiltonians exist.

Introduction

In 1950 Dirac [1] (see also [2], [3], [4]) presented a method which allows a Hamiltonian to be obtained for variational problems

$$\delta \int_{t_1}^{t_2} dt L(q_1 \ldots q_N, \dot{q}_1 \ldots \dot{q}_N, t) = 0 \ ,$$
$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \qquad i = 1, \ldots, N, \tag{1}$$

with non-standard Lagrangians L. For such problems the usual expression

$$H_p = \sum_{i=1}^{N} p_i \dot{q}_i \ - \ L \tag{2}$$

with

$$p_i = \partial L / \partial \dot{q}_i \qquad , \tag{3}$$

called primary Hamiltonian in the following, cannot serve to determine the solutions to the variational problem (1) via Hamilton's canonical equations. Situations of this kind occur when eq. (3) implies relations of the form

$$\Phi_n(q_1 \ldots q_N, p_1 \ldots p_N, t) = 0 \tag{4}$$

which are called primary constraints between the q_i's, p_i's and t. The procedure to construct a Hamiltonian can then reveal further constraints, called secondary and so on constraints, which all have to be taken into account.

An example is

$$L = \sum_{i=1}^{N} \dot{q}_i \, F_i(q_1 ... q_N, t) + G(q_1 .... q_N, t) \quad , \tag{5}$$

which yields the constraints

$$\Phi_n = p_n - F_n(q_1 .... q_N, t) = 0 \quad , n = 1, ..., N. \tag{6}$$

A Lagrangian of the form (5) occurs in the context of, for instance, the so-called guiding-centre motion of charged particles in strong magnetic fields [5], and it is useful to describe such motions by a Hamiltonian in order to formulate a kinetic guiding centre theory obeying all the necessary conservation laws, e.g. that for the total energy [6].

In the following a new and more explicit formulation of Dirac's method is presented which resembles to a certain degree the one found in Ref. [4]. It is based on a proper analysis of the structure of the Euler-Lagrange equations for the variational problem (1) and of the implications of the canonical momentum relation (3), and makes extensive use of the eigenvectors of $\partial^2 L/\partial \dot{q}_i \partial \dot{q}_k$. This formulation keeps the problem of finding a Hamiltonian and the question of secondary and so on and first and second-class constraints well separated. The latter question will be dealt with by using again certain eigenvector representations. It will also be shown that different but equivalent forms of the Hamiltonians exist that are not related to canonical transformations of the q_i 's and p_i 's. Finally, an example is given to illustrate the new formulation.

I. Structure of the Euler-Lagrange equations

The Euler-Lagrange equations for (1) are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0. \tag{7}$$

Expanding the total derivative with respect to t, one obtains (with the summation convention applied)

$$\ddot q_k \frac{\partial^2 L}{\partial \dot q_k \partial \dot q_i} + \dot q_k \frac{\partial^2 L}{\partial q_k \partial \dot q_i} + \frac{\partial^2 L}{\partial t \partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0. \tag{8}$$

It is useful as done also in Ref. [4] to introduce the eigenvectors $a_i^{(\nu)}$ and eigenvalues Λ_ν of the symmetric matrix $\partial^2 L/\partial \dot q_i \partial \dot q_k$:

$$\frac{\partial^2 L}{\partial \dot q_k \partial \dot q_i}\, a_i^{(\nu)} = \Lambda_\nu\, a_k^{(\nu)} \;,\; a_i^{(\mu)}\, a_i^{(\nu)} = \delta_{\mu\nu} \quad . \tag{9}$$

This allows eq. (8) to be written as

$$\ddot q_i\, \Lambda_\nu a_i^{(\nu)} + \dot q_k \frac{\partial^2 L}{\partial q_k \partial \dot q_i}\, a_i^{(\nu)} + \frac{\partial^2 L}{\partial t \partial \dot q_i}\, a_i^{(\nu)} - \frac{\partial L}{\partial q_i}\, a_i^{(\nu)} = 0 \quad . \tag{10}$$

Let us choose the ordering of the eigenvalues such that

$$\Lambda_{\hat\nu} \neq 0 \quad , \quad \hat\nu = 1, \dots\dots, m,$$

$$\Lambda_{\nu_o} = 0 \quad , \qquad \nu_o = m+1, ..., N. \tag{11}$$

Equation (10) then means m relations for the combinations $\ddot{q}_i \, a_i^{(\nu)}$ and N-m relations not containing the second derivatives of the q_i *'s* with respect to t. $m < N$ represents the non-standard cases for which the primary Hamiltonian (2) does not lead to the equivalence of the Euler-Lagrange equations (7) in the form of the canonical equations.

II. Structure of the canonical momentum relations

From eq. (3) we find

$$\delta p_i \;-\; \delta \dot{q}_k \,\frac{\partial^2 L}{\partial \dot{q}_k\, \partial \dot{q}_i} \;-\; \delta q_k \,\frac{\partial^2 L}{\partial q_k\, \partial \dot{q}_i} \;-\; \delta t\, \frac{\partial^2 L}{\partial t\, \partial \dot{q}_i} \;=\; 0 \quad . \tag{12}$$

By means of eq. (9) this can be decomposed into

$$\left[\delta p_i \;-\; \delta \dot{q}_i\, \Lambda_{\hat{\nu}} \;-\; \delta q_k \,\frac{\partial^2 L}{\partial q_k\, \partial \dot{q}_i} \;-\; \delta t\, \frac{\partial^2 L}{\partial t\, \partial \dot{q}_i}\right] a_i^{(\hat{\nu})} \;=\; 0 \tag{13}$$

and

$$\left[\delta p_i \;-\; \delta q_k \,\frac{\partial^2 L}{\partial q_k\, \partial \dot{q}_i} \;-\; \delta t\, \frac{\partial^2 L}{\partial t\, \partial \dot{q}_i}\right] a_i^{(\nu_o)} \;=\; 0 \quad . \tag{14}$$

Introducing

$$\Phi_{\nu_o} \;=\; a_i^{(\nu_o)} \left(p_i \;-\; \frac{\partial L}{\partial \dot{q}_i}\right) \quad , \tag{15}$$

we can formulate eq. (14) equivalently as

$$\delta \Phi_{\nu_o} \;=\; 0 \quad \text{at} \quad p_i \;=\; \frac{\partial L}{\partial \dot{q}_i} \tag{16}$$

At $p_i \;=\; \frac{\partial L}{\partial \dot{q}_i}$ the function Φ_{ν_o} therefore only depends on the p_i 's, q_i 's and t.

III. The primary Hamiltonian

The primary Hamiltonian (2) can always be written as a function of the p_i 's, q_i 's and t, which follows from

$$\begin{aligned} \delta H_p &= \delta p_i \dot{q}_i \ + \ p_i \, \delta \dot{q}_i \ - \ \delta \dot{q}_i \, \frac{\partial L}{\partial \dot{q}_i} \ - \ \delta q_i \, \frac{\partial L}{\partial q_i} \ - \ \delta t \, \frac{\partial L}{\partial t} \\ &= \delta p_i \, \dot{q}_i \ - \ \delta q_i \frac{\partial L}{\partial q_i} \ - \ \delta t \frac{\partial L}{\delta t} \quad . \end{aligned} \tag{17}$$

The latter is obtained by means of eq. (3). Relation (17) does not, however, allow one in the general case to obtain the partial derivatives of H_p because of the relations (14) between the δp_i 's, δq_i 's and δt. In order to find these derivatives, we first express $\delta p_i \, \dot{q}_i$ as

$$\begin{aligned} \delta p_i \, \dot{q}_i &= \sum_{\nu=1}^{N} \delta p_k \, a_k^{(\nu)} \, a_i^{(\nu)} \, \dot{q}_i \\ &= \sum_{\dot{\nu}=1}^{m} \delta p_k \, a_k^{(\dot{\nu})} \, a_i^{(\dot{\nu})} \, \dot{q}_i \ + \sum_{\nu_o=m+1}^{N} \delta p_k \, a_k^{(\nu_o)} a_i^{(\nu_o)} \, \dot{q}_i \quad . \end{aligned} \tag{18}$$

With eq. (14) this becomes

$$\delta p_i \, \dot{q}_i \ = \sum_{\dot{\nu}=1}^{m} \delta p_k \, a_k^{(\dot{\nu})} a_i^{(\dot{\nu})} \, \dot{q}_i \ + \sum_{\nu_o=m+1}^{N} \left(\delta q_k \, \frac{\partial^2 L}{\partial q_k \, \partial \dot{q}_l} \ + \ \delta t \, \frac{\partial^2 L}{\partial t \partial \dot{q}_l} \right) a_l^{(\nu_o)} \, a_i^{(\nu_o)} \, \dot{q}_i . \tag{19}$$

Using this expression in δH_p as given by eq. (17), we are now allowed to take δp_i , δq_i , δt as independent of each other, the components of δp_i relevant to relation (14) being eliminated. It therefore follows that

$$\frac{\partial H_p}{\partial p_i} = \sum_{\dot{\nu}=1}^{m} a_i^{(\dot{\nu})} a_k^{(\dot{\nu})} \dot{q}_k \quad , \tag{20}$$

$$\frac{\partial H_p}{\partial q_i} = \sum_{\nu_o=m+1}^{N} \frac{\partial^2 L}{\partial q_i \, \partial \dot{q}_k} a_k^{(\nu_o)} a_l^{(\nu_o)} \dot{q}_l - \frac{\partial L}{\partial q_i} \quad , \tag{21}$$

$$\frac{\partial H_p}{\partial t} = \sum_{\nu_o=m+1}^{N} \frac{\partial^2 L}{\partial t \, \partial \dot{q}_k} a_k^{(\nu_o)} a_l^{(\nu_o)} \dot{q}_l - \frac{\partial L}{\partial t}. \tag{22}$$

IV. The Hamiltonian

Let

$$H = H_p + \sum_{\nu_o=m+1}^{N} \gamma_{\nu_o} \Phi_{\nu_o} \quad , \tag{23}$$

where Φ_{ν_o} is defined in eq. (15) with $\delta\Phi_{\nu_o}$ as given by eq. (14). The γ_{ν_o} are quantities still to be determined. At $p_i = \partial L/\partial\dot{q}_i$ the variation of the function H is therefore given by

$$\delta H = \delta H_p + \sum_{\nu_o=m+1}^{N} \gamma_{\nu_o} a_i^{(\nu_o)} \left(\delta p_i - \delta q_k \frac{\partial^2 L}{\partial q_k \partial \dot{q}_i} - \delta t \frac{\partial^2 L}{\partial t \partial \dot{q}_i}\right) \quad . \tag{24}$$

Thus at $p_i = \partial L/\partial\dot{q}_i$ H, too, is a function of the p_i 's , q_i 's and t. For H to be a Hamiltonian it is required that

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad , \quad \frac{\partial H}{\partial q_i} = - \frac{\partial L}{\partial q_i} = - \dot{p}_i \quad , \tag{25}$$

where the last requirement ensures that the Euler-Lagrange equations hold. This yields

$$\frac{\partial H_p}{\partial p_i} = \dot{q}_i - \sum_{\nu_o=m+1}^{N} \gamma_{\nu_o} a_i^{(\nu_o)} \quad , \tag{26}$$

$$\frac{\partial H_p}{\partial q_i} = - \frac{\partial L}{\partial q_i} + \sum_{\nu_o=m+1}^{N} \gamma_{\nu_o} a_k^{(\nu_o)} \frac{\partial^2 L}{\partial q_i \partial \dot{q}_k} \tag{27}$$

Comparing eq. (26) with eq. (20), and eq. (27) with eq. (21) we obtain

$$\gamma_{\nu_o} = a_i^{(\nu_o)} \dot{q}_i \quad . \tag{28}$$

With this and eq. (22) it also follows that

$$\frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t} \tag{29}$$

and the Hamiltonian reads

$$\begin{aligned} H &= H\ (q_1, ..., q_N, p_1, ...p_N, t) \\ &= H_p + \sum_{\nu_o = m+1}^{N} \dot{q}_k\, a_k^{(\nu_o)}\, a_i^{(\nu_o)} \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right) \quad . \end{aligned} \tag{30}$$

This expression is to be taken at $p_i = \frac{\partial L}{\partial \dot{q}_i}$, i.e. when derivatives are to be taken, $p_i - \frac{\partial L}{\partial \dot{q}_i}$ is to be set equal to zero only afterwards. Furthermore, the N-m relations $\Phi_{\nu_o} = 0$ have to be used in order to determine the N-m additional constants of integration when solving the canonical instead of the Euler-Lagrange equations.

V. Non-Uniqueness of the Hamiltonian

By means of eq. (10) with $\nu = \nu_o$, $\Lambda_{\nu_o} = 0$ the p_i 's can be expressed in different ways without changing their numerical values. They then correspond to different forms of the Lagrangians which, however, remain also numerically unchanged. As a consequence, one obtains different forms of the Hamiltonians which are numerically identical with the original ones. The example in Sec. 7 will also illustrate this point.

VI. Determination of the quantities $\dot{q}_k \; a_k^{(\nu_o)}$

Because of the general proof that H can be written as a function of the q_i 's, p_i 's and t, it is clear that the quantities $\dot{q}_k \; a_k^{(\nu_o)}$ can be expressed in terms of these variables by using the canonical momentum relations (3) and the Euler-Lagrange equations (2). When analysing the corresponding procedure one usually arrives at the concepts of secondary and so on constraints and first and second-class constraints. With the representation introduced in this paper we have the following situation:

For the functions

$$\Phi_\nu = a_i^{(\nu)} \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right) \quad , \nu = 1....N \tag{31}$$

we find at $p_i = \partial L / \partial \dot{q}_i$

$$\frac{\partial \Phi_{\nu}}{\partial(\dot{q}_k a_k^{(\mu)})} = \frac{\partial \Phi_{\nu}}{\partial \dot{q}_k} a_k^{(\mu)} = a_i^{(\nu)} \left(- \frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_k} \right) a_k^{(\mu)}$$
$$= - \Lambda_{\nu} \, \delta_{\nu\mu} \quad ; \qquad \nu \, , \, \mu = 1....N \quad . \tag{32}$$

The quantities $\Phi_{\dot{\nu}}$ and Φ_{ν_o} thus do not depend on $\dot{q}_k \; a_k^{(\mu_o)}$. Since, in addition,

$$det \; \frac{\partial \Phi_{\dot{\nu}}}{\partial(\dot{q}_k \; a_k^{(\dot{\mu})})} = \prod_{\dot{\nu}=1}^{m} (-\Lambda_{\dot{\nu}}) \neq 0 \tag{33}$$

one can obtain from $\Phi_{\dot{\nu}} = 0$ the quantities $\dot{q}_k \; a_k^{(\dot{\mu})}$ as functions of the q_i 's , p_i 's and t. In order to obtain equations for the $\dot{q}_k \; a_k^{(\nu_o)}$'s, we observe that $\Phi_{\nu_o} = 0$ must hold for all times. Using $\dot{p}_i = \partial L / \partial q_i$, we therefore find at $p_i = \partial L / \partial \dot{q}_i$

$$\frac{d\Phi_{\nu_o}}{dt} = a_i^{(\nu_o)} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \equiv \Phi_{\nu_o}^{(2)} = 0 \quad , \tag{34}$$

which, of course, is just the projection of the Euler-Lagrange equations into the null vector space of the $a_i^{(\nu_o)}$'s, i.e. eq. (34) is the same relation as eq. (10) with $\nu = \nu_o$. It is a "secondary" constraint in addition to the "primary" constraint $\Phi_{\nu_o} \equiv \Phi_{\nu_o}^{(1)} = 0$. The solvability of eq. (34) with respect to $\dot{q}_k \; a_k^{(\mu_o)}$ is governed by the properties of the matrix

$$\frac{\partial \Phi_{\nu_o}^{(2)}}{\partial \dot{q}_k \; a_k^{(\mu_o)}} = \frac{\partial \Phi_{\nu_o}^{(2)}}{\partial \dot{q}_k} a_k^{(\mu_o)} \quad . \tag{35}$$

If its determinant does not vanish, we can obtain the $\dot{q}_k \; a_k^{(\mu_o)}$'s from eq. (34). Otherwise we introduce the left and right sided eigenvectors $\hat{b}_{\nu_o}^{(\lambda)}$ and $b_{\mu_o}^{(\lambda)}$ to the eigenvalues $\phi_{\lambda}^{(2)}$ of eq. (35):

$$
\begin{aligned}
\hat{b}^{(\lambda)}_{\nu_o}\ \frac{\partial \phi^{(2)}_{\nu_o}}{\partial \dot{q}_k}\ a^{(\mu_o)}_k \ &= \ \phi^{(2)}_{\lambda}\ \hat{b}^{(\lambda)}_{\mu_o} \quad , \\
\frac{\partial \phi^{(2)}_{\nu_o}}{\partial \dot{q}_k}\ a^{(\mu_o)}_k\ b^{(\lambda)}_{\mu_o} \ &= \ \phi^{(2)}_{\lambda}\ b^{(\lambda)}_{\nu_o} \ , \quad \hat{b}^{(\lambda)}_{\mu_o}\ b^{(\lambda')}_{\mu_o} \ = \ \delta_{\lambda\lambda'} \quad .
\end{aligned}
\tag{36}
$$

With these eigenvectors we form the quantities

$$
\Psi^{(2)}_{\lambda} \ = \ \hat{b}^{(\lambda)}_{\nu_o}\ \Phi^{(2)}_{\nu_o}
\tag{37}
$$

for which we find at $\Phi^{(2)}_{\nu_o} \ = \ 0$

$$
\frac{\partial \Psi^{(2)}_{\lambda}}{\partial \dot{q}_k\ a^{(\mu_o)}_k} \ = \ \phi^{(2)}_{\lambda}\ b^{(\lambda)}_{\mu_o} \quad .
\tag{38}
$$

Scalar multiplication of eq. (38) by $b^{(\lambda')}_{\mu_o}$ yields

$$
\frac{\partial \Psi^{(2)}_{\lambda}}{\partial \dot{q}_k\ a^{(\mu_o)}_k}\ b^{(\lambda')}_{\mu_o} \ = \ \frac{\partial \Psi^{(2)}_{\lambda}}{\partial (\dot{q}_k\ a^{(\mu_o)}_k\ \hat{b}^{(\lambda')}_{\mu_o})} \ = \ \phi^{(2)}_{\lambda}\ \delta_{\lambda\lambda'}
\tag{39}
$$

With the notation $\phi^{(2)}_{\dot{\lambda}} \ \neq 0$, $\phi^{(2)}_{\lambda_o} \ = \ 0$ we obtain the result that $\Psi^{(2)}_{\dot{\lambda}} \ = \ 0$ can be solved for the quantities $\dot{q}_k\ a^{(\mu_o)}_k\ \hat{b}^{(\dot{\lambda})}_{\mu_o}$. In order to obtain the rest, we introduce a new function

$$
\Phi^{(3)}_{\lambda_o} \ = \ \frac{d}{dt}\ \Psi^{(2)}_{\lambda_o}
\tag{40}
$$

which does not depend on $d(\dot{q}_k\ a^{(\mu_o)}_k\ \hat{b}^{(\lambda_o)}_{\mu_o})/dt$, and try to obtain the quantities $\dot{q}_k\ a^{(\mu_o)}_k\ b^{(\lambda_o)}_{\mu_o}$ from the tertiary constraint

$$\Phi^{(3)}_{\lambda_o} = 0 \quad . \tag{41}$$

One can now proceed in the same way as before. After a finite number of steps one comes to an end which might imply that some combinations of the $\dot{q}_k \; a_k^{(\mu_o)}$ can be chosen freely in agreement with the Euler-Lagrange equations. The number of these combinations is given by the number of the so-called first-class primary constraints which results from the following consideration:

Let F_n denote all the primary, secondary and so on constraints. We can then write down the conditions $dF_n/dt = 0$ at $F_n = 0$ by using the Hamiltonian (30) as

$$\frac{dF_n}{dt} = \frac{\partial F_n}{\partial t} + [H_p, F_n] + \sum_{\mu_o} \dot{q}_k \; a_k^{(\mu_o)} \; [\Phi_{\mu_o}, F_n] = 0 \quad , \tag{42}$$

where the brackets denote Poisson brackets. If

$$\sum_{\mu_o} \hat{c}^{(\rho_o)}_{\mu_o} \; [\Phi_{\mu_o}, \; F_n] = 0 \quad \text{for all n} \tag{43}$$

holds with certain coefficients $\hat{c}^{(\rho_o)}_{\mu_o}$ then

$$\sum_{\mu_o} \hat{c}^{(\rho_o)}_{\mu_o} \; \Phi_{\mu_o} \equiv \Phi^{(\rho_o)} \tag{44}$$

are called first-class primary constraints and all other combinations second-class primary constraints. Choosing a full set of coefficients $\hat{c}^{(\rho)}_{\mu_o}$, $c^{(\rho)}_{\nu_o}$ with

$$\sum_{\rho=m+1}^{N} \hat{c}^{(\rho)}_{\mu_o} \; c^{(\rho)}_{\nu_o} \;=\; \delta_{\mu_o \nu_o} \tag{45}$$

we can write

$$\dot{q}_k \; a_k^{(\mu_o)} \;=\; \sum_{\rho} \sum_{\nu_o} \dot{q}_k \; a_k^{(\nu_o)} \; \hat{c}^{(\rho)}_{\nu_o} \; c^{(\rho)}_{\mu_o} \tag{46}$$

Equation (42) thus does not contain the quantities

$$\dot{q}_k \; a_k^{(\nu_o)} \; \hat{c}^{(\rho_o)}_{\nu_o} \; , \tag{47}$$

which remain freely choosable, and their number is the same as the number of first-class primary constraints.

VII. Example

As a non-trivial simple example let us consider

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}(\dot{q}_1 \, q_2 - q_1 \, \dot{q}_2) - \frac{E_o}{2}\,(q_1^2 + q_2^2) \, , \quad E_o = const. \tag{48}$$

The Euler-Lagrange equations are

$$\ddot{q}_1 = - \dot{q}_2 - E_o \, q_1 \, , \quad 0 = \dot{q}_1 - E_o \, q_2 \, . \tag{49}$$

The matrix $\partial^2 L / \partial \dot{q}_i \partial \dot{q}_k$ is

$$\left(\frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_k} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad . \tag{50}$$

Its eigenvalues and eigenvectors are therefore

$$\begin{aligned} \Lambda_1 &= 1 \, , \quad \left(a_i^{(1)} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Lambda_2 &= 0 \, , \quad \left(a_i^{(2)} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \end{aligned} \tag{51}$$

We have further

$$p_1 = \dot{q}_1 + \frac{1}{2} q_2 \quad , \quad p_2 = -\frac{1}{2} q_1 \tag{52}$$

and

$$H_p = \frac{1}{2} \left(p_1 - \frac{1}{2} q_2 \right)^2 + \frac{E_o}{2} \left(q_1^2 + q_2^2 \right) \quad . \tag{53}$$

According to eq. (30) we find

$$H = \frac{1}{2} \left(p_1 - \frac{1}{2} q_2 \right)^2 + \frac{E_o}{2} \left(q_1^2 + q_2^2 \right) + \dot{q}_2 \left(p_2 + \frac{1}{2} q_1 \right) \quad . \tag{54}$$

From eq. (49) we have

$$\dot{q}_2 = -\ddot{q}_1 - E_o q_1 = -E_o \dot{q}_2 - E_o q_1$$

- where the second step corresponds to eq. (41) - or

$$\dot{q}_2 = -\frac{E_o}{E_o + 1} q_1 \quad . \tag{55}$$

The Hamiltonian written explicitly as a function of the q_i 's, p_i 's and t is therefore

$$H = \frac{1}{2}(p_1 - \frac{1}{2}q_2)^2 + \frac{E_o}{2}(q_1^2 + q_2^2) - \frac{E_o}{E_o+1}q_1(p_2 + \frac{1}{2}q_1) \quad . \qquad (56)$$

Hamilton's canonical equations with this H are

$$\dot{q}_1 = p_1 - \frac{1}{2}q_2 \quad , \dot{q}_2 = -\frac{E_o}{E_o + 1}q_1 \quad ,$$
$$\dot{p}_1 = -E_o q_1 + \frac{1}{2}\frac{E_o}{E_o + 1}q_1 \; , \; \dot{p}_2 = \frac{1}{2}(p_1 - \frac{1}{2}q_2) - E_o q_2 \quad . \qquad (57)$$

The equation for $\dot{q}_1$ is identical with the definition of p_1 in eq. (52). The equation for $\dot{q}_2$ is the same as eq. (55), which was derived from the Euler-Lagrange equations. The equation for $\dot{p}_1$ is identical with the first Euler-Lagrange equation. The $\dot{q}_1$, $\dot{q}_2$ and $\dot{p}_1$ equations can be combined into

$$\ddot{q}_1 - E_o \dot{q}_2 = 0 \quad ,$$

or

$$\dot{q}_1 - E_o q_2 = c = const.$$

The $\dot{p}_2$ equation can then be written as

$$\dot{p}_2 = -\frac{1}{2}\dot{q}_1 + c \quad .$$

The relation $\Phi_1 = p_2 + \frac{1}{2} q_1 = 0$ yields c=0 , from which it follows that

$$\dot{q}_1 - E_o\, q_2 = 0 \quad ,$$

this being the second Euler-Lagrange equation.

An equivalent expression for H is obtained by using the second of the equations (49) in the definition equations (52) for p_1 and p_2:

$$p_1 = (E_o + \frac{1}{2})\, q_2 \quad , \; p_2 = -\frac{1}{2} q_1 \quad . \tag{58}$$

This implies

$$L = (E_o + \frac{1}{2})\, q_2\, \dot{q}_1 - \frac{1}{2}\, q_1\, \dot{q}_2 - \frac{E_o}{2} \left(q_1^2 + (1 + E_o)\, q_2^2 \right) , \tag{59}$$

which is numerically identical with eq. (48) and leads to the Euler-Lagrange equations equivalent to eq. (49):

$$\begin{aligned} (E_o + \frac{1}{2})\, \dot{q}_2 &= -\frac{1}{2}\, \dot{q}_2 - E_o\, q_1 \quad , \\ -\frac{1}{2}\, \dot{q}_1 &= (E_o + \frac{1}{2})\, \dot{q}_1 - E_o\, (1 + E_o)\, q_2 \quad . \end{aligned}$$

The primary Hamiltonian is now

$$H_p = \frac{E_o}{2}\left(q_1^2 + (1 + E_o)\, q_2^2\right) . \tag{60}$$

Since the new Lagrangian has the property $\partial^2 L/\partial \dot{q}_i \partial \dot{q}_k = 0$, we have $\nu_o = 1,2$ and therefore

$$H = \frac{E_o^2}{2}\, q_2^2 + \frac{E_o}{2}\,(q_1^2 + q_2^2) + \dot{q}_1 \left(p_1 - (E_o + \frac{1}{2})\, q_2\right) + \dot{q}_2\,(p_2 + \frac{1}{2}\, q_1) \quad , \tag{61}$$

where again $\dot{q}_1$ is given by eq. (49), and $\dot{q}_2$ by eq. (55). In addition to these relations, we find from eq. (61) the canonical equations

$$\dot{p}_1 = -\, E_o\, q_1 - \frac{1}{2}\, \dot{q}_2 = (E_o + \frac{1}{2})\, \dot{q}_2 \quad ,$$

$$\dot{p}_2 = -\, E_o\,(1 + E_o)\, q_2 + (E_o + \frac{1}{2})\, \dot{q}_1 = -\frac{1}{2}\, \dot{q}_1 \quad .$$

Integrating these equations, we obtain

$$p_1 - (E_o + \frac{1}{2})\, q_2 = const_1 \quad , p_2 + \frac{1}{2}\, q_1 = const_2.$$

The two constants are to be determined by the two constraints which require them to be zero.

Summary

Analyzing the structure of the Euler-Lagrange equations and of the canonical momentum relations, it was found similar to a certain degree as in Ref. [4] that the N-m eigenvectors $(a_i^{(\nu_o)})$, $\nu_o = m+1, \ldots, N$, corresponding to the zero eigenvalues of the symmetric $N \times N$ matrix $\partial^2 L/\partial\dot{q}_i\partial\dot{q}_k$ play a central role in determining a Hamiltonian corresponding to a non-standard Lagrangian. Using $\Phi_{\nu_o} = a_i^{(\nu_o)} (p_i - \frac{\partial L}{\partial \dot{q}_i}) = 0$ as constraints, secondary and so on constraints do not occur. They appear to have only a more technical meaning as do the first class primary constraints the number of which being equal to the number of the freely choosable functions. The main result of the paper is the closed representation of a Hamiltonian as given by eq. (30) with H_p defined in eq. (2). In addition, it is found that this Hamiltonian is not the only possible one, the reason being that there is an ambiguity in expressing the canonical momenta as functions of the q_i's, $\dot{q}_i$'s and t because the relations $a_i^{(\nu_o)} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right) = 0$ do not contain the $\ddot{q}_i$'s.

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