Stochastic Runaway of Dynamical Systems

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Abstract:

One-dimensional, stochastic, dynamical systems are well studied with respect to their stability properties. Less is known for the higher dimensional case. This paper derives sufficient and necessary criteria for the asymptotic divergence of the entropy (runaway) and sufficient ones for the moments of n-dimensional, stochastic, dynamical systems. The crucial implication is the incompressibility of their flow defined by the equations of motion in configuration space. Two possible extensions to compressible flow systems are outlined.

Stochastic Runaway of Dynamical Systems

1. Introduction

In order to explain what is meant by stochastic runaway, let us consider a canonical system which is stirred up by some kind of external stochastic driving forces. One then expects in the average an input of energy into the system, i.e. the total energy content of the system will "run away".

We shall confirm this or similar behaviour for a rather general class of so-called dynamical systems. There will be no need for a Hamiltonian nor for the notation of energy, which may be replaced by the entropy. All that is required is incompressibility of the internal dynamic flow.

A dynamical system [1] belonging to this class is described by an n-dimensional vector

$$\underset{\sim}{u}(s) = (u_1(s), \ldots, u_n(s)) \tag{1}$$

s can often be the time t but in some applications can also represent a space variable. In the following, s is always called time. The time evolution of $\underset{\sim}{u}$ is given by

$$d\underset{\sim}{u}(s) = \underset{\sim}{b}(\underset{\sim}{u}(s), \underset{\sim}{v}(s))\,ds \tag{2}$$

The m-dimensional vector

$$\underset{\sim}{v}(s) = (v_1(s), \ldots, v_m(s)) \tag{3}$$

is some stochastic process and serves to make the flow field $\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v})$ in $\underset{\sim}{u}$ - space time-dependent in a stochastic way. $\underset{\sim}{b}$ is assumed to be divergence-free in u - space:

$$\frac{\partial}{\partial \underset{\sim}{u}} \cdot \underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v}) = \frac{\partial b_i}{\partial u_i} = 0 \tag{4}$$

Equation (2) therefore describes an incompressible flow in $\underset{\sim}{u}$ - space. (Throughout the paper the summation convention is applied).

For $\underset{\sim}{v}(s)$ we choose an Ornstein-Uhlenbeck process defined by its stochastic differential equation [2)]

$$dv_j(s) = a_j(\underset{\sim}{v})\,ds + A_{jk}(\underset{\sim}{v})\,dw_k(s) \tag{5}$$

$j = 1, \cdot\cdot, m$; summation over k from 1 to m.

$w_k(s)$ are Wiener processes defined by

$$dw_k(s) = \xi_k(s)\,ds \tag{6}$$

with

$$\langle \xi_k(s_1)\, \xi_l(s_2) \rangle = \delta_{kl}\, \delta(s_1 - s_2). \tag{7}$$

The brackets denote averages. A generalised telegraph process (Poisson process) for v is also considered (see Appendix). Since our results can be established for both of these rather different cases, we believe them to be valid even in more general cases such as are treated in connection with, for instance, stochastic curve integrals by Gichman and Skorohod [2)]

The class of systems we are investigating therefore belongs to the class of stochastic differential equations

$$d\underset{\sim}{x}(s) = \underset{\sim}{k}(\underset{\sim}{x}, s)\, ds + \underset{\sim}{g}_k(\underset{\sim}{x}, s)\, dw_k(s). \tag{8}$$

where k and g_k are nonstochastic functions of $\underset{\sim}{x}$ and s .

In our case we have

$$\underset{\sim}{x} = (\underset{\sim}{u}, \underset{\sim}{v}) \tag{9}$$

and there is assumed to be no influence of $\underset{\sim}{u}$ on the time evolution of $\underset{\sim}{v}$.

There exist a number of investigations of the asymptotic behaviour of the solutions $\underset{\sim}{x}(s)$ of equation (8) that are based on the corresponding Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + L f = 0 \tag{10}$$

The so-called L-harmonics h defined by

$$L h = 0 \tag{11}$$

are then of special interest. In fact, in the one-dimensional case ($\underset{\sim}{x}(s) = x(s)$) they provide a complete classification concerning the boundedness of $x(s)$ (see Gichman and Skorohod [2] chapter 2 part II. Compare also L. Arnold et al. [16) 17) 18)]).

For more than one dimension some general results are available for the dwelling time of any given path within a finite volume, especially through a certain relation to potential theory [3].

Using so-called L-superharmonics ($Lh_{super} \leqq 0$), Gichman and Skorohod [2] were able to study the asymptotic trend of the moments. For more than one dimension sufficient conditions for their boundedness were obtained which, however, can easily be shown to be far from necessary.

For arbitrary dimensions ergodic theory is also used which deals with the existence of time averages of measurable functions $\ell(\underset{\sim}{x})$ of the form [4) 5)]

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \ell(\underset{\sim}{x}(s))\, ds \tag{12}$$

for nearly all starting points $\underset{\sim}{x}(s = 0)$ and with their probability distribution. In this case the existence of a stationary measure, i.e. normalized and invariant to the flow, is necessary. One may then consider the above average as a Fejer mean for the function $\ell(\underset{\sim}{x})$ and draw conclusions on the behaviour of $\ell(\underset{\sim}{x})$ for growing s except for intervals of s of a relatively small measure. This therefore somewhat restricted asymptotics may nevertheless be useful in physical applications as stated by Sulem and Frisch in treating the reflection of light by a random medium [6) 7) 8) 9)].

Liapunov techniques might also be expected to be useful, but, as Kushner points out [1)], "it is not clear what the appropriate theorems are for such cases nor is it understood how to find (even in relatively simple cases) useful Liapunov functions".

In the present paper the following is done:

1) proof of an H theorem for systems described by eq. (2) in $\underset{\sim}{u}$, $\underset{\sim}{v}$ space on the assumption of stationary $\underset{\sim}{v}$ processes

2) derivation of necessary and sufficient conditions for the divergence of H

3) proof of the divergence of any moment of $\underset{\sim}{u}$ when H is divergent

4) explanation of the peculiar role of incompressibility and sketch of two ways of possibly achieving also results for compressible systems .

In the course of the proof certain Dirichlet forms are introduced.

The possibility to consider them as a metric of function spaces offers far-reaching connections with other topics [11] [12] [13] [14] [15]. The third point, of course, does not necessary imply that H diverges when there exist divergent moments. It would therefore be of interest to know what additional information could be extracted from the fact of diverging entropy.

2. Stirring forces derived from an Ornstein-Uhlenbeck process

2.1 Description by a Fokker-Planck equation

We discuss here the system (2) with $\underset{\sim}{v}(s)$ given by eqs. (5) to (7), whereas the discontinuous generalised telegraph or Poisson process is left to the Appendix. $\underset{\sim}{v}(s)$ is assumed to be in its stationary state.

One question we are interested in is whether

$$u = \sqrt{u_1^2 + \dots + u_n^2} \tag{13}$$

will assume larger and larger values in a statistical sense such that the average value of u or of some power of it increases infinitely as s goes to infinity, which we shall establish as a consequence of stochastic runaway.

In order to get an answer to this question, we describe the statistical properties of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ by a probability density $P(s: \underset{\sim}{u}, \underset{\sim}{v})$ with

$$\begin{gathered} \int P(s: \underset{\sim}{u}, \underset{\sim}{v})\, d^n u\, d^m v = 1 \\ d^n u = du_1 \dots du_n ; \qquad d^m v = dv_1 \dots dv_m \end{gathered} \tag{14}$$

In the case of the Ornstein-Uhlenbeck process P is a solution of a Fokker-Planck equation

$$\frac{\partial P}{\partial s} + \underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v}) \cdot \frac{\partial P}{\partial \underset{\sim}{u}} = W P \tag{15}$$

with

$$W = \frac{\partial}{\partial v_i} \left(\gamma_i(\underset{\sim}{v}) + \frac{1}{2} D_{ik}(\underset{\sim}{v}) \frac{\partial}{\partial v_k} \right) \tag{16}$$

where W is the Fokker-Planck operator corresponding to eqs. (5) to (7) with

$$D_{ik} = A_{i\mu}(\underset{\sim}{v}) A_{k\mu}(\underset{\sim}{v}) \tag{17}$$

$$\gamma_i = a_i + \frac{1}{2} \frac{\partial}{\partial v_k} D_{ik} \tag{18}$$

From this it follows that $D_{ik}(\underset{\sim}{v})$ is positive semi-definite; it is restricted here to being positive definite.

Integration of eq.(15)over $\underset{\sim}{u}$ yields an equation for

$$V(s:\underset{\sim}{v}) := \int P(s:\underset{\sim}{u}, \underset{\sim}{v})\, d^n u \tag{19}$$

namely

$$\frac{\partial V}{\partial s} + W V = 0 \tag{20}$$

We are interested in the stationary $\underset{\sim}{v}$-distributions, i.e. $\partial V/\partial s = 0$. Such distributions are possible only if certain"Einstein relations" between γ_i and D_{ik} hold. These are obtained by writing

$$V(\underset{\sim}{v}) \sim e^{-\beta(\underset{\sim}{v})} \tag{21}$$

with $\beta(\underset{\sim}{v})$ real and

$$\int_{\text{all } \underset{\sim}{v} \text{ space}} e^{-\beta(\underset{\sim}{v})} d^m v < \infty \tag{22}$$

From WV = 0 we find

$$\left(\gamma_i + \frac{1}{2} D_{ik} \frac{\partial}{\partial v_k} \right) e^{-\beta(\underset{\sim}{v})} = \frac{\partial}{\partial v_k} a_{ik} \, e^{-\beta(\underset{\sim}{v})} \tag{23}$$

with a certain antisymmetric a_{ik}:

$$a_{ik}(\underset{\sim}{v}) = - a_{ki}(\underset{\sim}{v}) \tag{24}$$

For

$$\hat{\gamma}_i = \gamma_i - \frac{\partial}{\partial v_k} a_{ik} \tag{25}$$

$$\hat{D}_{ik} = D_{ik} - 2 a_{ik} \tag{26}$$

we obtain the Einstein relation

$$\hat{\gamma}_i = \frac{1}{2} \hat{D}_{ik} \frac{\partial \beta}{\partial v_k} \tag{27}$$

It thus follows that

$$P_o(\underset{\sim}{u}, \underset{\sim}{v}) = U(\underset{\sim}{u}) \cdot V(\underset{\sim}{v}) \tag{28}$$

with

$$V(\underset{\sim}{v}) = e^{-\beta} / \int e^{-\beta} d^m v \tag{29}$$

is a distribution function for which

$$W P_o = 0 \tag{30}$$

holds. In general, however, it does not solve the whole Fokker-Planck equation (15). P_o can be a solution of this equation only if a solution $U(\underset{\sim}{u})$ of the equation

$$\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v}) \cdot \frac{\partial U(\underset{\sim}{u})}{\partial \underset{\sim}{u}} = 0 \tag{31}$$

exists with

$$U \geqq 0; \qquad \int_{\text{all } \underset{\sim}{u}\text{-space.}} U(\underset{\sim}{u})\, d^n u = 1 \tag{32}$$

The expressions (31) and (32) impose a condition on $\underset{\sim}{b}(\underset{\sim}{u},\underset{\sim}{v})$. Since this condition will play an important role later, we will illustrate it here by a 2-dimensional example for $u = (u_1, u_2)$. Because of eq. (4) we can express it as

$$b_1(\underset{\sim}{u}, \underset{\sim}{v}) = \frac{\partial \psi}{\partial u_2}(\underset{\sim}{u}, \underset{\sim}{v}); \qquad b_2(\underset{\sim}{u}, \underset{\sim}{v}) = -\frac{\partial \psi(\underset{\sim}{u}, \underset{\sim}{v})}{\partial u_1} \tag{33}$$

Inserting this in eq. (30), we obtain

$$\frac{\partial \psi}{\partial u_2} \cdot \frac{\partial U(\underset{\sim}{u})}{\partial u_1} - \frac{\partial \psi}{\partial u_1} \cdot \frac{\partial U(\underset{\sim}{u})}{\partial u_2} = 0 \tag{34}$$

from which a representation of ψ in terms of an assumed existing $U(\underset{\sim}{u})$

$$\psi(\underset{\sim}{u}, \underset{\sim}{v}) = g(U(\underset{\sim}{u}), \underset{\sim}{v}) \tag{35}$$

follows and therefore

$$b_1 = \frac{\partial g}{\partial U} \cdot \frac{\partial U}{\partial u_2}; \qquad b_2 = -\frac{\partial g}{\partial U} \frac{\partial U}{\partial u_1} \tag{36}$$

Thus b_1/b_2 is a function of $\underset{\sim}{u}$ only, which of course imposes a restriction. An example of $\underset{\sim}{b}$ not satisfying eq. (36) is the stochastic oscillator:

$$\frac{d^2 u_1}{ds^2} + \omega^2(\underset{\sim}{v}(s))\, u_1 = 0 \tag{37}$$

$$\frac{du_1}{ds} = u_2 = b_1 ; \qquad \frac{du_2}{ds} = -\omega^2 u_1 = b_2 \tag{38}$$

$$\psi = \frac{1}{2}\left(\omega^2(\underset{\sim}{v})\, u_1^2 + u_2^2 \right) \tag{39}$$

This contradicts eqs. (34) and (35). Of similar structure is the one-dimensional wave propagation in a stochastic medium. Related to the fact that a function of the form P_o does not solve eq. (15) if $\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v})$ does not allow the expression (31) and (32) to have solutions is an H-theorem for eq. (15) which will be helpful in discussing the question of stochastic runaway and which we shall prove in the next section.

2.2 Proof of an H-theorem

H is defined as

$$H = \int P \ln P \, d^n u \, d^m v \tag{40}$$

and we want to prove an H-theorem for solutions of (15) for which

$$\int P \, d^n u = V(\underset{\sim}{v}) \tag{41}$$

holds, where $V(\underset{\sim}{v})$ is defined in eq. (29)

Solutions satisfying eq. (41) initially do this for all s.

That an H-theorem is of interest in our context becomes clear from an auxiliary theorem which we shall prove first before turning to the H-theorem itself. The auxiliary theorem states that for any real $\alpha > 0$

$$\int u^\alpha P \, d^n u \, d^m v \to \infty \qquad for \;\; H \to -\infty \tag{42}$$

For the proof we assume the contrary, namely

$$H \to -\infty \qquad and \qquad \int P u^\alpha d^n u \, d^m v \to C < \infty \tag{43}$$

In order to find out whether this is possible we determine the maximum of -H under the constraints (41) and

$$\int P u^\alpha d^n u \, d^m v = C \tag{44}$$

The maximizing P is

$$P_{max} = V(\underset{\sim}{v}) \cdot \frac{e^{-\lambda u^\alpha}}{\int e^{-\lambda u^\alpha} d^n u} \tag{45}$$

with λ being determined by eq. (44). The integral appearing in eq. (45) exists for all positive non-vanishing α. With P_{max}

we obtain

$$(-H)_{max} = -\int V(\underset{\sim}{v}) \ln V(\underset{\sim}{v}) d^m v$$
$$+ \ln \int e^{-\lambda u^{\alpha}} d^n u \qquad (46)$$
$$+ \lambda \int u^{\alpha} P_{max} d^n u \, d^m v$$

This expression represents not merely a relative but an absolute maximum as can be shown by familiar methods of statistical mechanics (compare e.g. Zubarev [19]).

It is finite for finite C. Hence it contradicts the first of the two assumptions (43) and therefore proves theorem (42).

Let us now turn to the proof of the H-theorem.

With eq. (15) we have

$$\frac{dH}{ds} = \int W P \cdot \ln P \, d^n u \, d^m v \qquad (47)$$

We shall show that $\frac{dH}{ds} \leq 0$, where the equality sign holds only for solutions of eq. (15) of the form P_o (as defined in eq. (28)) which, however, might not exist. Partial integrations in eq. (47) yield

$$\frac{dH}{ds} = \int P \frac{\partial \gamma_i}{\partial v_i} d^n u \, d^m v - \frac{1}{2} \int D_{ik} \frac{1}{P} \frac{\partial P}{\partial v_i} \frac{\partial P}{\partial v_k} d^n u \, d^m v \qquad (48)$$

Since $\gamma(\underset{\sim}{v})$ does not depend on $\underset{\sim}{u}$, we obtain with eq. (41)

$$\frac{dH}{ds} = \int V(\underset{\sim}{v}) \frac{\partial \gamma_i}{\partial v_i} d^m v - \frac{1}{2} \int D_{ik} \frac{1}{P} \frac{\partial P}{\partial v_i} \frac{\partial P}{\partial v_k} d^n u \, d^m v \tag{49}$$

Only the second term depends on the special solution P. The maximum value of dH/ds is thus given by the minimum value of the second term under the constraints (41) and $P \gtreqless 0$. Indeed this term has a lower bound, since it is always positive because of D_{ik} being a positive definite matrix.

In order to guarantee $P \gtreqless 0$, we express P by

$$P = \phi^2 \tag{50}$$

where $\phi = \phi(\underset{\sim}{u}, \underset{\sim}{v})$ is a real function of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. This yields

$$\frac{dH}{ds} = \int V(\underset{\sim}{v}) \frac{\partial \gamma_i}{\partial v_i} d^m v - 2 \int D_{ik} \frac{\partial \phi}{\partial v_i} \frac{\partial \phi}{\partial v_k} d^m v \, d^n u \tag{51}$$

Using eqs. (24) till (29), we can also write this as a difference of two Dirichlet forms

$$= 2 \int D_{ik} \frac{\partial \sqrt{V}}{\partial v_i} \frac{\partial \sqrt{V}}{\partial v_k} d^m v - 2 \int D_{ik} \frac{\partial \phi}{\partial v_i} \frac{\partial \phi}{\partial v_k} d^m v \, d^n u \, . \tag{52}$$

We now represent ϕ by

$$\phi = \sum_q \sum_\nu c_{q\nu} \varphi_q(\underset{\sim}{v}) U_\nu(\underset{\sim}{u}) \tag{53}$$

with $\varphi_q(\underset{\sim}{v})$ and $U_\nu(\underset{\sim}{u})$ forming complete sets of real orthonormal functions:

$$\int \varphi_q(\underset{\sim}{v})\,\varphi_p(\underset{\sim}{v})\,d^m v = \delta_{qp} \tag{54}$$

$$\int U_\nu(\underset{\sim}{u})\,U_\mu(\underset{\sim}{u})\,d^n u = \delta_{\nu\mu} \tag{55}$$

The set $U_\nu(\underset{\sim}{u})$ can be chosen arbitrarily. But the φ_q are taken as the set of eigenfunctions of the Hermitian eigenvalue problem

$$-\frac{\partial}{\partial v_i} D_{ik} \frac{\partial \varphi_q}{\partial v_k} + \Lambda \varphi_q = E_q \varphi_q \tag{56}$$

with

$$\Lambda(\underset{\sim}{v}) = \frac{1}{\sqrt{V}} \frac{\partial}{\partial v_i} D_{ik} \frac{\partial \sqrt{V}}{\partial v_k} \tag{57}$$

The eigenvalues E_q have a lower bound because of the properties of D_{ik}. One eigenfunction, say φ_0, is given by

$$\varphi_0 = \sqrt{V} \qquad \text{with } E_0 = 0. \tag{58}$$

The eigenfunction to the lowest eigenvalue of a Hermitian eigenvalue problem is characterized by having no zeros. It holds further that the lowest eigenvalue is simple. φ_0 has the property of not vanishing except for $|\underset{\sim}{v}| \to \infty$. Thus $E_0 = 0$ is the lowest eigenvalue and there is no other eigenfunction besides φ_0 to this eigenvalue.

The series (53) is restricted by the condition (41), which yields

$$\sum_q \sum_p \sum_\nu c_{q\nu} c_{p\nu} \varphi_q(\underset{\sim}{v}) \varphi_p(\underset{\sim}{v}) = V(\underset{\sim}{v}) \tag{59}$$

Inserting eq. (53) in eq. (52), we obtain

$$\frac{dH}{ds} = 2\int D_{ik} \frac{\partial\sqrt{V}}{\partial v_i} \frac{\partial\sqrt{V}}{\partial v_k} d^m v - 2\int D_{ik} \sum_{q,p,\nu} c_{q\nu} c_{p\nu} \frac{\partial\varphi_q}{\partial v_i} \frac{\partial\varphi_p}{\partial v_k} d^m v$$

$$= 2\int D_{ik} \frac{\partial\sqrt{V}}{\partial v_i} \frac{\partial\sqrt{V}}{\partial v_k} d^m v \tag{60}$$

$$-\sum_{q,p,\nu} 2\, c_{q\nu} c_{p\nu} \left(E_p \delta_{qp} - \int \Lambda \varphi_q \varphi_p \, d^m v \right)$$

Because of eq. (59) the last term in eq. (60) containing Λ exactly compensates the first term and therefore

$$\frac{dH}{ds} = -2 \sum_q \sum_\nu c_{q\nu}^2 E_q \tag{61}$$

The coefficients $c_{q\nu}$ satisfy the condition

$$\sum_q \sum_\nu c_{q\nu}^2 = 1 \tag{62}$$

as follows from eq. (59).

Equation (61) proves our H-theorem, i.e., $dH/ds < 0$ except for $c_{q\nu} = 0$ for $q \geqq 1$. If this latter is true, we have $dH/ds = 0$ and

$$\phi = \sqrt{V} \sum_\nu c_{o\nu} u_\nu(\underset{\sim}{v}) \tag{63}$$

which corresponds to a distribution function P of the form P_o (eq. (28)).

2.3 Conditions for stochastic runaway

In case $\emptyset$ corresponds to a solution of the Fokker-Planck equation (15), the coefficients $c_{q\nu}$ become functions of the variable s:

$$c_{q\nu} = c_{q\nu}(s) \tag{64}$$

and therefore

$$\frac{dH}{ds} = -2 \sum_{q \geq 1} \sum_{\nu} c^2_{q\nu}(s) E_q \tag{65}$$

If $E_o = 0$ is a discrete eigenvalue and if $\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v})$ does not satisfy the conditions expressed by (31) and (32), the r.h.s. of eq. (65) is negative and not infinitesimally small, except possibly for an initial $s = s_o$ where all the $c_{q\nu}(s_o)$, $q \geq 1$, could, of course, be chosen zero. Under such conditions there is thus stochastic runaway.

Whether $E_o = 0$ is a discrete eigenvalue or not depends on the Ornstein-Uhlenbeck process. But there is obviously a large class of such processes for which at least the lowest eigenvalue is discrete. A simple example is

$$D_{ik} = D\,\delta_{ik}\,, \quad \frac{\partial D}{\partial \underset{\sim}{v}} = 0\,; \quad \beta = \underset{\sim}{v}^2 \tag{66}$$

In this case we find from eq. (57)

$$\Lambda = D(\underset{\sim}{v}^2 - m) \tag{67}$$

and eq. (56) becomes

$$-D\sum_i \frac{\partial^2 \varphi_q}{\partial v_i^2} + D(\underset{\sim}{v}^2 - m)\varphi_q = E_q \varphi_q \tag{68}$$

This is the Schroedinger equation of an m-dimensional oscillator of frequency 1 with energy eigenvalues $\frac{1}{2D} \cdot E_q + \frac{m}{2} = q + \frac{m}{2}$ from which it follows that

$$E_q = 2D \cdot q \tag{69}$$

Let us now also discuss the situation where $\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v})$ satisfies the conditions (31) and (32). To this end we first analyse the meaning of these conditions. They yield surfaces $U(\underset{\sim}{u})$ = const in u-space to which the vectors $\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v})$ are tangential for all $\underset{\sim}{v}$. "Particles" possessing "velocities" $\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v})$ in $\underset{\sim}{u}$-space thus cannot cross such surfaces and this holds for all values of $\underset{\sim}{v}$. The $\underset{\sim}{v}$-dependence of $\underset{\sim}{b}$ can only lead to a stochastic motion within such surfaces.

The surfaces $U(\underset{\sim}{u})$ = const which are possible are restricted by $\int U(\underset{\sim}{u}) d^n u < \infty$ with $U(\underset{\sim}{u}) > 0$. This can always be realised if the surfaces form nested, closed surfaces in some parts of $\underset{\sim}{u}$-space. But surfaces can also extend, for example, needle-like to infinity, so that the volume between two such surfaces is finite.

If $b(\underset{\sim}{u}, \underset{\sim}{v})$ is such that surfaces of this kind exist, then all "particles" initially in areas surrounded by such surfaces cannot leave these areas. If in an initial distribution function

$$P(s=0 : \underset{\sim}{u}, \underset{\sim}{v}) = P_1(s=0 ; \underset{\sim}{u}, \underset{\sim}{v}) + P_2(s=0 ; \underset{\sim}{u}, \underset{\sim}{v}) \tag{70}$$

P_2 describes such particles and P_1 those outside such surfaces, then $P_2(s : \underset{\sim}{u}, \underset{\sim}{v})$ and the corresponding coefficients $c^{(2)}_{q\nu}(s)$ in eq. (53) develop independently of P_1 and $c^{(1)}_{q\nu}$ in such a way that all $c^{(2)}_{q\nu}(s)$ with $q \gtrapprox 1$ approach zero as $s \rightarrow \infty$. This means that the "density" of "particles" becomes constant on each surface $U(\underset{\sim}{u}) = \text{const.}$

If the surfaces $U(\underset{\sim}{u}) = \text{const.}$ are closed, $u = |\underset{\sim}{u}|$ stays finite for these "particles" for $s \rightarrow \infty$, too. If the surfaces extend to infinity, infinitely large values of u are also realized, but for suitably shaped $U(\underset{\sim}{u})$ the function $H(s)$ stays finite and the probability of stochastic runaway converges to zero.

In the event that more than one function $U(\underset{\sim}{u})$ describing different surfaces exists, the possible "motions of the particles" are accordingly restricted.

2.4 The role of incompressibility

If incompressibility is absent, there are two ways out of the problem:

- a suitable nonlinear transformation
- the introduction of an additional variable.

Both methods have their merits. As an example of the former case let us consider the propagation of waves of fixed frequency in a medium with time independent but in space randomly varying refraction index. In this problem, x will play the role of "time" s. Writing the wave amplitude

$$\psi \sim e^{\, i\int_0^x n(x')dx' - \int_0^x \kappa(x')dx'} \tag{71}$$

with $n(x)$ and $\kappa(x)$ real we obtain from the wave equation

$$\psi'' + k^2(x)\,\psi = 0 \tag{72}$$

$$n' = 2n\kappa\,; \qquad \kappa' = k^2 - n^2 + \kappa^2 \tag{73}$$

With $n = u_1$, $\kappa = u_2$ this would not result in an incompressible flow. With $\frac{1}{n} = u_1$, $\kappa = u_2$ the flow field is, however, divergence-free (Frisch and Sulem[6]). These authors show "nearly" convergence of the reflection coefficient for a halfspace towards 1 in the sense of Fejer as mentioned in the introduction. They overcome the non-normalizability of the stationary solutions of the above continuity equation by some ergodic reasoning. Our method would yield e.g.

$$\left\langle \frac{1}{u^2} \right\rangle + \left\langle K^2 \right\rangle \to \infty \tag{74}$$

The second possibility - the introduction of an artificial variable - is obvious: Define (u_0, b_0) by

$$b_0 = - u_0 \sum_{i=1}^{n} \frac{\partial b_i}{\partial u_i} \tag{75}$$

Any divergence of moments we obtain would then apply to this extended system and must hence be projected into the original subsystem $(u_1 \ldots\ldots u_n)$. If the right-hand sum in the above formula is denoted by χ, we obtain explicitly

$$u_0 = C e^{-\int_0^s \chi(u_1 \ldots u_n) ds} \tag{76}$$

Assume χ to be strictly positive, we may then also conclude from the divergence of

$$\langle u_0^2 \rangle + \langle u_1^2 \rangle + \ldots + \langle u_n^2 \rangle \to \infty \tag{77}$$

that

$$\langle u_1^2 \rangle + \cdots \quad \langle u_n^2 \rangle \rightarrow \infty \tag{78}$$

diverges in the subspace of interest. Applied to the problem of stochastic wave propagation, we obtain from $K > 0$ (and hence $\chi > 0$) the additional relation

$$\langle n^2 \rangle + \langle K^2 \rangle \rightarrow \infty \tag{79}$$

For the reflection coefficient R = $\rho + i\sigma$ given by [9]

$$R = \frac{i(n-1) - K}{i(n+1) - K} \tag{80}$$

eq. (79) implies:

$$\left\langle \left| \frac{R-1}{R+1} \right|^2 \right\rangle = \langle n^2 \rangle + \langle K^2 \rangle \rightarrow \infty \tag{81}$$

Since R is restricted to $|R| \leqq 1$ we expect the probability distribution of R to be sharply peaked around $\rho \approx 1$, $\sigma = \pi$, the point of total reflection.

Summary

An H-theorem is derived for n-dimensional, incompressible, dynamical systems driven by external stationary stochastic forces. On this basis sufficient and necessary criteria for the asymptotic divergence of entropy (runaway) are obtained. These criteria represent at the same time sufficient ones for the divergence of moments. The stochastic forces considered are mappings of general Ornstein-Uhlenbeck processes as a continuous example and of telegraph processes as a discret one. It is believed that extension to more general stationary Markovian processes should be possible. The assumed incompressibility seems to be unavoidable as regards the present method of using an H-theorem. Sometimes one can nonlinearly transform a compressible system into an incompressible one, as, for instance, in the case of wave propagation in a random medium, and thus still obtain interesting results. Another way to generate incompressibility is to add an auxiliary dimension, which allows conclusions to be drawn for a semi-monotonically shrinking flow.

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Appendix

The generalised telegraph process

Markovian processes can be split essentially into continuous diffusion processes, which we have already treated in the text, and into those with discrete jumps. In order to deal with the latter, let us assume the stirring forces $\underset{\sim}{v}$ to jump with a certain probability "per second". There is no restriction if we confine ourselves to just one component of $\underset{\sim}{v}$. Let us denote the transition probabilities from $v = a_i$ to $v = a_j$ by S_{ij}. We then obtain for the joint probabilities V_i the following equation:

$$\frac{dV_i}{ds} = -\sum_j S_{ij} (V_i - V_j) , \qquad S_{ij} = S_{ji} > 0 \tag{A.1}$$

As a consequence the equidistribution is the only stationary solution for a finite number of states N, i.e.,

$$\begin{aligned} V_i &= 1/N \\ &=: \hat{V}_i \end{aligned} \tag{A.2}$$

remains constant in time.

This equation, however, does not yet imply the time-evolution of our system variables $\underset{\sim}{u}$. In the discontinuous case, our previous

$\underset{\sim}{b}(\underset{\sim}{u}, \underset{\sim}{v})$ may be represented as a finite set of vectors

$$\underset{\sim}{b}_i(\underset{\sim}{u}) := \underset{\sim}{b}(\underset{\sim}{u}, a_i) \tag{A.3}$$

and the corresponding equation for the simultaneous joint probability of $\underset{\sim}{u}$ and $v = a_i$ is

$$\frac{\partial P_i}{\partial s} + \underset{\sim}{b}_i(\underset{\sim}{u}) \cdot \frac{\partial P_i}{\partial \underset{\sim}{u}} = - \sum_j S_{ij}(P_i - P_j) \tag{A.4}$$

with the marginal distributions

$$\int P_i(s : \underset{\sim}{u})\, d^n u = V_i \tag{A.5}$$

as solutions of the above equation (A.1).

With

$$P_i^0(s : \underset{\sim}{u}) = \mathcal{U}(\underset{\sim}{u})\, \hat{V}_i \tag{A.6}$$

where $\hat{V}_i$ is the stationary solution (A.2), the right-hand side collision term in (A.4) vanishes. - Hence, in a way similar to that in the Ornstein-Uhlenbeck process, this ansatz can be made a solution of (A.4) if a simultaneous solution of the equation

$$\underset{\sim}{b}_i(\underset{\sim}{u}) \cdot \frac{\partial}{\partial \underset{\sim}{u}} \mathcal{U}(\underset{\sim}{u}) = 0 \tag{A.7}$$

for all i can be found, which is a normalised measure:

$$\mathcal{U}(\underset{\sim}{u}) \geq 0 \quad \text{and} \quad \int \mathcal{U} \, d^n u = 1 \tag{A.8}$$

The examples eqs. (35) and (71) are also consistent with this requirement as they were with the Ornstein-Uhlenbeck case. - Furthermore, we realise that the proof of the H-theorem, with H defined by

$$H = \sum_i \int P_i \ln P_i \, d^n u \tag{A.9}$$

can be taken over from Sec.2.2 quite literally without any change.

This remark also applies for the runaway conditions, and so we may conclude that the results should also hold in the more general case of a mixture of stationary continuous and discrete Markovian stirring forces, at least if they are independent of each other.

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