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Resistive Spectrum of Configurations
without Magnetic Shear

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RESISTIVE SPECTRUM OF CONFIGURATIONS WITHOUT MAGNETIC SHEAR

Abstract

The problem of determining the normal modes of a slightly resistive, incompressible plasma slab with unidirectional fields is reduced to quadratures. The eigenfrequencies are on a system of curves in the stable part of the complex plane which are independent of the resistivity η. For finite wave number the distance from the tip of the ideal Alfvén continuum of the nearest eigenvalue is $O(\eta^{1/3})$. The first correction to this is $O(1/|\ln \eta|)$, which thus only becomes small for unrealistically small values of η. Including this logarithmic correction yields quantitative agreement with numerical computations.

1. INTRODUCTION

The frequency spectrum of the linearized motion of a plasma is perhaps one of the most useful tools in a variety of applications such as stability, wave propagation and heating. In the present article we derive the influence of small resistivity on the Alfvén continua of ideal magnetohydrodynamics. Most previous investigations of this problem (for instance refs. [1-4]) focused on unstable, resistive modes whose eigenfrequencies emerge from the origin owing to the presence of "singular surfaces". More recently, various authors [5-9] have attempted to investigate the entire resistive spectrum numerically. Since introducing resistivity into the ideal equations is a singular perturbation which increases the order and therefore yields new eigenvalues, such numerical integrations are limited not only to not too large values of the mode numbers, but also to not too small values of the resistivity. It is therefore desirable to develop an analytical asymptotic theory which closes these gaps. We do this for the case that the ideal continuum does not contain the origin (there are no singular surfaces). Even though we only consider the simplest possible case, viz. the incompressible motion about a static, plane slab equilibrium, we believe that our results are representative.

2. BASIC EQUATIONS

The equations of resistive, incompressible magnetohydrodynamics, when linearized about a static equilibrium subject to $\nabla P + \vec{B} \times \text{curl}\, \vec{B} = 0$ and $\text{div}\, \vec{B} = 0$, are

$$\left.\begin{aligned} &\rho\, \partial\vec{u}/\partial t + \nabla p + \vec{B} \times \text{curl}\, \vec{b} + \vec{b} \times \text{curl}\, \vec{B} = 0, \\ &\partial\vec{b}/\partial t + \eta\, \text{curl curl}\, \vec{b} - \text{curl}\, (\vec{u} \times \vec{B}) = 0, \\ &\text{div}\, \vec{u} = 0, \quad \text{div}\, \vec{b} = 0. \end{aligned}\right\} \qquad (1)$$

Here p and $\vec{b}$ are the perturbations of the equilibrium pressure P and magnetic field $\vec{B}$, $\vec{u}$ is the flow velocity, and ρ and η (both assumed to be constants) are the mass density and resistivity. We impose the boundary conditions pertaining to a perfectly conducting rigid wall, $\vec{u}_n = 0, \vec{b}_n = 0$, and $\text{curl}_t\, b = 0$ (the subscripts n and t denote normal and tangential components). Since resistive diffusion is ignored, the system (1) is only meaningful for small η. Thus, terms $O(\eta^\alpha)$ with $\alpha \geqq 1$ will be neglected.

Considering a slab equilibrium, characterized by $\partial/\partial y = \partial/\partial z = 0, B_x = 0$, and $P + \frac{1}{2}B^2 = \text{const}$, we Fourier decompose the system (1) by putting the perturbations proportional to $\exp(\sigma t + ik_y y + ik_z z)$. With the abbreviations $\vec{k} = (0, k_y, k_z), \vec{n} = (1,0,0)$, $F = \vec{k} \cdot \vec{B}$, and $\Delta = d^2/dx^2 - k^2$ we then write the system (1) as

$$\rho\sigma\, \vec{u} + p'\vec{n} + i\, \vec{k}\, p + \vec{n}(\vec{B} \cdot \vec{b})' + i\, \vec{k}\, \vec{B} \cdot \vec{b} - \vec{B}'\, \vec{n}\cdot\vec{b} - iF\vec{b} = 0, \qquad (2a)$$

$$\sigma\vec{b} - \eta\Delta\vec{b} + \vec{B}\, \vec{n}\cdot\vec{u}' + \vec{B}'\, \vec{n}\cdot\vec{u} - i\, \vec{u}\, F + i\, \vec{B}\, \vec{k}\cdot\vec{u} = 0, \qquad (2b)$$

$$\vec{n} \cdot \vec{u}' + i\, \vec{k} \cdot \vec{u} = 0, \qquad (2c)$$

$$\vec{n} \cdot \vec{b}' + i\, \vec{k} \cdot \vec{b} = 0, \qquad (2d)$$

where primes denote derivatives with respect to x.

For $\vec{k} = 0$ the system (2) is uninteresting because it yields $\sigma \sim O(\eta)$. It is thus assumed that $\vec{k} \neq 0$. Dotting eqs. (2a) and (2b) with $\vec{n}$, $\vec{k}$, and $\vec{k}\times\vec{n}$, we then obtain

$$\rho\sigma\, \vec{n} \cdot \vec{u} + p' + (\vec{B}\cdot\vec{b})' - i\, F\, \vec{n} \cdot \vec{b} = 0, \qquad (3a)$$

$$\rho\sigma\, \vec{k} \cdot \vec{u} + i\, k^2\, p + i\, k^2\, \vec{B} \cdot \vec{b} - F'\vec{n}\cdot\vec{b} - i\, F\, \vec{k}\cdot\vec{b} = 0, \qquad (3b)$$

$$\rho\sigma\ (\vec{k}\times\vec{n})\cdot\vec{u} - (\vec{k}\times\vec{n})\cdot\vec{B}'\ \vec{n}\cdot\vec{b} - i\ F(\vec{k}\times\vec{n})\cdot\vec{b} = 0, \quad (3c)$$

$$\sigma\ \vec{n}\cdot\vec{b} - \eta\ \Delta\ \vec{n}\cdot\vec{b} - i\ F\ \vec{n}\cdot\vec{u} = 0, \quad (4a)$$

$$\sigma\ \vec{k}\cdot\vec{b} - \eta\ \Delta\ \vec{k}\cdot\vec{b} + (F\ \vec{n}\cdot\vec{u})' = 0, \quad (4b)$$

$$\sigma\ (\vec{k}\times\vec{n})\cdot\vec{b} - \eta\ \Delta\ (\vec{k}\times\vec{n})\cdot\vec{b} + \left[(\vec{k}\times\vec{n})\cdot\vec{B}\ \vec{n}\cdot\vec{u}\right]'$$
$$- i\ F(\vec{k}\times\vec{n})\cdot\vec{u} + i(\vec{k}\times\vec{n})\cdot\vec{B}\ \vec{k}\cdot\vec{u} = 0. \quad (4c)$$

With the aid of eqs. (2c), (2d), (3b) and (3c) the quantities $\vec{k}\cdot\vec{u}$, $\vec{k}\cdot\vec{b}$, $(\vec{k}\times\vec{n})\cdot\vec{u}$, and p can be eliminated algebraically. Then eq. (4b) is omitted because it is a consequence of eq. (4a), and we are left with the system

$$\rho\sigma\ \Delta u - F\Delta\ b + F''b = 0, \quad (5a)$$

$$\sigma\ b - \eta\ \Delta\ b + F\ u = 0, \quad (5b)$$

$$\sigma^2\rho a - \eta\ \sigma\ \rho\ \Delta\ a + F^2 a + G'(Fb - u) = 0, \quad (5c)$$

where $u = -\ i\ \vec{n}\cdot\vec{u}$, $b = \vec{n}\cdot\vec{b}$, $a = i(\vec{k}\times\vec{n})\cdot\vec{b}$, and $G = (\vec{k}\times\vec{n})\cdot\vec{B}$. The boundary conditions are $u = 0$, $b = o$, and $a' = 0$ at $x = 0$ and $x = L$.

In what follows it is assumed that there are no singular surfaces, i.e. that $F(x)$ vanishes nowhere. If there is no magnetic shear (i.e. if $\vec{B}$ is unidirectional), F is either identically zero or it has no zeroes at all. For $F \equiv 0$, however, the system (5) is again uninteresting because $\sigma = 0(\eta)$.

The system (5) yields two families of eigenvalues σ: The first is governed by the sub-system (5a), (5b), thus having eigenfunctions $u(x)\not\equiv 0$, $b(x)\not\equiv 0$; the function $a(x)$ is determined afterwards by the inhomogeneous equation (5c). The second family is governed by eq. (5c) with $u(x)\equiv 0$ and $b(x)\equiv 0$, thus having eigenfunctions $a(x)$.

Introducing dimensionless variables by $x = L\ \bar{x}$, $k^2=\bar{k}^2/L^2$, $F = F_o\bar{F}$, $\sigma = F_o\rho^{-1/2}\ \bar{\sigma}$, and $\eta = L^2F_o\rho^{-1/2}\bar{\eta}$ (F_o is a characteristic value of F) and omitting the bars, we have the fourth-order eigenvalue problem

$$\left[\eta\ \Delta\ F^{-1}\ \Delta\ F + (d/d\ x)\ D\ d/d\ x - k^2 D\right]\ F^{-1}b = 0 \quad (6a)$$

with boundary conditions

$$b = 0,\ b'' = 0 \quad \text{at } x = 0,\ x = 1 \quad (6b)$$

and the second-order eigenvalue problem

$$\eta \,\Delta\, a + D\, a = 0 \qquad (7a)$$

with boundary conditions

$$a' = 0 \text{ at } x = 0,\ x = 1, \qquad (7b)$$

where $D = -\,\sigma - F^2/\sigma$.

3. LIMITING CASES

The problems (6) and (7) can be completely solved in the following two limiting cases: 1. $\eta = 0$ (ideal limit), 2. $F(x)\equiv 1$ (homogeneous magnetic field).

In the ideal case ($\eta = 0$), the zeroes of $D(x)$ are singular points of eq. (6a). Correspondingly, the ranges of the functions

$$\sigma(x) = \pm\, i\, F(x) \qquad (8)$$

form continuous spectra (the Alfvén continua). Discrete eigenvalues do not exist. In contrast, the resistive spectrum is always purely discrete because eqs. (6a), (7a) have no singular points if $\eta \neq 0$.

If $F \equiv 1$, eqs. (6), (7) have constant coefficients and are readily solved by

$$a = \cos n \pi x\ ,\ n = 0, 1, 2, 3, \ldots$$

$$b = \sin n \pi x\ ,\ n = 1, 2, 3, \ldots$$

The eigenvalues σ_n of both problems satisfy the dispersion relation

$$\sigma_n^{\,2} + \eta\, \sigma_n K^2 + 1 = 0 \quad , \quad K^2 = n^2 \pi^2 + k^2 \qquad (9)$$

They are located on the left half of the unit circle or on the negative real axis, depending on whether ηK^2 is smaller or greater than 2. For $n \to \infty$ they accumulate at both $\sigma = 0$ and $\sigma = -\infty$. Each eigenvalue approaches the ideal continuum (now shrunk to the points $\sigma = \pm i$) as η approaches zero. Nevertheless, the whole system of curves (semi-circle plus negative real axis) is filled with eigenvalues for arbitrarily small values of η because ηK^2 attains arbitrarily large values.

Obviously, the two limits $\eta \to 0$ and $F' \to 0$ are not interchangeable. If we first let $\eta \to 0$ and then $F' \to 0$, the spectrum consists of the points $\sigma = \pm i$. On the other hand, if we first let $F' \to 0$ and then $\eta \to 0$, the spectrum densely fills the entire negative real axis and the semicircle.

Note that if there is no shear then for the homogeneous field one has $G' \equiv 0$. The interaction term in eq. (5c) thus vanishes and for every solution σ_n of the dispersion relation (9) there are two eigenfunctions, e.g. either $a \not\equiv 0$, $b \equiv 0$ or $a \equiv 0$, $b \not\equiv 0$. In contrast, in the non-homogeneous case $F' \not\equiv 0$, $G' \not\equiv 0$, which will be treated in the next section, the eigenvalues of problems (6) and (7) coincide only asymptotically for $\eta \to 0$.

4. SMALL RESISTIVITY AND ARBITRARY PROFILE

In this section we restrict attention to eigenvalues σ which are at a distance $0(1)$ from the ideal continuum, so that $D(x) = 0(1)$ throughout (the boundary layer case $\mathrm{Re}\,\sigma \to 0$ since $\eta \to 0$ will be treated separately). Equation (6a) then has two types of solutions: Either $b(x)$ varies on the equilibrium scale so that the first term can be neglected; or $b(x)$ varies fast compared with $F(x)$ so that multiplication by $F(x)$ can be interchanged with differentiation. In either case eq. (6a) reduces to

$$(\eta\, \Delta^2 + \frac{d}{dx} D \frac{d}{dx} - k^2 D)\, \frac{b}{F} = 0. \tag{10}$$

Multiplying eq. (10) by the complex conjugate of b/F and integrating with respect to x, we obtain ($<\ldots> = \int \ldots dx$)

$$\sigma^2 <|(\frac{b}{F})'|^2 + k^2|\frac{b}{F}|^2> + \sigma\,\eta <|\Delta \frac{b}{F}|^2> + <F^2 [|(\frac{b}{F})'|^2 + k^2 |\frac{b}{F}|^2]> = 0. \tag{11}$$

Analogously, problem (7) yields

$$\sigma^2 <|a|^2> + \sigma\eta <|a'|^2 + k^2 |a|^2> + <F^2 |a|^2> = 0. \tag{12}$$

Conditions (11) and (12) imply that the spectrum is restricted to the negative real axis and to the semi-annulus $F_{min}^2 \leq |\sigma|^2 \leq F_{max}^2$, $\mathrm{Re}\,\sigma < 0$. In particular, they imply stability.

For the rest of this section it is assumed that $\eta k^2 \ll 1$ (for instance $k^2 = 0(1)$). Then in eq. (7) and in the first term of eq. (10) k^2 can be neglected. Let b_i ($i=1,..,4$) be four independent solutions of eq. (10) and let b_3, b_4 satisfy the ideal equation (i.e. eq. (10) with $\eta = 0$), thus varying on the equilibrium scale. The remaining two solutions b_1, b_2 can then be chosen as $b = \int y\, d x$, where y satisfies

$$\eta\, y'' + D\, y = 0\,. \tag{13}$$

Then in the dispersion relation

$$\begin{vmatrix} b_1''(0) & b_2''(0) & b_3''(0) & b_4''(0) \\ b_1''(1) & b_2''(1) & b_3''(1) & b_4''(1) \\ b_1(0) & b_2(0) & b_3(0) & b_4(0) \\ b_1(1) & b_2(1) & b_3(1) & b_4(1) \end{vmatrix} = 0$$

the terms b_1'', b_2'' are large compared with the other terms, so that its dominant part is

$$\begin{vmatrix} b_1''(0) & b_2''(0) \\ b_1''(1) & b_2''(1) \end{vmatrix} \begin{vmatrix} b_3(0) & b_4(0) \\ b_3(1) & b_4(1) \end{vmatrix} = 0 \qquad (14)$$

The second factor on the l.h.s. of eq. (14) does not vanish because the ideal spectrum contains no discrete eigenvalues. Hence, the first factor must be zero,

$$y_1'(0)\, y_2'(1) - y_1'(1)\, y_2'(0) = 0.$$

Thus, the eigenvalues are determined from eq. (13) with the boundary conditions $y' = 0$. Since this problem is identical to problem (7), the eigenvalues of eqs. (5 a, b) and (5c) coincide asymptotically for $\eta \to 0$.

5. MONOTONIC PROFILE

If the pressure profile $P(x)$ is monotonic, $B(x)$ and $F(x)$ are monotonic too, and the ideal continua (8) are simply covered as x varies from zero to unity. For definiteness, let F and F' be positive and restrict attention to the upper half-plane $\text{Im}\ \sigma > 0$ (the spectrum is symmetric to the real axis).

A new independent variable z is introduced by [10]

$$\left.\begin{aligned} z &= \eta^{-1/3} \left(\frac{3}{2}\, i\, \alpha\right)^{2/3}, \\ \alpha(x,\sigma) &= \int_{x_o(\sigma)}^{x} dx \left[D(x,\sigma)\right]^{1/2}, \end{aligned}\right\} \qquad (15)$$

where the complex α - plane is cut along the positive imaginary axis, so that $|\arg z| \leq 2\pi/3$ and x_o solves the equation $\sigma = i\, F(x)$. With $w = \sqrt{z'}\, y$ as a new dependent variable, eq. (13) becomes

$$\frac{d^2w}{dz^2} - z\, w = - \eta^{2/3} \left(\frac{3}{2}\right)^{-4/3} i^{2/3}\, \psi^3\, \psi''\, w, \qquad (16)$$

where $\psi = [(\alpha^{2/3})']^{-1/2}$. The function $\psi(x)$ has no singularities near the interval $0 \leqq x \leqq 1$. Hence, the r.h.s. of eq. (16) can be neglected for $\eta \to 0$ and the equation reduces to the Airy equation

$$\frac{d^2w}{dz^2} - z\, w = 0. \tag{17}$$

Since y(x) varies fast compared with z(x), the boundary condition is $d\, w / d\, z = 0$, and the dispersion relation is

$$\frac{dw_1}{dz}(z_o)\frac{dw_2}{dz}(z_1) - \frac{dw_1}{dz}(z_1)\frac{dw_2}{dz}(z_o) = 0, \tag{18}$$

where w_1, w_2 is any pair of independent Airy functions, and $z_\nu = z(\alpha_\nu), \alpha_\nu(\sigma) = \alpha(\nu,\sigma), \nu = 0,1$.

There are three particular Airy functions, denoted by A_i (i = 0, +, -). A_o is defined by [11,12]

$$A_o(z) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{r \exp(-2\pi i/3)}^{r \exp(2\pi i/3)} dt \exp\left(-\frac{t^3}{3} + z\, t\right). \tag{19}$$

The functions A_+ and A_- are obtained from A_o by rotation:

$$A_\pm(z) = \exp(\mp 2\pi i/3)\, A_o\,(z \exp(\mp 2\pi i/3)). \tag{20}$$

The three Airy functions satisfy the identity

$$A_o(z) + A_+(z) + A_-(z) = 0 \tag{21}$$

They are entire functions, i.e. they only have singularities at infinity. $A_o(z)$ is real and oscillatory along the ray $\arg z = -\pi$ (its "Stokes line"). It has infinitely many zeroes there, but none alsewhere. Correspondingly, $A_\pm$ has the Stokes line $\arg z = \mp \pi/3$. For large $|z|$ there are the asymptotic representations

$$\left.\begin{aligned} A_o &\simeq (4\pi)^{-1/2} z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right), -\pi < \arg z < \pi \\ A_+ &\simeq -i(4\pi)^{-1/2} z^{-1/4} \exp\left(\frac{2}{3}z^{3/2}\right), -\frac{\pi}{3} < \arg z < \frac{5\pi}{3} \\ A_- &\simeq i(4\pi)^{-1/2} z^{-1/4} \exp\left(\frac{2}{3}z^{3/2}\right), -\frac{5\pi}{3} < \arg z < \frac{\pi}{3} \end{aligned}\right\} \tag{22}$$

At its Stokes line, each of the A_i can be obtained from the other two functions by using eq. (21). The asymptotic representation of A_i is thus discontinuous at its Stokes line even though A_i itself is analytical.

Obviously one has $\alpha_\nu(\sigma) \sim O(1)$ unless the eigenvalue $\sigma(\eta)$ moves to a zero of $\alpha_\nu(\sigma)$ as $\eta \to 0$. It will turn out that the latter only happens in the boundary layer case Re $\sigma \to 0$. Since it holds that $z_\nu \sim O(\eta^{-1/3})$, the asymptotic representations (22) may be used in the dispersion relation (18). Since these representations are different in the three sectors of the complex plane and on the three Stokes lines, one has to distinguish various cases depending on where z_o and z_1 are located. It turns out that relation (18) cannot be satisfied if the points z_o, z_1 are in different sectors or if one of them is on a Stokes line while the other is in one of the adjacent sectors. If it is taken into account that Re $(\alpha_1-\alpha_o) > 0$, there are the following three possibilities:

$$\left.\begin{array}{l} 1.\ |\arg z_o| < \pi/3 \text{ and } |\arg z_1| < \pi/3 \\ 2.\ \arg z_o = -\pi/3 \text{ and } \arg z_1 > \pi/3 \\ 3.\ \arg z_o < -\pi/3 \text{ and } \arg z_1 = \pi/3 \end{array}\right\} \qquad (23)$$

or, equivalently,

1. Im $\alpha_o < o$ and Im $\alpha_1 < o$,
2. $\alpha_o < o$ and Im $\alpha_1 > o$,
3. Im $\alpha_o > o$ and $\alpha_1 > o$.

The dispersion relation, therefore, has the following three branches:

$$\left.\begin{array}{ll} 1.\ \alpha_1 - \alpha_o = \eta^{1/2}\, n\, \pi, & \text{Im } \alpha_\nu \leqq 0 \\ 2.\ \alpha_o = -\eta^{1/2}\, n\, \pi, & \text{Im } \alpha_1 \geqq 0 \\ 3.\ \alpha_1 = \eta^{1/2}\, n\, \pi, & \text{Im } \alpha_o \geqq 0 \end{array}\right\} \qquad (24)$$

where n is a positive integer. In branches 2 and 3 n is limited from above, while in branch 1 it is limited from below. The eigenvalues of branches 2 and 3 are on two curves (see fig. 1) which start at the two edges of the ideal continuum (solid line), where they form angles of $\pi/6$ with the imaginary axis. They meet at the "triple point" where both α_o and α_1 are real. The eigenvalues of branch 1 (larger n) are on a curve which starts at the triple point to meet its complex conjugate somewhere at the negative real axis, and then continues along the real axis to both sides. For $n \to \infty$ the eigenvalues accumulate at both the origin and infinity according to $\sigma \to - \langle F^2 \rangle / (\eta n^2 \pi^2)$ and $\sigma \to - \eta n^2 \pi^2$. For the case plotted in fig. 1 ($\eta = 10^{-5}$ and $F = 1 + 3x$) numerical integration of eqs. (6) is difficult because η is too small. However, we found excellent agreement between numerical evaluation of eq. (6) and the asymptotic formulae (24) for $\eta = 1.25 \times 10^{-4}$ even for eigenvalues with small n ($n \geqq 2$).

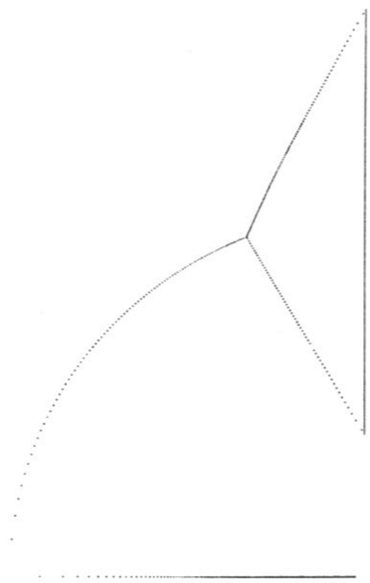

Fig. 1 Eigenvalues in upper left quadrant of the complex σ - plane.

6. BOUNDARY LAYER

The dispersion relations (24) are not valid near the ideal continua (i.e. for $\eta n^2 \ll 1$) because z_o and z_1 are both not large there. The exact Airy functions must be used there instead of their asymptotic forms.

If it holds that $\sigma \simeq i F(o)$, then it follows that $x_o \simeq 0$, $\alpha_o \simeq 0$, and arg $\alpha_1 \approx \pi/4$. Hence arg $z_1 \approx \pi/2$, and $A_+(z)$, being exponentically small for $z = z_1$, is the appropriate eigenfunction of eq. (17). The dispersion relation (18) thus reduces to $d A_+(z_o)/d z = 0$. Since $d A_+/d z$ has its zeroes at its Stokes line arg $z_o = - \pi/3$, the eigenvalues σ are still on the curve $\alpha_o < o$. Similarly, if it holds that $\sigma \approx i F(1)$, it is found that arg $z_1 = \pi/3$, and the eigenvalues are on the curve $\alpha_1 > 0$.

To determine the eigenvalues quantitatively, let

$$\sigma = i\,F(0) + \eta^{1/3}\,\lambda,\ \pi/2 < \arg\lambda < \pi \qquad (25)$$

If one then expands about x = 0 it is found that

$$z_o = -2^{1/3}\,F'(0)^{-2/3}\,\lambda\,. \qquad (26)$$

The dispersion relation $dA_+(z_o)/dz = 0$ now implies $\arg\lambda = 2\pi/3$ and $\lambda \sim 0(1)$. Hence, the curve $\alpha_o < 0$ forms an angle of $\pi/6$ with the imaginary axis, and the spacing of these eigenvalues is $0(\eta^{1/3})$. Similarly, near the other tip of the continuum it is found that $\arg\lambda = -2\pi/3$, where λ determines σ through

$$\sigma = i\,F(1) + \eta^{1/3}\,\lambda.$$

The foregoing analysis is the correct asymptotic description for problem (7). However, for problem (6) a correction due to the slowly varying solutions arises which turns out to be $0(1/|\ln\eta|)$, which thus only becomes small for values of η which are so small that a numerical integration of eqs. (6) is impossible.

7. LOGARITHMIC CORRECTION

For $\eta\,k^2 \ll 1$ the reduced form (10) of eq. (6) reads

$$\eta\,b'''' + F\left[D(F^{-1}b)'\right]' - k^2 D\,b = 0. \qquad (27)$$

Here, only the boundary layer formation near x = 0 is considered. The ideal equation ($\eta = 0$) then has a solution which vanishes at x = 1 and has a logarithmic singularity near x = 0. This solution has to be matched with the boundary layer solution for eq. (27). If one again expands about x = 0:

$$\sigma \approx i\,F_o + \eta^{1/3}\,2^{-1/3}\,F_o'^{\,2/3}\,\lambda\,,$$

$$F_o = F(0),\ F_o' = F'(0),\ D \approx 2\,iF_o'x - \eta^{1/3}\,(2\,F_o')^{2/3}\,\lambda,$$

it is found that the boundary layer equation is of the form

$$\frac{d}{d\xi}\left(\frac{d^2}{d\xi^2} + i\,\xi - \lambda\right)\frac{d}{d\xi}\,b = 0, \qquad (28a)$$

where ξ is a stretched variable defined by

$$x = \varepsilon\, \xi \;,\; \varepsilon = \left(\frac{\eta}{2\,F_o{}'}\right)^{1/3} .$$

For large ξ the function b must behave as $b \sim \gamma + \ln \varepsilon + \ln \xi$, where the constant γ is determined by the ideal solution.The boundary conditions for eq. (28a) can then be written

$$\xi = 0 : b = d^2b/d\xi^2 = 0 \tag{28b}$$

$$\xi \to \infty : b \to 1 - \delta \ln \xi,\; \delta = -\, 1/(\gamma + \ln \varepsilon). \tag{28c}$$

Let the general solution of eq. (28a) be

$$b = C_0 b_0 + C_1\, b_1 + C_2 b_2 + C_3 .$$

Then by putting $db_i/d\xi = y_i$, $i = 0,\, 1,2$ the y's satisfy the equations

$$(d^2/d\xi^2 + i\,\xi - \lambda\,)\; y_0 = i, \tag{29}$$

$$(d^2/d\xi^2 + i\,\xi - \lambda\,)\; y_{1,2} = 0. \tag{30}$$

With

$$z = i\,\xi - \lambda \tag{31}$$

eq. (29) becomes the inhomogeneous Airy equation

$$d^2w/dz^2 - z\,w = 1/\pi\;, \tag{32}$$

where $y_0 = -\,i\,\pi\,w$. The solution [11]

$$w = Hi(z) = \frac{1}{\pi}\int_0^\infty dt\; e^{-\frac{1}{3}t^3}\, e^{zt}$$

has the desired behaviour at infinity

$$w \to -\,\frac{1}{\pi z} + 0\left(\frac{1}{z^2}\right) ,\; z \to \infty$$

and yields

$$b_0(z) = \ln z - i\,\frac{\pi}{2} + \int_0^\infty dt\,(2t - t^4)\,e^{-\frac{1}{3}t^3}\left[-\,\frac{e^{zt}}{z} - t\,E_1(-zt)\right], \tag{33}$$

where E_1 is the exponential integral defined by

$$-E_1(-zt) = \int_{i\infty}^{z} \frac{dz}{z} e^{zt} = \int_{-i\infty t}^{-zt} d\,x \frac{e^{-x}}{x} \quad .$$

For $z \to \infty$ one finds $b_0 \to \ln(-iz)$.

For $y_1(z)$ a solution of the Airy equation

$$d^2y/d\,z^2 - z\,y = 0$$

has to be found which decays for λ finite and $\xi \to \infty$. The asymptotic representations (22) show that $y_1 = A_+$ has the desired property. With the choice

$$C_o = -\delta,\ C_2 = 0,\ C_3 = 1 \tag{34}$$

condition (28c) is thus satisfied. Conditions (28b) are then

$$C_1\, b_1(0) + 1 - \delta\, b_0(0) = 0,$$

$$C_1\, d^2b_1(0)/d\,\xi^2 - \delta\, d^2b_0(0)/d\,\xi^2 = 0,$$

which finally yields the dispersion relation

$$\delta\left[b_o \frac{d^2b_1}{dz^2} - b_1 \frac{d^2b_0}{dz^2}\right] - \frac{d^2b_1}{dz^2} = 0,\ z = -\lambda. \tag{35}$$

The complex roots λ of eq. (35) were computed numerically with expression (33) for b_0 and

$$\frac{d^2b_1}{dz^2} = \frac{d}{dz}A_+ \ ,\ b_1(z) = \int_{i\infty}^{z} dz\, A_+$$

for b_1. The result, shown in fig. 2 for the lowest eigenvalue, is compared with direct numerical solution of problem (6). The solid line is the evaluation of the analytic theory. The points × are obtained by applying a matrix eigenvalue solver to 200 Fourier-coefficients. It is seen that the latter method breaks down for a δ-value lower than 0.3, while for $\delta > 0.5$ there is no boundary layer, corresponding to the fact that the branches 2 and 3 of relations (24) disappear. For $\delta \to 0$, one gets $\lambda \to \frac{1}{2}(-1 + \sqrt{3})$. In this limit the eigenvalue curve forms an angle of $\pi/6$ with the imaginary axis, which was the result of sec. 6. Note, however, that the limit $\delta \to 0$ is reached extremely slowly as $\eta \to 0$. For instance, for $F_o' = .5$, $\gamma = 1$, an η-value of 5×10^{-4} yields $\delta = 0.112$ while $\eta = 10^{-10}$ yields $\delta = 0.0454$.

8. THE BALLOONING CASE

Again considering monotonic profiles, let us suppose that k^2 is so large that $\eta k^2 = 0(1)$ and assume that

$$\text{Re } \sigma \sim 0(1), \tag{36}$$

so that b rapidly oscillates everywhere. Then in eq. (6a) the functions D, F may be interchanged with differentiation, yielding

$$\eta \Delta y + D y = 0, \tag{37a}$$

where $y = \Delta b$ satisfies the boundary condition

$$x = 0, 1 : y = 0. \tag{37b}$$

From

$$\sigma^2 <|y|^2> + \sigma\eta <|y'|^2 + k^2|y|^2> + <F^2|y|^2> = 0 \tag{38}$$

it is concluded that for complex eigenvalues σ it holds that

$$\text{Re } \sigma = - \frac{\eta}{2} \left(\frac{<|y'|^2>}{<|y|^2>} + k^2\right) < - \frac{1}{2} \eta k^2 , \tag{39}$$

which justifies the assumption (36). Although the boundary conditions $y' = 0$ for eq. (13) are different from conditions (37b), asymptotically there is no difference in the dispersion relation, i.e. the derivatives in eq. (18) can be omitted without changing the asymptotic spectrum. This also means that problems (7) and (37) yield the same spectra.

Replacing D by $D - \eta k^2$ the formulae of sec. 5 can be used to solve problem (37) for the case of a monotonic profile. With

$$\alpha_\nu = \int_{x_o}^{\nu} d x \, (D - \eta k^2)^{1/2} \quad , \nu = 0, 1$$

the spectrum is again described by eqs. (24). Relation (39)shows that the branches 2 and 3 move away from the tips of the continuum. The computation of the end points of these branches has to be done with exact Airy functions, as in sec. 6. However, an estimate of the position of these end points is found by putting α_o, α_1 equal to zero, because the number n in formulae (24) is an independent parameter. Thus, $\alpha_o = 0$ implies $x_o = 0$, or

$$\sigma^2 + \sigma \eta k^2 + F_o^2 = 0 \tag{40}$$

and

$$\alpha_1 = \sqrt{-\frac{1}{\sigma}} \int_0^1 d\,x \sqrt{F^2 - F_0^2} . \qquad (41)$$

Formula (41) shows that $\mathrm{Im}\,\sigma > o$ implies $\mathrm{Im}\,\alpha_1 > o$. The other end point, defined by $\alpha_1 = o$ yields the condition

$$\sigma^2 + \sigma\,\eta\,k^2 + F_1{}^2 = 0, \quad F_1 = F(1).$$

Correspondingly, the formula

$$\alpha_0 = -\sqrt{\frac{1}{\sigma}} \int_0^1 d\,x \sqrt{F_1{}^2 - F^2}$$

shows that $\mathrm{Im}\,\sigma > o$ implies $\mathrm{Im}\,\alpha_0 > 0$. For $\eta k^2 = 2\,F_o$ the first end point reaches the real axis, and for $\eta k^2 = 2\,F_1$, this happens for the second end point, so that for $\eta k^2 \geqq 2\,F_1$ all eigenvalues are real. This latter result can also be derived directly from eq. (38).

9. CONCLUSION

The spectral problem of a plasma slab with a monotonic profile without shear has been reduced to quadratures, yielding a complete qualitative description of the entire spectrum. In particular, it has been shown in this geometry that shearless equilibria (without magnetic nulls) are resistively stable, and that the damping of the Alfvén modes is $O(1)$ as $\eta \to 0$. This supports the conjecture [13] that ideally stable equilibria without magnetic shear (and without magnetic nulls) remain resistively stable, and it may have an impact on the theory of Alfvén wave heating. In addition to the results in [14] (see also the announcement in [15]), the boundary layer modes ($\mathrm{Re}\,\sigma \to 0$) as well as the ballooning modes ($k^2 \to \infty$) have been discussed, the main result being that the ideal spectrum is not approximated by the resistive spectrum and that the leading-order asymptotics is insufficient for obtaining the spectrum quantitatively.

The present analysis has been generalized to non-monotonic profiles. Here, an additional branch appears which emerges at an angle of $\pi/4$ with the imaginary axis from the point $\sigma = i\,F(\xi)$, where $F'(\xi) = 0$. This will be published elsewhere.

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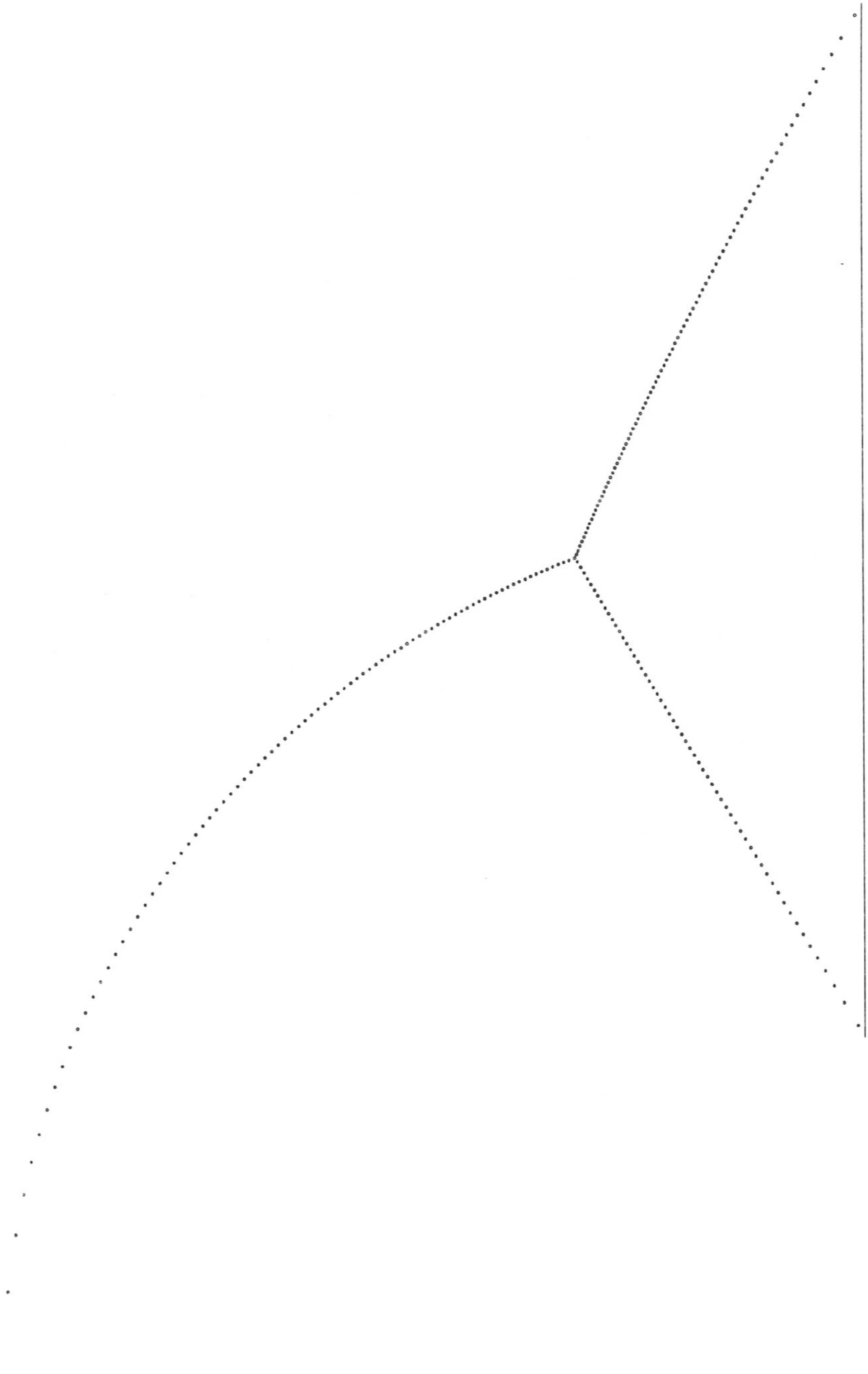
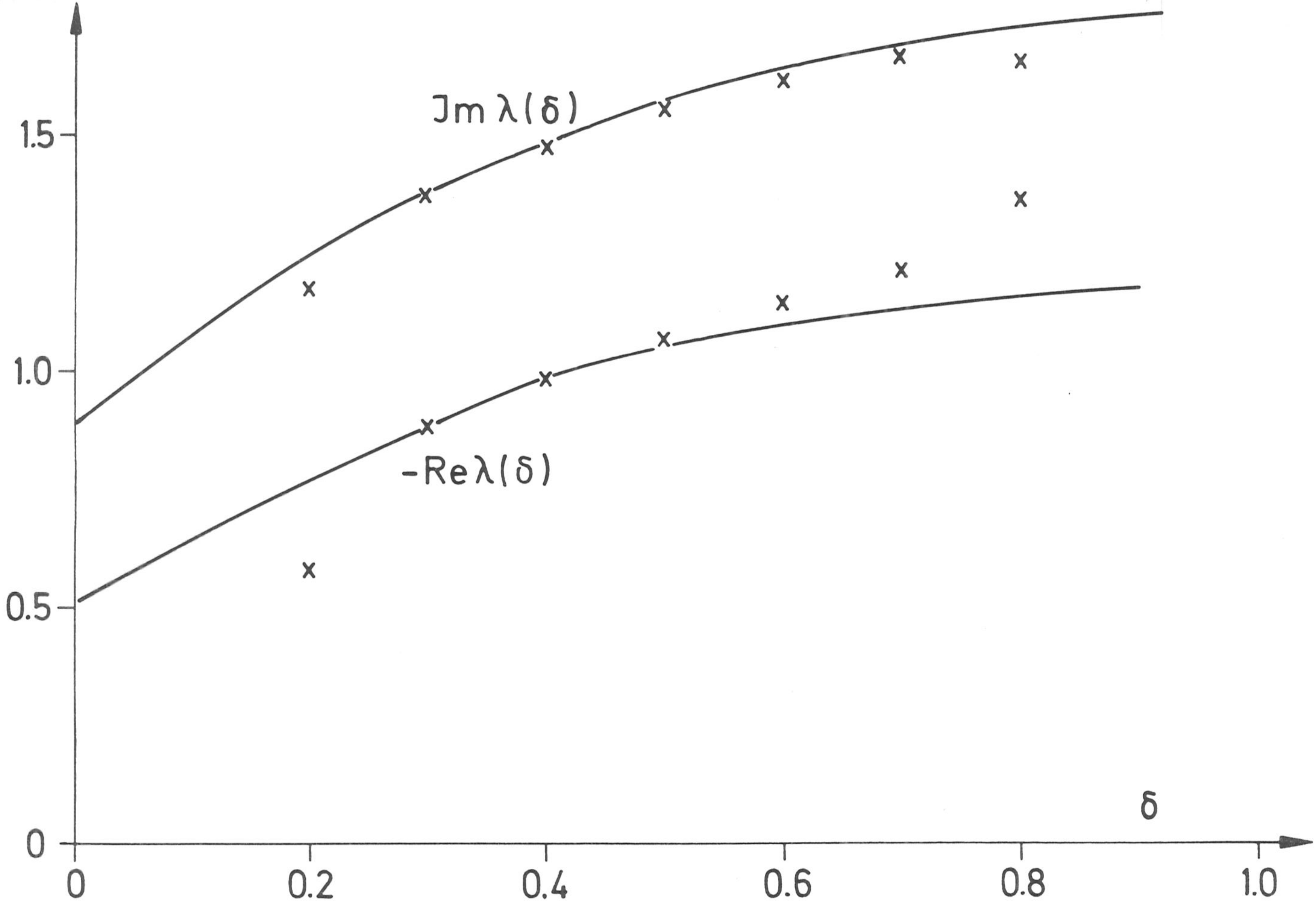
FIGURE CAPTIONS

Fig. 1 Eigenvalues in upper left quadrant of the complex σ - plane

Fig. 2 Lowest Eigenvalue $\lambda(\delta)$ for $k^2 = 0$, $F = 1 + x$




Jm λ(δ)
-Re λ(δ)
δ
1.5
1.0
0.5
0
0
0.2
0.4
0.6
0.8
1.0