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Fluctuation Spectrum for Linear Gyroviscous MHD

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(in English)

Abstract:

The influence of gyroviscosity on the fluctuations of an MHD plasma is investigated by the method reported in a previous note [1]. The main result is that gyroviscosity does not help to remove ultraviolet divergences. For a sub-class of observables it does not even show up. The full non-linear problem may be needed.

A general formalism for obtaining the fluctuation spectrum of plasmas and fluids in statistical equilibrium has been proposed in Ref. 1 . It is valid for linearized equations of conservative systems in a heat bath which allows the use of Gibbs statistics. The application of this formalism to gyroviscous one-fluid homogeneous plasmas is discussed in this letter. For general observables the spectrum depends on the eigenvalues of a symmetric operator containing gyroviscous and ideal MHD contributions. For a more restricted class of fluctuations such as density fluctuations, however, the spectrum only depends on the eigenvalues of the MHD operator.

The linearized equations of motion of non-dissipative gyroviscous one-fluid plasmas can be written [2] in the form

$$\rho_0 \ddot{\vec{v}} + \nabla . \delta \dot{\overleftrightarrow{\Pi}} + Q \vec{v} = 0 \tag{1}$$

where $\vec{v}$ is the perturbed fluid velocity, ρ_0 is the mass density in equilibrium, Q the MHD operator [3] and $\delta \overleftrightarrow{\Pi}$ the perturbed gyroviscous tensor [4] .

It is assumed that the unperturbed plasma is in homogeneous static equilibrium with a constant magnetic field $\vec{B}_0$, and that the perturbations are two-dimensional with velocities perpendicular to $\vec{B}_0$. For

$$\vec{B} = B_0 \vec{e}_z \tag{2}$$

the components of the tensor $\delta \overleftrightarrow{\Pi}$ are [4]

$$\delta\Pi_{xx} = -\delta\Pi_{yy} = -\alpha\Gamma_{xy} \quad ,$$
$$\delta\Pi_{xy} = \delta\Pi_{yx} = \frac{-\alpha}{2}\left(\Gamma_{yy} - \Gamma_{xx}\right) \quad , \tag{3}$$

with

$$\Gamma_{xy} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \quad ,$$
$$\Gamma_{yy} - \Gamma_{xx} = \frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial x} \quad , \tag{4}$$

$$\alpha = \frac{P_0}{\omega_{ic}} \tag{5}$$

P_0 and ω_{ic} being the pressure and the ion cyclotron frequency in equilibrium, respectively.

It is a matter of simple algebra to write

$$\nabla . \delta\overleftrightarrow{\Pi} = \frac{\alpha}{2}\nabla^2\left(\vec{e}_z \times \vec{v}\right) \tag{6}$$

and

$$Q\vec{v} = -\left(\frac{B_0^2}{\mu_0} + \gamma P_0\right)\nabla\left(\nabla . \vec{v}\right) \tag{7}$$

where γ is the ratio of specific heat capacities.

Equation (1) can now be written as

$$\ddot{\vec{v}} + E\dot{\vec{v}} + F\vec{v} = 0 \quad , \tag{8}$$

where

$$E = a\nabla^2\vec{e}_z \times \tag{9}$$

is an antisymmetric operator and

$$F = -b\nabla(\nabla. \ldots) \tag{10}$$

is a symmetric operator.

a and b are given by

$$a = \frac{\alpha}{2\rho_0} \quad , \quad b = \frac{1}{\rho_0}\left(\frac{B_0^2}{\mu_0} + \gamma P_0\right) \tag{11}$$

As discussed in Ref. 1 the Hamiltonian of eq. (8) can be written as

$$H = \frac{1}{2}(Y, MY) \tag{12}$$

with

$$Y = \begin{pmatrix} \vec{p} \\ \vec{v} \end{pmatrix} \quad , \quad \vec{p} = \dot{\vec{v}} + \frac{E}{2}\vec{v} \tag{13}$$

and the symmetric operator

$$M = \begin{pmatrix} I & -E/2 \\ E/2 & F - E^2/4 \end{pmatrix} . \tag{14}$$

Y is expanded in terms of the eigenfunctions of M

$$Y = \sum_{i=1}^{\infty} a_i Y_i \quad , \tag{15}$$

where

$$Y_i = \begin{pmatrix} \vec{A} \\ \vec{C} \end{pmatrix} e^{i\vec{k}_i \cdot \vec{r}} \tag{16}$$

in view of the homogeneity of the equilibrium. $\vec{A}$ and $\vec{C}$, as $\vec{p}$ and $\vec{v}$, are vectors perpendicular to $\vec{B}_0$. The eigenvalues λ_i can be obtained from

$$M Y_i = \lambda_i Y_i \tag{17}$$

which by substitution of eq. (16) in eq. (17) reduces to

$$\begin{pmatrix} \vec{A} + a k_i^2 \vec{e}_z \times \vec{C} \\ -a k_i^2 \vec{e}_z \times \vec{A} + (a^2 k_i^4 + b k_i^2) \vec{C} \end{pmatrix} = \lambda_i \begin{pmatrix} \vec{A} \\ \vec{C} \end{pmatrix} \tag{18}$$

After crossing the first equation by $\vec{e}_z$ the characteristic equation leads to

$$\lambda_i = \frac{1}{2}\left(1 + a^2 k_i^4 + b k_i^2 \pm \sqrt{(1 + a^2 k_i^4 + b k_i^2) - 4 b k_i^2}\,\right) \tag{19}$$

For large k_i the λ_i behave as

$$\lambda_i \approx a^2 k_i^4 \tag{20}$$

and

$$\lambda_i \approx \frac{b}{a^2} k_i^{-2} \tag{21}$$

The formula for the expectation values of a_i^2 derived in 1 is

$$\langle a_i^2 \rangle = \frac{1}{\beta \lambda_i} \tag{22}$$

Its application leads here to a spectrum having contributions in k_i^{-4} and k_i^2 for large k_i. The latter contribution is obviously not acceptable. Without gyroviscous effects the contributions would be like

k_i^{-2} and 1 for large k_i. The latter contribution, though not divergent, is not acceptable either.

This "ultraviolet" catastrophe well known in other areas such as field theory [5] seems to become worse when gyroviscous effects are taken into account. This is true of a Gaussian distribution and is an open question for the full nonlinear problem. Non-Gaussianity may be the key answer as demonstrated for the Korteweg-de Vries equation in [6].

An improvement can also be achieved if the observables are restricted to being functions of $\vec{v}$ only (not of $\vec{p}$). In this case $\langle a_i^2 \rangle$ depend on the eigenvalues of F only, which behave as k_i^2. This is easy to see from the general Hamiltonian introduced in [1]:

$$H = \frac{1}{2}\left[\left(\vec{p} - \frac{\epsilon}{2}\vec{v}\right), \left(\vec{p} - \frac{\epsilon}{2}\vec{v}\right)\right] + \frac{1}{2}\left(\vec{v}, F\vec{v}\right) \tag{23}$$

If

$$\langle f(\vec{v}) \rangle = \frac{\int D(\vec{p})\, D(\vec{v})\, f(\vec{v})\, e^{-\beta H}}{\int D(\vec{p})\, D(\vec{v})\, e^{-\beta H}} \tag{24}$$

the integration over $\vec{p}$ can be done separately, leaving

$$\langle f(\vec{v}) \rangle = \frac{\int D(\vec{v})\, f(\vec{v})\, e^{-\beta H}}{\int D(\vec{v})\, e^{-\beta H}} \tag{25}$$

For such observables the expectation values are thus unaffected by gyroviscosity and behave as k_i^{-2}.

As noticed in 1, the expectation value of the total Hamiltonian diverges for a Gaussian in any case and leads us to the conclusion that non-Gaussianity combined with gyroviscous effects will have to be studied next. This is a very difficult problem in more than one dimension, as discussed in 1.

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