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Abstract
The behaviour of IC waves near resonances in tokamak geometry is investigated in details. For this purpose, a one-dimensional model is proposed, which takes into account the orientation of the incident wavefronts with respect both to the singular layer and to the magnetic surfaces. The differential equations describing the waves are derived again from Vlasov-Maxwell equations in the finite Larmor radius approximation; they are shown to conserve the wave power flux in the absence of dissipation, and to reproduce the local dispersion relation in the WKB limit. These equations are solved exactly in some important situations, and with the Green-function technique in the general case. The amount of power coupled to Bernstein waves and absorbed by cyclotron damping is explicitly evaluated.

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C O N T E N T

§ 1 - Introduction

The description of the propagation and absorption of Ion Cyclotron waves in a Tokamak plasma is made difficult by the fact that Maxwell equations do not admit separate-variable solutions in this geometry. Approximate methods of solution, such as inverse aspect ratio expansion or geometric optics, fail in the vicinity of resonances (ion cyclotron harmonics and two-ion hybrid), i.e. precisely where most of the absorption is expected to take place. Hence a separate investigation of the singular layers is of the utmost importance.

In practice, such an investigation is only feasible in a plane-layered geometry, where the problem can be reduced to the solution of an appropriate set of ordinary differential equations /1-7/. Perkins /8/ has suggested how the results obtained with this approach can be applied to a tokamak. Observing that near a resonance the gradients of the dielectric tensor are essentially horizontal, so that $k_\perp \to \infty$ implies $k_X \gg k_Z$, he writes

$$k_{/\!/} \simeq \frac{n_\varphi}{R} + \frac{B_P}{B_T} k_X \qquad \vec{k}_\perp \simeq k_X \vec{e}_X \tag{1}$$

(here X, Z are cartesian coordinates in a meridian plane, $\vec{k}$ is the wavevector, B_p and B_T the poloidal and toroidal magnetic field, respectively). Eqs. (1) imply a linear relation between $k_{/\!/}$ and $k_\perp$, the wavevector components parallel and perpendicular to the static magnet field, which, together with the conservation of the toroidal wavenumber n_φ, permits the reduction of the tokamak problem to one in a single space dimension.

The relation between $k_{/\!/}$ and $k_\perp$ following from Eq. (1) is, however, surprising at first, since by definition $\vec{k}_\perp$ cannot have a component along the static magnetic field. In the next section, we will show that Eq. (1) is a consequence of a special assumption about the incident wave. Nevertheless, a simple generalization of this equation can indeed be used to construct a one-dimensional model of the waves behaviour near singular layers, much along the lines suggested by Perkins. This behaviour depends, however, on the orientation of the incident wavefronts with respect both to the resonance layer and the magnetic surfaces. Thus quantitative results can only be obtained by combining the local analysis of the resonance with ray tracing (or some other global method of solution) between the antenna and the singularity.

Section 3 is devoted to the derivation of the appropriate differential equations for waves in an inhomogeneous plasma. This is immediate only in the cold plasma limit. When finite Larmor radius effects have to be taken into account, additional space derivatives arise from the non-locality (dispersion) of the conductivity tensor. For these terms the simple prescription of substituting the gradient operator for the wavevector , $\vec{k} \rightarrow i\vec{\nabla}$, is not sufficient to define the differential equations uniquely. In the absence of a better criterion, the differential form of the warm plasma terms was therefore mainly chosen in order to obtain a soluble set of equations. These "standard" wave-transformation equations, however, do not strictly conserve the power flux in the absence of dissipation /7/.

Recently, Colestock and Kashuba /9/ and Swanson /5/, /6/ have rederived the differential equations for waves in an inhomogeneous plasma, starting from Vlasov equation. In Section 3 we do the same with an euristic but very simple procedure. The equations obtained in Section 3 are specialized to I.C. waves in a tokamak plasma in Section 4. Finally, we use these equations in Section 5 to discuss ion cyclotron heating in a few situation of fusion interest.

We have been able to solve the wave equations "exactly" in two cases. The first is near a Two Ion Hybrid resonance, when the cyclotron resonance of the minority ions does not coincide with the first harmonic of the main plasma (e.g., a He_3^{++} or D^+ minority in an H^+ plasma). In this case, the problem is much simplified by the fact that it can be adequately treated within the cold plasma approximation, and it reduces to a slightly modified form of the classic Budden tunneling problem /10/.

The second soluble case is near the first harmonic resonance $\omega = 2\Omega_c$, in a single species plasma. It is fair to say that our results in this case are simpler than, but numerically very close to, the approximate results of Swanson /6/.

On the other hand we were not able to obtain a closed solution for the popular case of an H^+ minority in a D^+ plasma. Of course, this problem reduces to the second of the above mentioned cases for extremely small H^+ concentration, and, more interesting, to the first one when the H^+ concentration exceeds a certain threshold. We have obtained an approximate solution, valid for small optical thickness of the two-ion hybrid layer, which suggests how to interpolate between these limiting cases.

There is no space in this paper for sistematic applications. We may mention, however, that we have coupled the results obtained here with a ray-tracing code capable of following IC waves in tokamak plasmas of arbitrary meridian shape /11/. In spite of the fact that a number of interesting situations remain unaccessible to this approach (notably, all cases in which high quality cavity modes are excited), the resulting code represents a useful tool for the investigation of IC heating of tokamak plasmas.

§ 2 - Modelling resonance, layers in one dimension

Let us describe the plasma equilibrium configuration parametrically as

$$X = X(\psi,\vartheta) \qquad Z = Z(\psi,\vartheta) \tag{2}$$

ψ, a function of the magnetic flux, labels magnetic surfaces, while ϑ is the usual poloidal angle. ψ,ϑ are a natural set of coordinates, they are, however, not orthogonal, except in the case of circular concentric magnetic surfaces. We can nevertheless define at each point a triad of orthogonal unit vectors as

$$\vec{e}_\psi = \frac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|} \qquad \vec{e}_\tau = \vec{e}_\varphi \times \vec{e}_\psi \qquad \vec{e}_\varphi = R_T \vec{\nabla}\varphi \tag{3}$$

where $R_T = R_o + X$ is the distance from the vertical axis. The components of the wavevector with respect to this reference frame are /11/

$$k_\psi = (\vec{k}\cdot\vec{e}_\psi) = k_X \cos\tau + k_Z \sin\tau$$
$$k_\tau = (\vec{k}\cdot\vec{e}_\tau) = -k_X \sin\tau + k_Z \cos\tau \qquad (4)$$
$$k_\varphi = n_\varphi / R_T$$

where

$$\cos\tau = \frac{1}{N_\tau}\frac{\partial Z}{\partial\theta} \qquad \sin\tau = -\frac{1}{N_\tau}\frac{\partial X}{\partial\theta} \qquad (5)$$
$$N_\tau^2 = \left(\frac{\partial X}{\partial\theta}\right)^2 + \left(\frac{\partial Z}{\partial\theta}\right)^2$$

(in the limit of circular concentric magnetic surfaces, $\tau = \vartheta$).

The dispersion relation, on the other hand, depends only on the components of $\vec{k}$ parallel and perpendicular to the local static magnetic field. The perpendicular index of the fast magnetosonic wave depends only weakly on $k_\parallel$:

$$n_\perp^2 = -\frac{(n_\parallel^2 - R)(n_\parallel^2 - L)}{(n_\parallel^2 - S)} \qquad (6)$$

where R, L, and $S = \frac{1}{2}(R+L)$ are the elements of the dielectric tensor in the familiar notations of Stix /12/ (crf. Section 4) and $\vec{n} = \vec{k}c/\omega$. In the ion cyclotron frequency range R, L, S are of order $\omega_{pi}^2/\Omega_{ci}^2 \gg 1$, while typically $n_\parallel^2$ is of order unity. On the other hand, $n_\parallel$ plays a fundamental role in determining absorption, through the Doppler broadening of the ion cyclotron resonances. $k_\parallel$ and $k_\perp$ can be expressed in terms of k_ψ, k_τ, k_φ as follows:

$$k_\parallel = k_\tau \sin\Theta + \frac{n_\varphi}{R_T}\cos\Theta$$
$$k_\perp^2 = k_\psi^2 + k_\eta^2 \qquad (7)$$
$$k_\eta = k_\tau \cos\Theta - \frac{n_\varphi}{R_T}\sin\Theta$$

where $\Theta = \mathrm{atan}\,(B_p/B_T)$, and k_η is the component of $\vec{k}$ orthogonal to $\vec{B}$ and lying within the magnetic surface.

By combining Eqs. (4) and (7) we obtain the desired relation between k_X, k_Z on the one hand, $k_\parallel$ and $k_\perp$ on the other hand. This relation becomes very simple if we neglect terms of order Θ^2 or $r\Theta/R$, where r is the distance from the magnetic axis:

$$k_\perp^2 \cong k_X^2 + k_Z^2$$

$$k_{/\!/} \cong \frac{m\rho}{R} + \left(-k_X \sin\tau + k_Z \cos\tau\right) \sin\Theta \qquad (8)$$

Let us note first of all that we obtain Eq. (1) if we neglect k_Z, and we put $\sin\tau \cong \pm 1$ in Eqs. (8). In words, we are assuming that the incident wavefronts are essentially parallel to the resonance layer, and that the latter cuts magnetic surfaces at a right angle. A somewhat more general relation can be obtained by maintaining the assumption that wavefronts are vertical cylinders ($k_X \gg k_Z$), but allowing for an arbitrary position of the resonance layer with respect to the magnetic axis:

$$k_{/\!/} \cong \frac{m\rho}{R} - k_\perp \sin\tau \sin\Theta \qquad (9)$$

Note that $k_{/\!/}$ is then independent from $k_\perp$ for the equatorial ray ($\sin\tau = 0$).

From these examples we conclude that any relation between $k_{/\!/}$ and $k_\perp$, obtained by specializing Eqs. (8) as it has been done to write Eq. (1) or (9) are completely inadequate. Thus for example $k_X \gg k_Z$ cannot hold if the wave is approaching the two ion hybrid layer from the low magnetic field side, since then it encounters first a cut-off ($k_X \to 0$). Even in the opposite case, the Eikonal approximation in most cases breaks down before such a condition is well satisfied.

The values of k_Z and τ for a wave incident on a singular layer are determined both by the boundary conditions near the antenna (initial shape of the wavefronts) and by the refraction due to gradients of the dielectric tensor along the wavefronts. They can only be determined from the solution of Maxwell equations between the antenna and the singularity, e.g. using ray tracing. If k_Z and τ can be determined, on the other hand, Eqs. (8) provide a perfectly adequate starting point for a local one-dimensional analysis of the singular layer along the lines suggested by Perkins. Namely, the goal will be achieved by neglecting the variations of k_Z and τ in traversing the singular layer, while substituting $-i d/dX$ for k_X in all terms of the dispersion relation which contain $k_\perp^2$ (cfr. the first of Eqs. (7)):

$$\vec{k}_\perp \rightarrow -i\left(\frac{d}{dx}\right)\vec{e}_x + k_z\vec{e}_z \tag{10}$$

The order of the resulting system of differential equations is then determined by the highest term retained in the finite Larmor radius expansion of the Dispersion Relation.

The role of $k_{\parallel}$ is completely different: no finite-order expansion in $k_{\parallel}$ makes sense near ion cyclotron resonances. Thus a description of wave resonances with differential equations is justified only if these are not too close to cyclotron resonances, say if

$$\left|\frac{\omega - n\Omega_{ci}}{k_{\parallel} v_{th_i}}\right| \gtrsim 1.5 - 2$$

throughout the singular layer. Under this condition, it seems more appropriate to consider $k_{\parallel}$ as a parameter in the equations. We only note here that if condition (11) is not satisfied, the wave resonance is washed out, and absorption can be easily estimated by integrating the power transport equation /14/ through the Doppler broadened cyclotron resonance.

§ 3 - The differential equation for e.m. waves in an inhomogeneous, warm plasma

The differential equation describing the propagation of e.m. waves in a plane layered plasma near ion-cyclotron harmonic or two ion-hybrid resonances has been carefully derived by Swanson /5/ starting from the set of Vlasov-Maxwell equations. In principle, we should generalize this derivation to the somewhat more complex situation described in the previous section, in which the wavevector is not parallel to the space gradients ($k_z \neq 0$), and $k_{\parallel}$ is not a constant. To avoid the rather cumbersome algebra required for this purpose, however, we propose a less explicit but very simple derivation, which also provides some immediate insight into the conditions for the validity of the obtained differential equation. It is convenient to deal first with the general case; application to the tokamak geometry will be made in the next section.

The constitutive relation in a locally Maxwellian plasma can be written

$$\vec{J}(\vec{r},t) = \sum_j \frac{2N_j(\vec{r})e_j^2}{m_j} \int d_3\vec{v}\, F_{M_j}(\vec{r},\vec{v}) \cdot \frac{\vec{v}}{v_{th_j}} \int_{-\infty}^{t} dt' \left[\frac{\vec{v}'}{v_{th_j}} \cdot \vec{E}(\vec{r}',t')\right] \tag{11}$$

where $\vec{r}', \vec{v}'$ are the solutions of

$$\frac{d\vec{r}'}{dt'} = \vec{v}' \qquad \frac{d\vec{v}'}{dt'} = \frac{e_j}{m_j}\left[\vec{E}(\vec{r}',t') + \frac{\vec{v}'}{c} \times \vec{B}_0(\vec{r}',t')\right] \tag{12}$$

which satisfy the "final" conditions

$$\vec{r}' = \vec{r}, \quad \vec{v}' = \vec{v} \quad \text{at } t' = t \tag{13}$$

If the plasma is uniform and steady state, the "conductivity kernel" in Eq. (11) depends only on t-t', so that $\vec{J}$ and $\vec{E}$ are algebraically related in the Fourier-transformed space-time ($\vec{k},\omega$). If the plasma is nonhomogeneous, on the other hand, a finite-order system of differential equations for the propagation of waves can be obtained from the integral relation (11) only by assuming weak dispersion:

$$\frac{k_\perp v_{th}}{\Omega_c} \ll 1 \qquad \left|\frac{\omega - n\Omega_j}{k_\parallel v_{th_j}}\right| \gtrsim 2 \tag{14}$$

for all species of particles, and n = 0, $\pm$ 1, $\pm$ 2, ... On can then distinguish in Eqs. (11) the local, short scale dependency due to the thermal motion of the particles, still of the form t-t', and the slower general dependencies due to the gradients of the plasma parameters. This justifies the ansatz

$$\vec{E}(\vec{r},t) = \vec{E}_{\vec{k}}(\vec{r}) \exp i\left(S(\vec{r}) - \omega t\right) \tag{15}$$

with

$$|\vec{k}| \equiv |\vec{\nabla} S| \gg \left| \frac{S}{E_k} \vec{\nabla} E_k \right| \tag{16}$$

Substituting (15) into (11) gives

$$J(\vec{r},t) = e^{i(S(\vec{r}) - \omega t)} \sum_j \frac{2N_j e_j^2}{m_j} \int d_3\vec{v}\, F_{M_j}(\vec{r},\vec{v}) \cdot$$

$$\frac{\vec{v}}{v_{th_j}} \int_{-\infty}^{t} dt' \left[\frac{\vec{v}'}{v_{th_j}} \cdot \vec{E}_k(\vec{r}') \right] \exp i \left[\omega(t-t') - (S(\vec{r}) - S(\vec{r}')) \right] \tag{17}$$

In the dispersionless (cold) limit

$$F_M(\vec{r},\vec{v}) = \frac{e^{-v^2/v_{th_j}^2}}{\pi^{3/2} v_{th_j}^3} \rightarrow \delta(\vec{v}) \tag{18}$$

Since to lowest order in the inhomogeneity

$$S(\vec{r}') - S(\vec{r}) \cong \vec{k} \cdot (\vec{r}' - \vec{r}) \cong \vec{k}(\vec{r}) \cdot \underline{\underline{M}}(\vec{r}, t'-t) \cdot \vec{v} \tag{19}$$

where $\underline{\underline{M}}$ is an appropriate matrix independent from $\vec{v}$, the space dependent part of the phase factor in Eq. (17) disappears in this limit. Eq. (17) then easily leads to the usual cold plasma conductivity tensor, with plasma parameters dependent on space. To next order in the dispersion, under conditions (14) and (16), $|\vec{r} - \vec{r}'|$ is uniformly bounded, and the largest space derivatives arise from $S(\vec{r})$. It will therefore be enough to expand

$$\exp i\left[S(\vec{r}') - S(\vec{r})\right] \cong \exp i\vec{k}(\vec{r}' - \vec{r}) \cdot$$

$$\cdot \left\{ 1 + \frac{i}{2} (\vec{r}' - \vec{r})(\vec{r}' - \vec{r}) : \vec{\nabla}\vec{k} \right\} \tag{20}$$

Substituting into (17), we immediately obtain

$$J(\vec{r},t) = \underline{\underline{\sigma}}^{cold} \cdot \vec{E} - \frac{1}{2}\frac{\partial}{\partial x_j}\left\{\left[\frac{\partial^2 \underline{\underline{\sigma}}^{hot}}{\partial k_j \partial k_\ell}\right]_{k=0} \frac{\partial \vec{E}}{\partial x_\ell}\right\} \tag{21}$$

where $\underline{\underline{\sigma}}^{cold}$ and $\underline{\underline{\sigma}}^{hot}$ are the cold and hot local conductivity tensor, respectively, and in the second term the limit $\vec{k} \to 0$ has to be made after taking the $\vec{k}$ - derivatives. Hote that in spite of the appearance of two space derivatives, this expression arises only from first order terms, one space derivative being introduced by the identity

$$k_\ell \vec{E} = -i \frac{\partial \vec{E}}{\partial x_\ell} \quad + \text{ higher order terms} \tag{22}$$

which follows from (15) under condition (16). Substituting into Maxwell equations, we finally obtain the desired form of the full wave equation in a warm, weakly inhomogeneous plasma

$$\mathrm{rot}\,\mathrm{rot}\,\vec{E} = \frac{\omega^2}{c^2}\left\{\underline{\underline{\varepsilon}}^{cold} \cdot \vec{E} - \frac{1}{2}\frac{\partial}{\partial x_j}\left\{\left[\frac{\partial^2 \underline{\underline{\varepsilon}}^{hot}}{\partial k_j \partial k_\ell}\right]_{k=0} \frac{\partial \vec{E}}{\partial x_\ell}\right\}\right\} \tag{23}$$

where $\underline{\underline{\varepsilon}}(\vec{r})$ is the local dielectric tensor.

From Eq. (23) one immediately obtains the power transport equation of the Eikonal approximation (Brambilla and Cardinali /14/). This makes it almost selfevident that for negligible dissipation Eq. (23) implies the conservation of the total power flux (electromagnetic plus kinetic):

$$\mathrm{div}\,\vec{S} = 0$$

$$\vec{S} = \frac{c}{8\pi} \mathrm{Re}\left(\vec{E}^* \times \vec{B}\right) - \frac{\omega}{16\pi} \vec{E}^* \cdot \frac{\partial \underline{\underline{\varepsilon}}^H}{\partial \vec{k}} \cdot \vec{E} \tag{24}$$

where $\underline{\underline{\varepsilon}}^H$ is the hermitean part of $\underline{\underline{\varepsilon}}$. We can, however, give a simple direct proof of this statement as follows:

The most general differential operator which can appear on the rhs of Eq. (23) is

$$\underline{\underline{\varepsilon}}^{op.}\vec{E} = \underline{\underline{\varepsilon}}^{cold}\cdot\vec{E} - a\,\underline{\underline{\varepsilon}}''(i,j)\,\partial_i\partial_j\vec{E}$$

$$-b\,\partial_i\left[\underline{\underline{\varepsilon}}''(i,j)\,\partial_j\vec{E}\right] - c\,\partial_i\partial_j\left[\underline{\underline{\varepsilon}}''(i,j)\,\vec{E}\right] \qquad (25)$$

where for brevity $\partial_i = \partial/\partial x_i$ and $\underline{\underline{\varepsilon}}''(i,j) = \frac{1}{2}(\partial^2\underline{\underline{\varepsilon}}/\partial k_i\partial k_j)_{k=0}$ a,b,c are real constants. In the first place, in order to recover the correct dispersion relation in the limit of a uniform plasma, we must choose

$$a + b + c = 1 \qquad (26a)$$

If moreover we require that (24) holds, we easily obtain

$$a = c \qquad (26b)$$

Eq. (23) satisfies both (26a) and (26b), with the particular choice

$$a = c = 0 \qquad (26c)$$

This further condition excludes terms arising only from spatial derivations of the dielectric tensor, and ensures the validity of the local dispersion relation in a cold plasma even in the presence of a weak non-uniformity in agreement with the above considerations (Eqs. 18-19).

While Eqs. (14)-(16) look formally the same as the Eikonal ansatz, it is important to realize that for the validity of the Eikonal approximation one has to impose the further condition

$$|k^{-2}\,\vec{\nabla}\cdot\vec{k}| \ll 1 \qquad (27)$$

In deriving Eq. (23), no use has been made of this condition, but only of the much less restrictive conditions (16). Eq. (23) can accordingly be expected to be valid under more general conditions than the Eikonal approximation, namely whenever the warm plasma approximation is locally valid. In particular, it will be adequate to describe the wave behaviour near a hybrid resonance, in the absence of strong superposed ion cyclotron damping. Of course, far from the resonance, where (27) is also satisfied, Eq. (23) asymptotically admits the Eikonal solutions, as it should.

Eq. (23) is identical with the equations derived by Swanson (5-6/ and Colestock and Kashuba /9/ when $k_z = 0$, i.e. for wavefronts parallel to the singular surface. It differs slightly from these equations when $k_z \neq 0$; the reason of the difference is not completely understood. In most cases this difference is irrelevant as far as the physical results are concerned.

In one important respect, however, Eq. (23) is less general than the equations obtained in reference /5-6/ and /9/. By considering only a slab geometry, in which $k_{\parallel}$ is strictly constant, the authors cited can retain the full dispersion along the static magnetic field. Hence the complex plasma dispersion function (Eq. 32, below) appears in the coefficients of their wave differential equation. Thus this equation also covers the case of strong ion cyclotron absorption superposed on the two-ion hybrid resonance.

It should be clear from the above discussion, however, that such a procedure cannot be generalized to the more general geometry of a tokamak. The necessity of the second of conditions (14), as an integral part of the weak dispersion condition, is beyond doubt. To make this point even more evident, we can remark that, in the presence of rotational transform, $k_{\parallel}$ is itself a differential operator, as shown by Eq. (8). Hence a finite-order differential equation for the wave field can only be valid where the large argument asymptotic expansion of the transcendent Z-function is acceptable. This again excludes the immediate vicinity of cyclotron harmonic resonances, as required by the second of conditions (14).

§ 4 - The wave equations in Tokamak geometry

In order to specialize Eq. (23) to ion cyclotron waves in the tokamak model suggested in section 2, we have to specify the appropriate dielectric tensor. There is obviously a premium in making from the start all approximations allowed by the physics of the present problem. Thus under the unrestrictive condition

$$\beta_{plasma} \equiv \frac{8\pi n T}{B_0^2} \gtrsim \frac{m_e}{m_i} \tag{28}$$

the slow cold-plasma wave (shear Alfvén wave) can be factorized out by letting $|\varepsilon_{zz}| \sim \omega_{pe}^2/\omega^2 \to \infty$. Moreover, we note that for ions near the cyclotron resonance, $\omega = \Omega_{ci}$, the Finite Larmor Radius (FLR) corrections to the dielectric tensor can be neglected compared with the cold plasma contribution, which are also resonant. This is the case for the minority species, since the two-ion hybrid resonance occurs close to the cyclotron resonance of the minority ions. Hence among the FLR terms we need to retain only the contributions from those ions which are close to the first harmonic resonance, $\omega = 2\,\Omega_{ci}$, if present. With these simplifications, the reduced 2x2 dielectric tensor becomes ($\underline{\underline{\varepsilon}}$ is written as customary, in a reference frame such that $\vec{k} = k_\perp \vec{e}_x + k_\parallel \vec{e}_z$):

$$\underline{\underline{\varepsilon}} = \begin{vmatrix} S - \sigma n_\perp^2 & -i\left(D - \sigma n_\perp^2\right) \\ +i\left(D - \sigma n_\perp^2\right) & S - \sigma n_\perp^2 \end{vmatrix} \tag{29}$$

Here S, D are the zero Larmor radius terms,

$$S = \frac{1}{2}(R+L) \qquad D = \frac{1}{2}(R-L)$$

$$R = 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} - \Sigma_j \frac{\omega_{pj}^2}{\omega^2}\left[\frac{\omega}{\omega+\Omega_j} - \frac{\omega}{\Omega_j}\right] \tag{30}$$

$$L = 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} - \Sigma_j \frac{\omega_{pj}^2}{\omega^2}\left[-x_{0j} Z(x_{1j}) + \frac{\omega}{\Omega_j}\right]$$

while

$$\sigma = \frac{1}{4} \Sigma_{(2)} \frac{\omega_{pj}^2}{\omega^2} \frac{v_{thj}^2}{c^2} \left(-x_{0j} Z(x_{2j})\right) \tag{31}$$

embodies the FLR corrections. In σ, the sum extends over all species of ions for which $\omega = 2\,\Omega_{ci}$ near the point under investigation. Z is the Plasma Dispersion function for real argument,

$$Z(x) = \frac{1}{\sqrt{\pi}} P \int \frac{e^{-u^2}\,du}{u-x} + i \frac{k_{/\!/}}{|k_{/\!/}|} \sqrt{\pi}\, e^{-x^2} \tag{32}$$

$$x_{nj} = \frac{\omega - n\Omega_j}{k_{/\!/} v_{thj}} \qquad v_{thj}^2 = \frac{2\kappa T_j}{m_j}$$

The reduced dielectric tensor (29) has the decisive advantage of possessing rotational symmetry around the direction of the local magnetic field (this is not the case for the complete dielectric tensor, for which $\varepsilon_{xx} \neq \varepsilon_{yy}$). On the other hand, the resulting dispersion relation

$$0 = \sigma n_\perp^4 + \left[(n_{/\!/}^2 - S) + 2\sigma(n_{/\!/}^2 - R)\right] n_\perp^2 + (n_{/\!/}^2 - R)(n_{/\!/}^2 - L) \tag{33}$$

reproduces the results of the exact dispersion relation over the whole IC frequency range with very good accuracy /16/. In particular, we recover Eq. (6) for the fast magnetosonic wave, provided

$$\left| \sigma \frac{n_{/\!/}^2 - R}{n_{/\!/}^2 - S} \right| \ll 1 \tag{34}$$

Singular layers occur when this condition is violated: either because $\omega = 2\,\Omega_{ci}$ for some species, in which case σ becomes very large, or because a two-ion hybrid resonance makes S very small. Both causes for singularity occur simultaneously in a D^+ plasma with H^+ minority, due to the "degeneracy" $\Omega_{cH} = 2\,\Omega_{cD}$.

Guided by the considerations of the previous sections, we now proceed to write the field equations near these singularities. To this end, it is also important to choose the appropriate field components as dependent variables. It turns out that only two choices avoid the appearence of spurious singularities in the differential equations. These are either the couple E_y, B_x, or the circularly polarized components $E_{\pm}$, where

$$E_{\pm} = \frac{1}{\sqrt{2}}\left(E_x \pm i E_y\right) \tag{35}$$

For the present problem, the latter choice is to be preferred, since then $\sigma n_{\perp}^2$ applies only to E_+. The resulting equations, measuring lengths in units of c/ω and neglecting small terms, are

$$\begin{aligned}
&\frac{d}{dX}\left(\sigma\frac{dE_+}{dX}\right) - \sigma n_z^2 E_+ + \left(\frac{d^2E_+}{dX^2} - n_z^2 E_+\right) - 2\left(n_{\parallel}^2 - L\right)E_+ \\
&\qquad - \left(\frac{d^2E_-}{dX^2} - 2n_z\frac{dE_-}{dX} + n_z^2 E_-\right) = 0 \\
&- \left(\frac{d^2E_+}{dX^2} + 2n_z\frac{dE_+}{dX} + n_z^2 E_+\right) + \\
&\qquad + \left(\frac{d^2E_-}{dX^2} - n_z^2 E_-\right) - 2\left(n_{\parallel}^2 - R\right)E_- = 0
\end{aligned} \tag{36}$$

Power flux conservation for these equations reads

$$\begin{aligned}
&\frac{dS_X}{dX} = 0 \\
&S_X = \frac{c}{8\pi}\,\mathrm{Im}\left\{E_y^*\frac{dE_y}{dX} + n_z E_+^* E_- + 2\,\mathrm{Re}(\sigma)\,E_+\frac{dE_+}{dX}\right\}
\end{aligned} \tag{37}$$

From Eqs. (36) we can obtain a fourth-order equation for E_+ alone if we note that R is never resonant, so that its space derivatives near a singular layer can be neglected. Then

$$\frac{d^3}{dX^3}\left(\sigma\frac{dE_+}{dX}\right) - \frac{d^2}{dX^2}\left\{\left[(n_{/\!/}^2 - S) + n^2\sigma\right]E_+\right\} +$$

$$-\left[n_z^2 + 2(n_{/\!/}^2 - R)\right]\frac{d}{dX}\left(\sigma\frac{dE_+}{dX}\right) + \tag{38}$$

$$+\left\{(n_{/\!/}^2 - R)(n_{/\!/}^2 - L) + n_z^2\left[(n_{/\!/}^2 - S) + \sigma\left(n_z^2 + 2(n_{/\!/}^2 - R)\right)\right]\right\}E_+ = 0$$

Now, near a cyclotron resonance $\omega = n\,\Omega_{ci}$, but excluding the layer of strong dispersion (condition (11) above), we have

$$-x_{0j}\,Z(x_{nj}) \cong \frac{\omega}{\omega - n\Omega_j} \cong 1 + \frac{R_T}{X} \tag{39}$$

where for simplicity we have shifted the origin of the X-axis at the cyclotron resonance, and R_T denotes its distance from the vertical axis. We can then write

$$\sigma \cong \frac{\sigma'}{X} \qquad S \cong S_0 - \frac{S'}{X} \tag{40}$$

where S_0 (and similarly L_0, $R_0 = R$) are the "asymptotic" values of S(L,R), or, more precisely, the values of these quantities without the resonant terms; while

$$S' = \frac{1}{2}\sum_{(1)} \frac{\omega_p^2}{\Omega_c^2}\left(\frac{\omega}{c}R_T\right)$$

$$\sigma' = \frac{1}{4}\sum_{(2)} \frac{\omega_p^2}{\Omega_c^2}\frac{v_{th}^2}{c}\left(\frac{\omega}{c}R_T\right) \tag{41}$$

(S' includes contributions from those ions for which $\omega = \Omega_{ci}$ at X = 0). Equation (38) then becomes

$$-\frac{d^3}{dX^3}\left(\frac{\varepsilon}{X}\frac{dE_+}{dX}\right)+\frac{d^2}{dX^2}\left[\left(1-\frac{\alpha}{X}\right)E_+\right]$$

$$-h q_X^2 \frac{d}{dX}\left(\frac{\varepsilon}{X}\frac{dE_+}{dX}\right)+q_X^2\left(1-\frac{\beta}{X}\right)E_+=0 \tag{42}$$

where

$$\varepsilon=\frac{\sigma'}{n_{\parallel}^2-S_0} \qquad \alpha=X_S-\varepsilon n_Z^2$$

$$\beta=\frac{1}{q_X^2}\left\{Q_X^2 X_L-n_Z^2\left(X_S+\varepsilon p_X^2\right)\right\} \tag{43}$$

$$h=p_X^2/q_X^2=-\frac{1}{q_X^2}\left(n_Z^2+2\left(n_{\parallel}^2-R_0\right)\right)$$

X_S, X_L are the cold-plasma two-ion hybrid resonance and cut-off, respectively:

$$X_S=-\frac{S'}{n_{\parallel}^2-S_0} \qquad X_L=-\frac{2S'}{n_{\parallel}^2-L_0} \tag{44}$$

while $Q_F^2=-(n_{\parallel}^2-R_0)(n_{\parallel}^2-L_0)/(n_{\parallel}^2-S_0)$ and $q_X^2=Q_F^2-n_Z^2$ are the "asymptotic" values of $n_{\perp}^2$ and n_X^2 for the fast wave, respectively, and finally p_X^2 is the value of $n_X^2=n_{\perp}^2-n_Z^2$ at the cyclotron resonance, X = 0. Note that, in order of magnitude,

$$\varepsilon=O\left(\frac{m_e}{m_i}\cdot\beta_{plasma}\cdot\frac{\omega}{c}R_T\right)$$

$$\alpha\sim\beta\sim X_L\sim X_S=O\left(f_{Min}\cdot\frac{\omega}{c}R_T\right) \tag{45}$$

$$Q_F^2\sim q_X^2\sim p_X^2=O\left(\frac{m_i}{m_e}\right)$$

where $f_{Min}=n_{Min}/n_e$ is the concentration of the minority ions.

§ 5 - Applications

i) Two-ion hybrid resonance, non degenerate case

If the cyclotron resonance of the minority species does not coincide with the first harmonics of the majority ions, FLR corrections to the dielectric tensor and to the dispersion relation are very small, of order β_{plasma}. An example of dispersion diagrams near such a resonance is shown in Fig. 1. At the hybrid resonance, for $k_{\parallel} = 0$, the fast wave actually couples with the acoustic wave, whose perpendicular index is however so large, $n_{\perp}^2 \gtrsim 10^5$, to be out of the scale of the figure. Even a very small damping therefore suffices to decouple the two waves altogether. Under these conditions, FLR corrections can be safely neglected, and the hybrid resonance can be treated as a true cold-plasma resonance.

Examples of this situation are a He_3^{++} of D^+ minority in a hydrogen plasma. Such a plasma composition can be of temporary interest in large tokamaks, when routine work with Deuterium as majority species might give rise to contamination problems. It presents the unfavourable property that the hybrid layer screens the cyclotron resonance of the minority ions for waves coming from the low magnetic field side.

In the following, suffixes 1,2 will denote the majority and minority species, respectively. The concentration of the latter will be treated as small, say $Z_2 n_2/n_e \lesssim 0.1$.

Equation (42) reduces in this case to

$$\frac{d^2}{dX^2}\left[\left(1-\frac{X_S}{X}\right)E_+\right] + q_X^2\left(1-\frac{\beta}{X}\right)E_+ \qquad (46)$$

$$\beta = X_L\left[1+\frac{n_Z^2}{2q_X^2}\left(\frac{n_{\parallel}^2-R_0}{n_{\parallel}^2-S_0}\right)\right]$$

Note that

$$\Delta = \beta - X_S = \frac{S'}{q_X^2}\left(\frac{n_{\parallel}^2 - R_0}{n_{\parallel}^2 - S_0}\right)^2 \tag{47}$$

is always positive, i.e., as it is well known, the cut-off is always to the low-field side of the resonance. Moreover, a finite n_Z^2 displaces the effective cut-off farther away from the resonance: the point $X = \beta$ is the reflection caustic corresponding to the vanishing of $n_X^2 = n_\perp^2 - n_Z^2$.

Equation (46) is solved by noting that the quantity

$$F = \left(1 - \frac{X_S}{X}\right) E_+ \tag{48}$$

satisfies the Whittaker equations

$$\frac{d^2F}{dX^2} + q_X^2\left(1 - \frac{\Delta}{X - X_S}\right) F = 0 \tag{49}$$

The general solution of Eq. (46) can therefore be written in terms of Whittaker functions /17/:

$$E_+ = \frac{X}{X - X_S}\left\{ C_1 W_{-i\kappa,\frac{1}{2}}\left(2iq_X(X - X_S)\right) + \right. \left. + C_2 W_{+i\kappa,\frac{1}{2}}\left(-2iq_X(X - X_S)\right)\right\} \tag{50}$$

Here $X = \eta_1/\pi$, where

$$\eta_1 = \frac{\pi}{2} q_X \Delta = \frac{\pi}{2}\left(\frac{n_{\parallel}^2 - R_0}{n_{\parallel}^2 - S_0}\right)^2 \frac{S'}{q_X} \tag{51}$$

is the effective optical thickness of the evanescent layer between the cut-off and the resonance. The constants C_1 and C_2 have to be determined from the radiation conditions far from the singularity. In this way the coefficients for power transmission T, reflection R, and absorption A are found:

a) wave incident from the low magnetic field side:

$$T = e^{-2\eta_1} \qquad R = \left(1 - e^{-2\eta_1}\right)^2$$
$$A = e^{-2\eta_1}\left(1 - e^{-2\eta_1}\right) \tag{52}$$

b) wave incident from the high magnetic field side:

$$T = e^{-2\eta_1} \qquad R = 0 \qquad A = 1 - e^{-2\eta_1} \tag{53}$$

As well known, situation b), in which the wave encounters the resonance first, is more favorable for absorption than situation a), in which the wave encounters the cut-off first. Note also that η_1 increases with increasing n_Z^2: the decrease of the effective index $q_X = (Q_F^2 - n_Z^2)^{1/2}$ is more than offset by the enlargement of the evanescence layer due to the displacement of the reflection caustic when n_Z^2 increases. This enhances absorption in case b), but (if $\eta_1 \gtrsim \frac{1}{2}\ln 2 = 0.3466\ldots$) decreases it in case a). Finally, since η_1 is proportional to the toroidal radius of the plasma, single transit absorption of waves excited from the low-field side will be relatively inefficient in large devices.

The results (52)-(53) hold if the wave resonance $X = X_S$ lies outside the Doppler-broadened cyclotron resonance of the minority ions, $|X_S| \gtrsim 2|n_{\parallel}|\, v_{th_2}(\omega R_T/c)$ or

$$\frac{n_2}{n_e} \gtrsim 4|n_{/\!/}|\frac{v_{th_2}}{c} \tag{54}$$

Under this condition, the power absorbed by the minority ions is easily estimated by integrating the power transport equation through the cyclotron resonance layer. The result is

$$\frac{\Delta S_X}{S_X} \cong \pi \frac{\omega}{c} R_T \frac{(n_{/\!/}^2 - R_0)^2}{|P_X|(\omega_{p2}^2/\omega^2)} n_{/\!/}^2 \frac{v_{th_2}^2}{c^2} \tag{55}$$

where we have used the identity

$$\int_{-\infty}^{+\infty} \frac{e^{-\xi^2} d\xi}{|Z(\xi)|^2} = \frac{\sqrt{\pi}}{2} \tag{56}$$

In Eq. (55) S_X is the power flux of the incident wave if this wave comes from the high magnetic field side, of the transmitted wave in the opposite case (this statement refers to the case $Z_2/A_2 < Z_1/A_1$; it must be reversed if this inequality is reversed). Finally, p_X is defined in Eq. (43) as the WKB value of n_X at the cyclotron resonance.

Equation (55) predicts that cyclotron absorption by the minority ions is weak, and inversely proportional to their concentration. Screening of the left-hand component of the electric field, E_+, by the minority ions is very efficient, being a zero Larmor radius effect. Equation (48), together with the fact that the solutions of Eq. (49) are regular at X = 0, shows that E_+ vanishes at the resonance in the limit $n_{/\!/} \sqrt{T_2} = 0$; more generally, $|E_+|^2$ ./. $(n_e/n_2)^2 n_{/\!/}^2 T_2$. Since the antihermitean part of the dielectric tensor is proportional to n_2/n_e, the dependence of $\Delta S_X/S_X$ on the minority parameters is easily understood. Of course, if n_2/n_e is so small that condition (54) is violated, these arguments do not apply, and $\Delta S_X/S_X$ ./. n_2/n_e

when $n_2 \to 0$. It follows that ion cyclotron heating of the minority ions is most efficient just near the transition where Eq. (54) is only marginally satisfied. Since most antenna excite a broad n_φ--spectrum, and refraction further broadens the spectrum of $n_{\parallel}$, however, this condition has only an indicative value. For waves incident from the low magnetic field side, the minority concentration should rather be chosen to satisfy the condition $\eta_1 \simeq \frac{1}{2}\ln 2$, which maximizes the power absorption at the hybrid resonance. This restricts n_2/n_e to rather low values in large devices.

ii) First harmonic heating of a single species plasma

Figure 2 shows examples of dispersion diagrams near the first harmonic of the cyclotron resonance in a pure Deuterium plasma, $\omega = 2\Omega_D$. As first noted by Weynants /18/, for $k_{\parallel} = 0$ there are two confluences between the fast and Bernstein waves, separated by a region in which the two roots of the dispersion relation are complex conjugate. This evanescence layer merges progressively with the layer of cyclotron harmonic damping when $|k_{\parallel}|$ increases.

In this case we have been able to find an exact solution of the wave equations only in the limit $n_z = 0$ (wavefronts parallel to the resonance layer). The case $n_z \neq 0$ will be treated approximately below.

With $n_z = 0$, in the present situation Eq. (42) becomes

$$-\frac{d^3}{dX^3}\left(\frac{\varepsilon}{X}\frac{dE_+}{dX}\right) - hq_X^2\frac{d}{dX}\left(\frac{\varepsilon}{X}\frac{dE_+}{dX}\right) + \frac{d^2E_+}{dX^2} + q_X^2 E_+ = 0 \tag{57}$$

where $q_X^2 = Q_F^2$ and $h = 2(n_{\parallel}^2-S)/(n_{\parallel}^2-L)$. The position $F = X^{-1}(dE_+/dX)$ transforms this equation into a system of two second-order equations, which is in a form appropriate for solution with the Laplace integral method. In this way we obtain

$$E_+(X) = \int_C \frac{p(p+hq_X^2)}{p^2+q_X^2} \cdot \exp\left\{pX + \varepsilon q_X^3\left[(1-h)\left(\frac{p}{q_X} - \operatorname{atan}\frac{p}{q_X}\right) - \frac{1}{3}\frac{p^3}{q_X^3}\right]\right\} dp \tag{58}$$

Four independent contours producing solutions with a well-definite asymptotic behaviour for $X \to +\infty$ and $X \to -\infty$ are shown in Figs. 3a and 3b, respectively. The asymptotic behaviour of the solutions representing Bernstein waves is easily evaluated with the saddle-point method. It is important to note that on the propagating side ($X < 0$) these waves are found to behave as

$$E_+^B \sim |X|^{1/4} \exp\left(\pm i \frac{|X|^{3/2}}{\sqrt{\varepsilon}}\right) \tag{59}$$

Thus Eq. (57) predicts that Bernstein waves can carry a finite amount of energy to infinity, a result which makes it possible to conserve the total power flux.

The saddle-point method cannot be applied to the solutions representing fast waves, because the corresponding saddle-points are too close to the essential singularities $p = \pm iq_X$. However, the asymptotic behaviour of these solutions can be evaluated using a method related to the Hankel representation of the Γ-function /17/.

The knowledge of the asymptotic behaviours allows the derivation of the connection formulae. Omitting the details of the calculations, we list the results for the two cases of interest:

a) Fast wave incident from the low magnetic field side: the power transmission coefficient on the fast wave, T_F, on the Bernstein wave, T_B, and the reflection coefficient on the fast wave, R_F, are respectively

$$T_F = e^{-2\eta_2} \qquad T_B = e^{-2\eta_2}\left(1 - e^{-2\eta_2}\right)$$

$$R_F = \left(1 - e^{-2\eta_2}\right)^2 \tag{60}$$

b) Fast wave incident from the high magnetic field side: the power transmission coefficient on the fast wave, T_F, the reflection coefficient on the fast wave, R_F, and the reflection coefficient on the Bernstein wave (or conversion factor), R_B, are respectively

$$T_F = e^{-2\eta_2} \qquad R_F = 0 \qquad R_B = 1 - e^{-2\eta_2} \tag{61}$$

Here

$$\eta_2 = \frac{\pi}{2}(1-h)\,\varepsilon q_x^3 \qquad 1-h = -\frac{n_{\parallel}^2 - R}{n_{\parallel}^2 - L} > 0 \tag{62}$$

is the optical thickness of the evanescence layer.

Equation (57) coincides with the equation used for the same problem by Swanson /6/ in the same limit $n_z = 0$, and of course under the additional condition of sufficiently small Doppler broadening of the cyclotron harmonic resonance, Eq. (15). In the present case, this condition can be written

$$|n_{\parallel}|\frac{v_{thD}}{c} \lesssim \beta_{pl} \tag{63}$$

The results (60) and (61), obtained from the exact solution (58), are similar, but simpler, to those obtained iteratively by Swanson. Remarkably, they are formally identical to those of a cold-plasms Budden tunneling problem with the same effective optical thickness, except that T_B and R_B, the fraction of power carried away by the Bernstein wave, would appear as absorbed at the resonance itself in the cold problem.

iii) The general case

In the general case (two-ion hybrid resonance in a D^+ plasma with H^+ minority; first harmonic resonance in a single species plasma at oblique incidence, $R_Z \neq 0$), we have not been able to find analytic solutions for Eq. (42). We did therefore turn to the Green function technique familiar from the work of Swanson /3/. The above remark about the form of Eqs. (60) and (61) suggests however to apply this technique starting with the following very simple form of the wave equation:

$$\frac{d^2E_+}{dX^2} + q_X^2 E_+ = - H E_+ \qquad (64)$$

The fourth-order differential operator H can be immediately identified by comparison with Eq. (42).

Equation (64) can be interpreted as describing the scattering of fast magnetosonic waves by the two-ion hybrid singularity as a whole, and the solution obtained by the Green function approach can be loosely compared to the lowest-order Born approximation. We can therefore expect it to be a good approximation if the effective optical thickness of the singularity is small.

Due to the simple nature of the differential operator on the rhs and the assumption of small optical thickness, applying the Green function method to Eq. (64) is straightforward. Details of the procedure are given in Ref. /19/. The results are:

a) fast wave incident from the low magnetic field side:

$$
\begin{aligned}
T_F &= e^{-2(\eta_1+\eta_2)} \\
R_F &= \left(e^{-2\eta_1} - e^{-2\eta_2}\right)^2 \\
T_B &= 1 + e^{-2(\eta_1+\eta_2)} - e^{-4\eta_1} - e^{-4\eta_2}
\end{aligned}
\tag{65}
$$

b) fast wave incident from the high magnetic field side:

$$
\begin{aligned}
T_F &= e^{-2(\eta_1+\eta_2)} \\
R_F &= 0 \\
R_B &= 1 - e^{-2(\eta_1+\eta_2)}
\end{aligned}
\tag{66}
$$

T,R have the same meaning as above, while η_1 and η_2 generalize Eq. (51) and (62) to the general case:

$$
\begin{aligned}
\eta_1 &= \frac{\pi}{2}\left(\frac{n_{/\!/}^2 - R_0}{n_{/\!/}^2 - S_0}\right)^2 \frac{1}{|q_x|}\left(S' + \sigma' n_z^2\right) \\
\eta_2 &= \frac{\pi}{2}\left(\frac{n_{/\!/}^2 - R_0}{n_{/\!/}^2 - S_0}\right)^2 \sigma' q_x
\end{aligned}
\tag{67}
$$

Equations (65) and (66) have been obtained under the explicit condition

$$
\eta_1 \ll 1, \quad \eta_2 \ll 1 \tag{68}
$$

We note, however, that they reduce to the correct expressions in both cases in which the exact solution is known, independently from the magnitude of η_1 and η_2: namely, in the cold limit ($\sigma' = 0$, Eqs. (52)-(53)), and in a single species plasma near the first cyclotron harmonic for perpendicular incidence ($S' = 0$ and $n_Z^2 = 0$, Eqs. (60)-(61)). It is therefore tempting to suggest that they represent an acceptable approximation in all cases, as implicitly assumed in the literature for results derived with the Green-function method /3, 6/.

Equation (65) can be loosely interpreted by saying that the two-ion hybrid cut-off (more precisely, the reflection caustic $n_X^2 = n_\perp^2 - n_Z^2 = 0$) and the confluence with the Bernstein wave act as two independent scattering centers. The expression for R_F is particularly interesting, since it shows that for $\eta_1 = \eta_2$ the two reflected waves cancel each other by destructive interference. For a single species plasma near $\omega = 2\Omega_c$ ($S' = 0$) this occurs when $n_Z^2 = Q_X^2$, i.e. when the wavefronts are inclined by 45° with respect to the resonance layer. For the same condition, the power transmitted to Bernstein waves is maximized. Thus for oblique incidence, waves launched from the low-field side can be considerably more efficiently coupled into Bernstein waves than for perpendicular incidence. This possibility is, however, of interest only for a single species plasma, since in the case of minority heating the inequality $\eta_1 \gg \eta_2$ usually holds.

We should also mention that in the case of a single species plasma ($S' = 0$), the equation of Swanson /5, 6/ and Colestock and Kashuba /9/ can be integrated exactly also for $n_Z^2 \neq 0$,
and gives, instead of (65), the reflection coefficient $R_F =$ $= (1 - \exp(-2(\eta_1+\eta_2))^2$ (with the corresponding modification to T_B). In the presence of minority ions, however, the form (65) is obtained again, except for an irrelevant rearrangement of terms in the definitions of η_1 and η_2.

It remains to discuss the conditions for the validity of Eqs. (65)-(66) and their particular case (60),(61), and to estimate cyclotron absorption. To this end, one must distinguish two regimes, depending on whether the wave behaviour near the singularity is dominated by the confluence with the Bernstein wave or by the minority ions. The latter situation occurs as soon as

$$\frac{n_2}{n_e} \gtrsim \beta_{plasma} \approx \frac{\omega_{p1}^2}{\Omega_{c1}^2} \frac{v_{th_1}^2}{c^2} \qquad (69)$$

If this condition is not satisfied, the situation is essentially the same as in a single species plasma. Since the confluence with the Bernstein wave occurs quite close to the cyclotron harmonic resonance $\omega = 2\Omega_c$, the condition of small Doppler broadening is correspondingly severe:

$$|n_{/\!/}| \frac{v_{th_1}}{c} \lesssim \frac{1}{4} \beta_{plasma} \qquad (70)$$

In the case of minority heating near $\omega = \Omega_1 = 2\Omega_2$, on the other hand ($H^+$ minority in D^+) condition (69) is always satisfied in practice by a comfortable margin. As the minority concentration n_2/n_e increases, the hybrid layer moves away from the cyclotron resonance, as shown in the examples of Fig. 4. The condition that there is no overlapping becomes

$$\frac{n_2}{n_e} \gtrsim |n_{/\!/}| \frac{v_{th_2}}{c} \approx 2 \cdot 10^{-3} |n_{/\!/}| \sqrt{T_{2\,(keV)}} \qquad (71)$$

which is easily satisfied over most of the $n_{/\!/}$-spectrum from usual antennas. Under this condition, screening of E_+ by the minority ions is just as efficient as in the cold plasma limit discussed above, and the resulting cyclotron absorption is

$$\frac{\Delta S_X}{S_X} \approx \pi \frac{(n_{/\!/}^2 - R)^2}{|P_X|(\omega_{p2}^2/\omega^2)} \left(\frac{\omega}{c} R_T\right) \frac{n_{/\!/}^2 v_{th2}^2}{c^2} \cdot \tag{72}$$

$$\cdot \left\{ \frac{\omega_{p2}^2}{\omega^2} + \frac{\omega_{p1}^2}{\Omega_{c1}^2} \frac{v_{th1}^2}{c^2} |n_{/\!/}^2 - R| \cdot I\left(\sqrt{\frac{T_1 A_2}{T_2 A_1}}\right) \right\}$$

The first term, representing cyclotron absorption by the minority species, is identical with Eq. (55). The second term represents the (weaker) first harmonic absorption by the majority ions. I(a) is defined as

$$I(a) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\xi^2}}{|Z(a\xi)|^2} d\xi \tag{73}$$

$I(a) \rightarrow 2/\pi$ for $a \rightarrow 0$ (main species much colder than the minority), and increases as a^2 when $a \rightarrow \infty$ (opposite situation). For equal temperatures, $a = 1/2$, $I(a) \simeq 0.7$.

In Eq. (72), S_X represents the incident power flux if the waves are launched from the l.m.f.side. This feature makes minority heating of H^+ in a D^+ plasma much more efficient than the schemes considered in subsection i) above, with $Z_2/A_2 < Z_1/A_1$. In particular, there is no need to restrict the H^+ concentration to the very low values which maximize coupling to Bernstein waves at the two-ion hybrid layer. In spite of the inverse dependence from n_{H^+}/n_e, $\Delta S_X/S_X$ remains sufficiently large to exclude any cavity mode excitation (except at most the mode with $n_\varphi = 0$) up to much higher minority concentrations.

Acknowledgement

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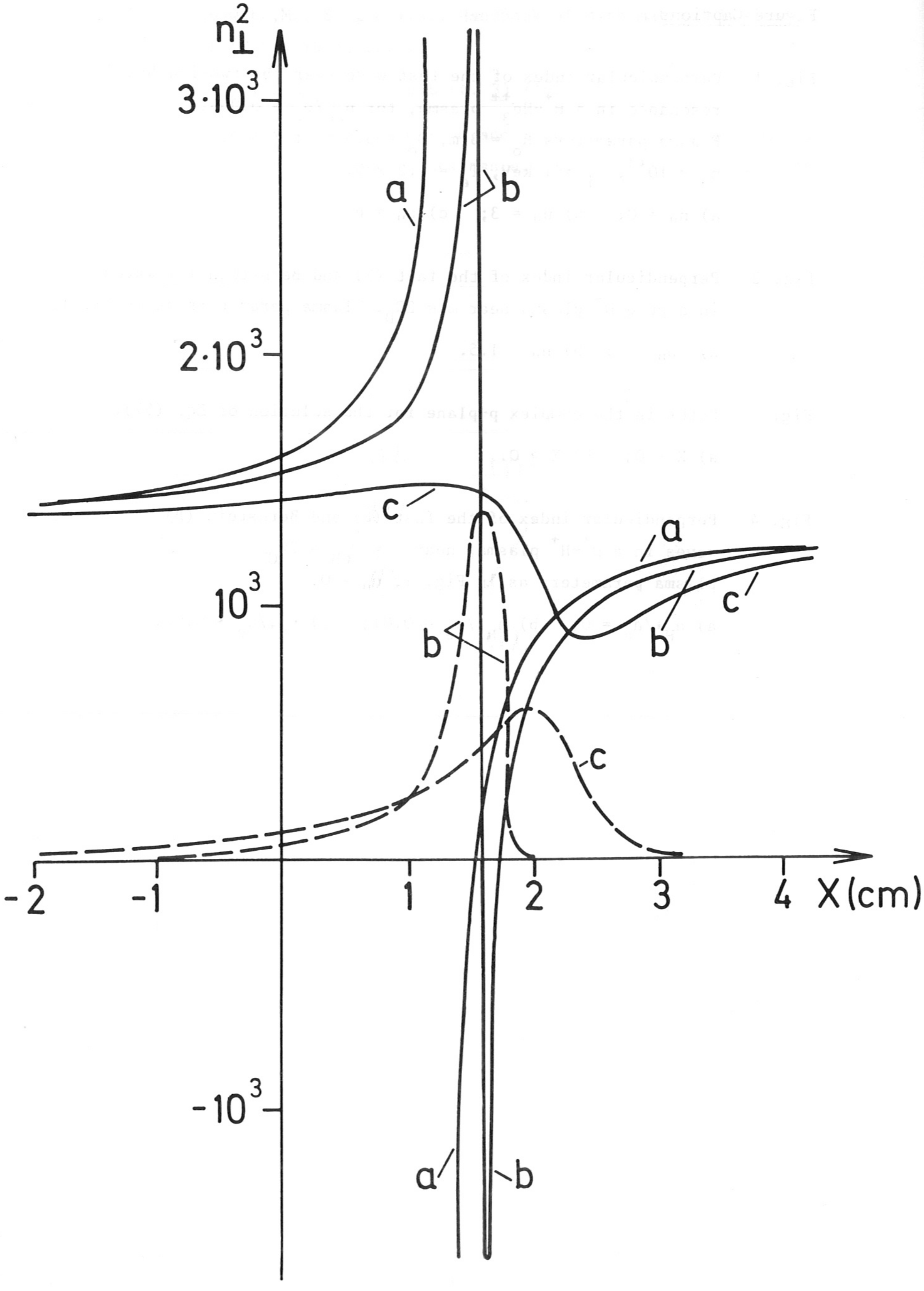
Figure Captions

Fig. 1 Perpendicular index of the fast wave near the two-ion hybrid resonance in a $H^{+}-He_3^{++}$ plasma, for $n_{He}/n_e = 0.005$.
Plasma parameters $R_o = 3$ m, $B_o = 3.5$ T at $X = 0$, $n_e = 10^{13}$, $T_i = 1$ keV, $T_e = 1.2$ keV.
a) $n_{\parallel} = 0$; b) $n_{\parallel} = 3$; c) $n_{\parallel} = 6$.

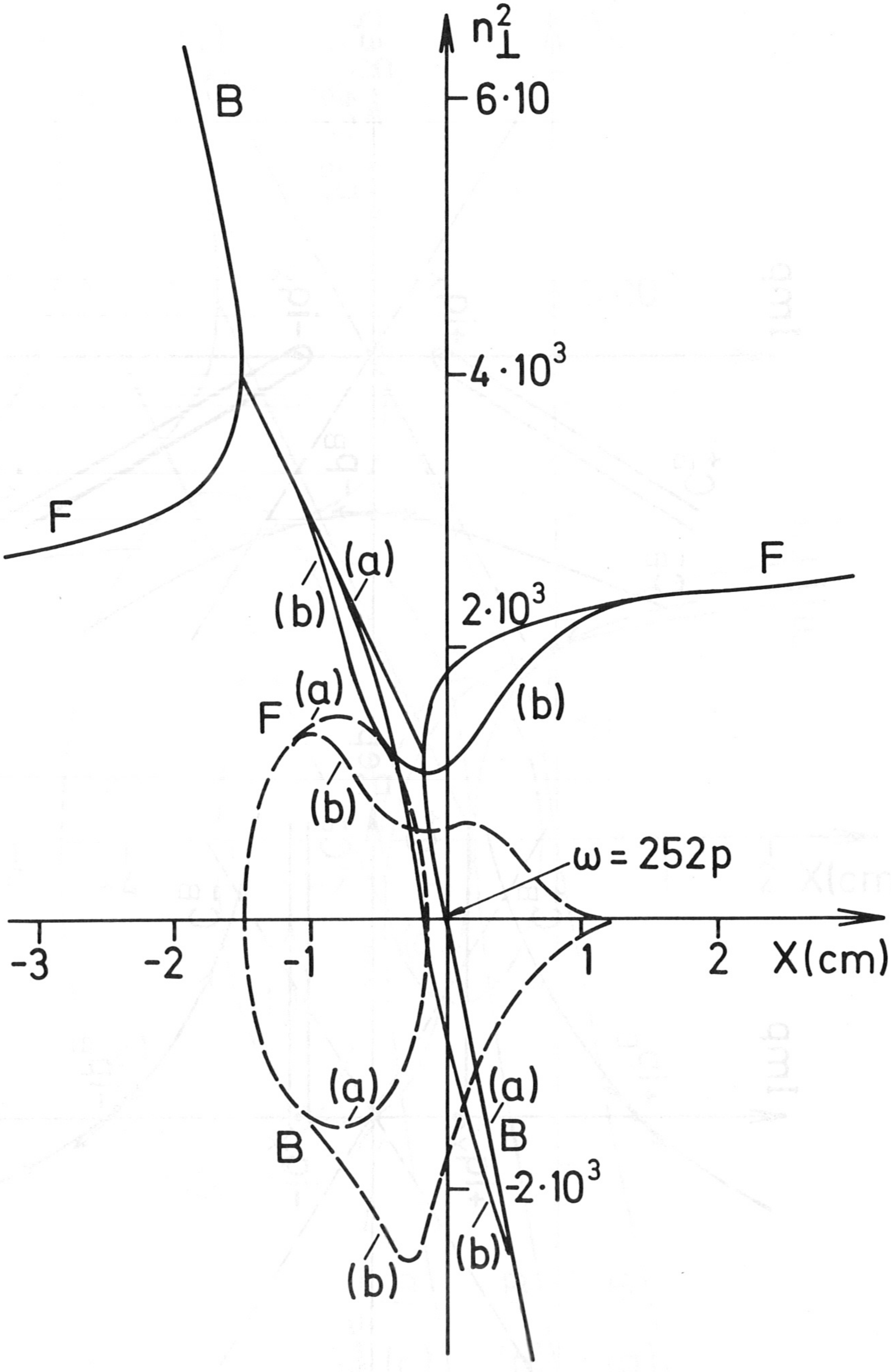
Fig. 2 Perpendicular index of the fast (F) and Bernstein (B) waves in a pure D^{+} plasma near $\omega = 2\Omega_D$. Plasma parameters as in Fig.1.
a) $n_{\parallel} = 0$; b) $n_{\parallel} = 1.5$.

Fig. 3 Paths in the complex p-plane for the solution of Eq. (57).
a) $X < 0$; b) $X > 0$.

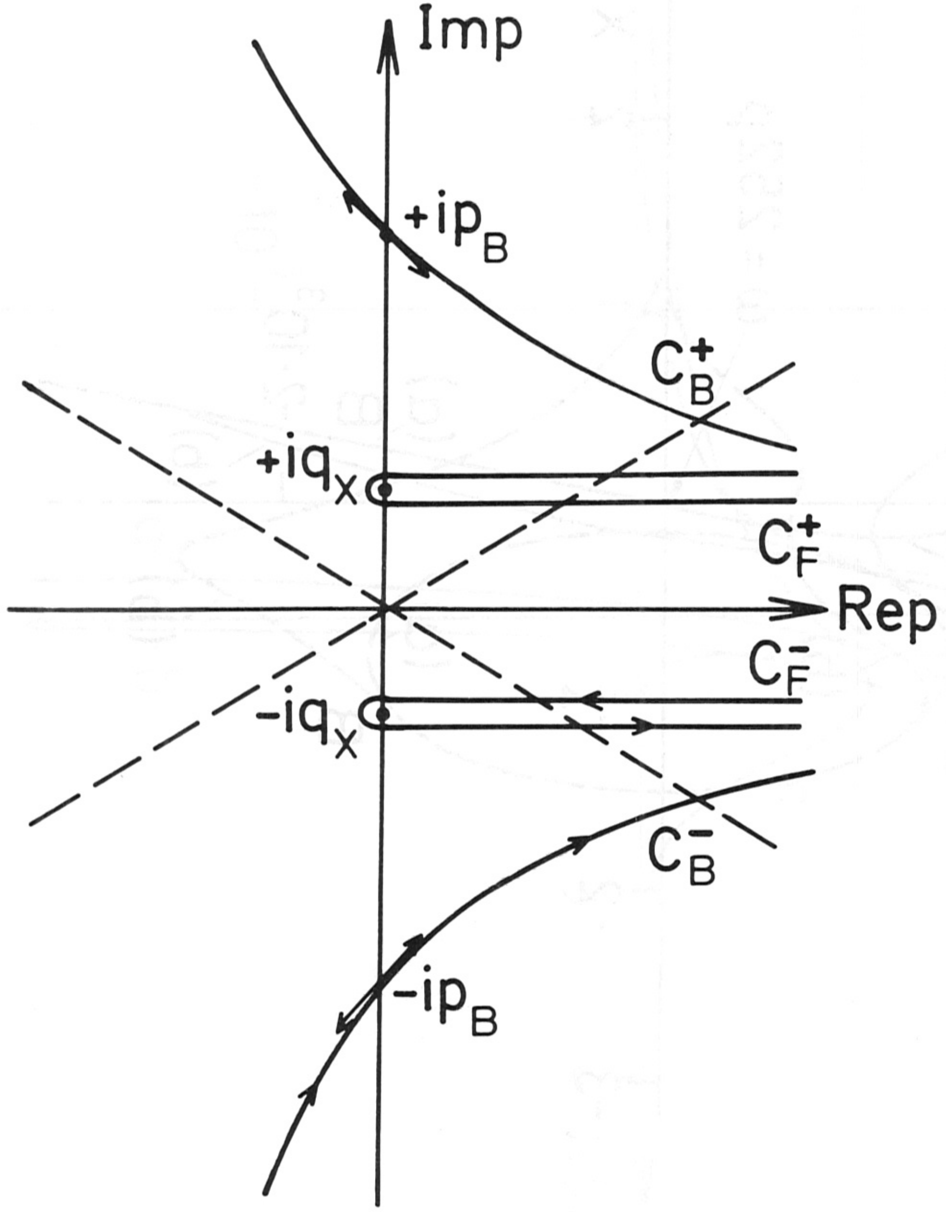
Fig. 4 Perpendicular index of the fast (F) and Bernstein (B) waves in a $D^{+}-H^{+}$ plasma, near $\omega = \Omega_{CH} = 2\Omega_{CD}$; plasma parameters as in Fig. 1, $n_{\parallel} = 0$.
a) $n_{H^+}/n_e = 0$; b) $n_{H^+}/n_e = 0.01$; c) $n_{H^+}/n_e = 0.02$.

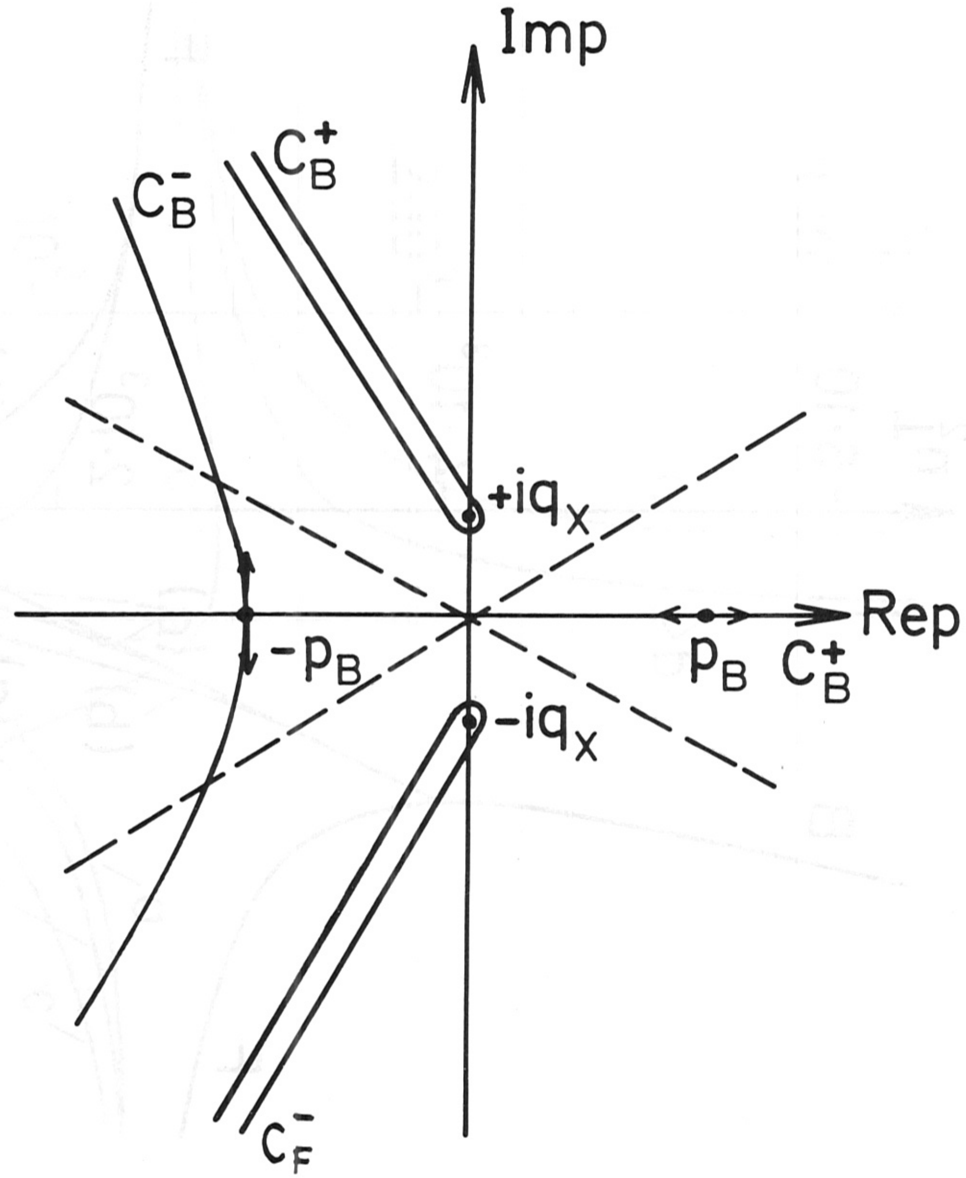


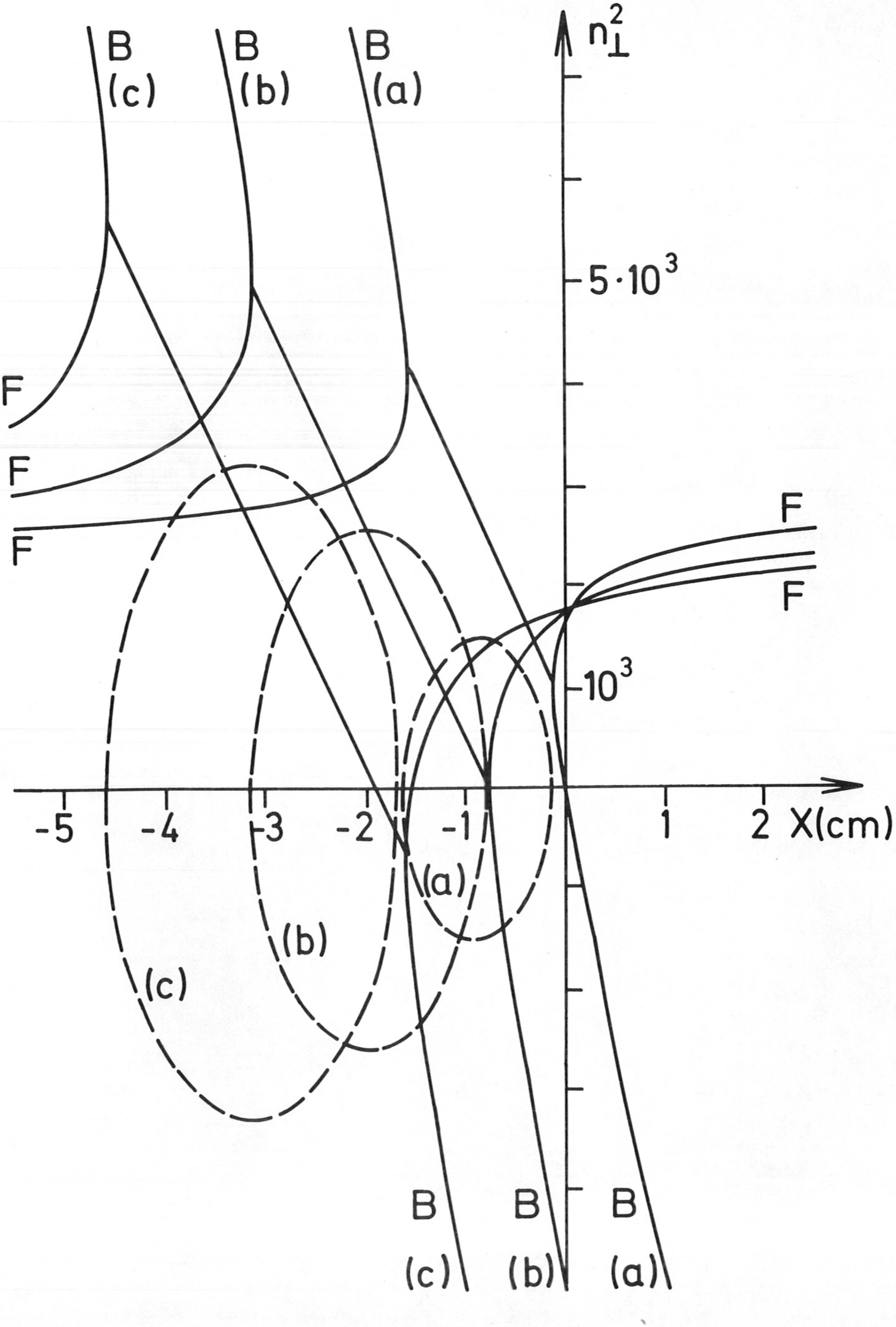
$n_\perp^2$
$3\cdot10^3$
$2\cdot10^3$
10^3
-10^3
a
b
c
a
c
b
b
c
a
b
-2
-1
1
2
3
4
X(cm)



$n_\perp^2$
$6 \cdot 10$
$4 \cdot 10^3$
$2 \cdot 10^3$
$-2 \cdot 10^3$
B
F
F
(a)
(b)
(b)
F
(a)
(b)
$\omega = 252p$
-3
-2
-1
1
2
X(cm)
(a)
(a)
B
B
(b)
(b)


Imp
+ip_B
C_B^+
+iq_X
C_F^+
Rep
C_F^-
-iq_X
C_B^-
-ip_B


Imp
C_B^-
C_B^+
+iq_X
Rep
-p_B
p_B
C_B^+
-iq_X
C_F^-



B
(c)
B
(b)
B
(a)
$n_\perp^2$
$5 \cdot 10^3$
10^3
F
F
F
F
F
-5
-4
-3
-2
-1
1
2
X(cm)
(a)
(b)
(c)
B
(c)
B
(b)
B
(a)