Mode-Conversion and Tunneling at the Two-Ion Hybrid and Ion Cyclotron resonances

Maurizio Ottaviani

IPP 4/211 July 1983

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A b s t r a c t

This report is devoted to the propagation of electromagnetic waves in the I.C. frequency range and is intended to complete a work by M. Brambilla and the present authors/1/.

In addition to a derivation of the coupling equations, we present an evaluation of the coupling coefficients in many of the current I.C. heating scenarios.

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CONTENT

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1 Introduction

It is known that coupling of two waves propagating in inhomogenous plasmas can occur in a well defined region of space where the values of the plasma parameters let the wave vectors to become equal.

This fact may be useful for R.F. heating purposes, when a relatively undamped wave can propagate until the coupling region, and is converted there into another wave, which can eventually be easily absorbed. Typical examples are the IC heating schemes, in which the fast wave launched by the antenna couples with an ion Bernstein wave; this happens in proximity of the ion cyclotron harmonics or in proximity of the two-ion hybrid "resonance", according to the actual scenario.

This paper is written with the purpose to complete the work on the subject by Brambilla and Ottaviani |1| , by including the calculations there omitted.

The problem was initially approached |2-6| by starting directly from the dispersion relation and interpreting the wave vector as a differential operator: $\vec{k} \rightarrow -i\vec{\nabla}$. This method, when applied to hot plasmas, leads however to ambiguities concerning the position of the derivatives. The resulting equations usually do not satisfy energy conservation, or eventually conserve quantities which are not clearly related to the physical ones |7| .

Recently Swanson |8| and Colestock-Kashuba |9| derived the electric field equations directly from the Vlasov equation. The approach followed in Ref. |1| is however simpler than the one of Ref. |8-9| , and leads to similar results. The difference is discussed in Ref. |1| .

Sec 2 of the present work partially reproduces the derivation of the field equations of Ref |1| , while their explicit solution in slab geometry is obtained in Sec 3 in the small Doppler broadening limit.

The author wishes to thank Dr. M. Brambilla for addresing him to the subject of the present paper and for continuous advice during this work.

2 Electric field equations

The electric field equations for a collisionless hot plasma may be in principle obtained by solving the coupled set of the Vlasov equation and the Maxwell equations.

The resulting equation will be in any case of the form:

$$1) \qquad \mathrm{rot}\,\mathrm{rot}\,\vec{E} - \underline{\underline{\varepsilon}}_{op}\,\vec{E} = 0$$

where $\underline{\underline{\varepsilon}}_{op}$ must reduce to the usual collisionless hot plasma dielectric tensor $\underline{\underline{\varepsilon}}$ in absence of inhomogenities; the lengths are measured in units c/ω .

Restricting ourselves to the warm plasma approximation, the homogeneous $\underline{\underline{\varepsilon}}$ becomes:

$$2) \qquad \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}_{\circ} + \underline{\underline{\sigma}}(i,j)\, n_i n_j$$

where $\underline{\underline{\varepsilon}}_{o}$ is the cold plasma dielectric tensor and:

$$3) \qquad \underline{\underline{\sigma}}(i,j) = \frac{1}{2}\left(\frac{\partial^2 \underline{\underline{\varepsilon}}}{\partial n_i\, \partial n_j}\right)_{n_i = 0}$$

In order to find the inhomogeneous $\underline{\underline{\varepsilon}}_{op}$ we must allow $\underline{\underline{\varepsilon}}_o$ and $\underline{\underline{\sigma}}(i,j)$ to vary with $\vec{x}$ according to the implicit dependence through the plasma parameters, and we must substitute n_i with $-i\partial_i$.

This procedure leaves some uncertainties in the order of the derivatives, so the most general, warm plasma $\underline{\underline{\varepsilon}}_{op}$ will be of the form:

$$4)\quad \underline{\underline{\varepsilon}}_{op} = \underline{\underline{\varepsilon}}_o - a\,\underline{\underline{\sigma}}(i,j)\,\partial_i\partial_j - b\,\partial_i\,\underline{\underline{\sigma}}(i,j)\,\partial_j - c\,\partial_i\partial_j\,\underline{\underline{\sigma}}(i,j)$$

where a, b, c are real and

$$5)\quad a + b + c = 1$$

in order to obtain Eq. (2) in the homogeneous case. Further progress is gained by imposing energy conservation. In absence of absorption, when $\underline{\underline{\varepsilon}}_o$ and $\underline{\underline{\sigma}}(i,j)$ are hermitian, we must have

$$6)\quad \mathrm{div}\,(\vec{P} + \vec{T}) = 0$$

where $\vec{P}$ is the Poynting vector:

$$7)\quad \vec{P} = \frac{c}{8\pi}\,\mathrm{Re}\,(\vec{E}^* \times \vec{B}) = \frac{c}{8\pi}\,\mathrm{Im}\,(\vec{E}^* \times \mathrm{rot}\,\vec{E})$$

and $\vec{T}$ is the Kinetic flux.
Taking the divergence of (7) and using (1):

$$8)\quad \mathrm{div}\,\vec{P} = -\frac{c}{8\pi}\,\mathrm{Im}\,(\vec{E}^+\,\underline{\underline{\varepsilon}}_{op}\,\vec{E})$$

By Eq. (6), the r.h.s. of (8) must still be a divergence. It is easily shown that this is obtained only by imposing a = c in Eq.(4). Eq.(6) is thus satisfied by a Kinetic flux of the form:

$$9) \qquad \vec{T} = -\frac{c}{8\pi} \, Im \left[\vec{E}^{+} \underline{\underline{\sigma}}^{H}(i,j) \, \partial_j \vec{E} \right]$$

where only the hermitian part of $\underline{\underline{\sigma}}(i,j)$ must be considered. Eq.(9) coincides with the homogeneous result given by Stix |10| .

The above procedure leaves undetermined only c , which however we set equal to zero in order to obtain the simplest equations.
The dielectric tensor becomes:

$$10) \qquad \underline{\underline{\varepsilon}}_{op} \vec{E} = \underline{\underline{\varepsilon}}_{o} \vec{E} - \partial_i \left[\underline{\underline{\sigma}}(i,j) \, \partial_j \vec{E} \right]$$

Let's now specialize Eq (10) to slab geometry, introducing a system of coordinates (x, y, z) with the magnetic field directed along z and all the gradient along x . It is appropriate to perform the expansion (2) only in the perpendicular components of the wave vector; moreover, we set $E_{\parallel} = 0$ so that $\vec{E} = (E_x, E_y)$ and $\underline{\underline{\varepsilon}}$ reduces to a 2x2 matrix ; finally, among the warm plasma terms, we retain only those relative to the first harmonic resonance, since warm terms at the fundamental frequency are negligible with respect to the cold ones. The result is:

$$11) \qquad \underline{\underline{\varepsilon}} = \begin{vmatrix} S & -iD \\ iD & S \end{vmatrix} - \sigma \begin{vmatrix} 1 & i \\ -i & 1 \end{vmatrix} (n_x^2 + n_y^2)$$

where |II| :

$$12) \qquad S = \frac{1}{2}(R+L)$$

$$D = \frac{1}{2}(R-L)$$

$$R = 1 - \Sigma_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \left[-x_{0,\alpha} \, Z\left(x_{-1,\alpha}\right) \right]$$

$$L = 1 - \Sigma_\alpha \frac{\omega^2_{p\alpha}}{\omega^2}\left[-x_{0,\alpha} Z(x_{1,\alpha})\right]$$

$$\sigma = \Sigma_\alpha^{(2)} \frac{\omega^2_{p\alpha}}{\Omega^2_\alpha} \frac{v^2_{th\alpha}}{c^2}\left[-x_{0,\alpha} Z(x_{2,\alpha})\right]$$

$$x_{m,\alpha} = \frac{\omega - m\Omega_\alpha}{k_{\parallel} v_{th\alpha}}$$

Here α refers to both ions and electrons, and $\Sigma_\alpha^{(m)}$ denotes a sum only over the species that exhibit a resonance at $\omega = m\Omega_\alpha$ in the region of interest .

Eqs.(1) - (10) and (11) are better written in the variables E_+ and E_- defined as:

$$13) \quad \begin{vmatrix} E_+ \\ E_- \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} \cdot \begin{vmatrix} E_x \\ E_y \end{vmatrix}$$

The result is the system:

$$14) \quad \begin{cases} \left[\frac{1}{2}\left(\frac{d^2}{dx^2} - m_y^2\right) - (m_\parallel^2 - L) + 2\left(\frac{d}{dx}\sigma\frac{d}{dx} - \sigma m_y^2\right)\right]E_+ - \frac{1}{2}\left(\frac{d}{dx} - m_y\right)^2 E_- = 0 \\ -\frac{1}{2}\left(\frac{d}{dx} + m_y\right)^2 E_+ + \left[\frac{1}{2}\left(\frac{d^2}{dx^2} - m_y^2\right) - (m_\parallel^2 - R)\right]E_- = 0 \end{cases}$$

The advantage of using this variables is that the resonant terms apply only to E_+ . Moreover, since R does not contain resonant terms, its derivatives can be neglected. This permits, by multiplying to the left the first of Eqs.(14) for $\left[\frac{1}{2}\left(\frac{d^2}{dx^2} - m_y^2\right) - (m_\parallel^2 - R)\right]$

and the second one for $\frac{1}{2}\left(\frac{d}{dx} - m_y\right)^2$ and summing, to obtain

a single equation for E_+ , which describes the coupling between the fast wave and the (ion) Bernstein wave:

$$15)\quad \frac{d^3}{dx^3}\left(\sigma \frac{dE_+}{dx}\right) - \frac{d^2}{dx^2}\left\{\left[(n_{\parallel}^2 - S) + \sigma n_y^2\right]E_+\right\} -$$

$$+ \left[n_y^2 + 2(n_{\parallel}^2 - R)\right] \cdot \frac{d}{dx}\left(\sigma \frac{dE_+}{dx}\right) +$$

$$+ \left\{(n_{\parallel}^2 - R)(n_{\parallel}^2 - L) + n_y^2\left[(n_{\parallel}^2 - S) + \sigma\left(n_y^2 + 2(n_{\parallel}^2 - R)\right)\right]\right\}E_+ = 0$$

In order to simplify this equation, although retaining all the relevant physical features, we assume a B dependence of the form:

$$16)\quad B = B_o \frac{R_T}{R} = \frac{B_o}{1 + x/R_T}$$

where $x=0$ is the position of the cyclotron resonance.
So, for all the resonant species:

$$17)\quad x_{n,j} = \frac{x}{R_T n_{\parallel} \frac{V_{th\alpha}}{c}}$$

Thus the relevant dependence on x is through the Z function relative to the resonant species, while all the slowly varying terms may be kept constant. Separating resonant and not resonant terms, and using the asymptotic expansion for the Z function (a good approximation when the coupling layer is well separated from the Doppler broadened resonance) we get:

$$18)\quad R \equiv R_o \simeq 1 - \Sigma_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \frac{\omega}{\omega + \Omega_\alpha}$$

$$L \equiv L_o + L_1$$

$$S \equiv S_o + S_1$$

$$L_o \simeq 1 - \Sigma_\alpha^{(o)+(2)} \frac{\omega_{p\alpha}^2}{\omega^2} \frac{\omega}{\omega - \Omega_\alpha}$$

$$L_1 = - \Sigma_j^{(1)} \frac{\omega_{pj}^2}{\omega^2}\left[-x_{oj} Z(x_{1,j})\right] \simeq - \frac{2S'}{x}$$

$$S_0 \equiv \frac{1}{2}(R_0 + L_0)$$

$$S_1 \equiv \frac{1}{2} L_1 \simeq -\frac{S'}{x}$$

$$\sigma \simeq \frac{\sigma'}{x}$$

$$S' = \frac{1}{2} \Sigma_J^{(1)} \frac{\omega^2_{pJ}}{\omega^2} R_T$$

$$\sigma' = \Sigma_J^{(2)} \frac{\omega^2_{pJ}}{\Omega^2_j} \frac{V^2_{thJ}}{c^2} R_T = \Sigma_J^{(2)} \beta_j R_T = \frac{1}{4} \Sigma_J^{(2)} \frac{\omega^2_{pJ}}{\omega^2} \frac{V^2_{thJ}}{c^2} R_T$$

We thus obtain the equation which describes the propagation:

19) $$-\frac{d^3}{dx^3}\left(\frac{\mathcal{E}}{x}\frac{dE_+}{dx}\right) + \frac{d^2}{dx^2}\left[\left(1-\frac{\alpha}{x}\right)E_+\right] - h q^2_x \frac{d}{dx}\left(\frac{\mathcal{E}}{x}\frac{dE_+}{dx}\right) + q^2_x\left(1-\frac{\beta}{x}\right)E_+ = 0$$

where:

20) $$\mathcal{E} \equiv \frac{\sigma'}{(m^2_{\parallel} - S_0)}$$

$$h \equiv -\frac{m^2_y + 2(m^2_{\parallel} - R_0)}{q^2_x}$$

$$\alpha \equiv x_s - \mathcal{E} m^2_y$$

$$\beta \equiv h\alpha = \frac{1}{q^2_x}\left[Q^2_F x_L - m^2_y x_s + \mathcal{E} m^2_y \left(m^2_y + 2(m^2_{\parallel} - R_0)\right)\right]$$

And:

21) $$x_s = -\frac{S'}{(m^2_{\parallel} - S_0)}$$ two-ion hybrid "resonance"

$$x_L = -\frac{2S'}{(m^2_{\parallel} - L_0)}$$ L-cut off

$$Q^2_F = -\frac{(m^2_{\parallel} - R_0)(m^2_{\parallel} - L_0)}{(m^2_{\parallel} - S_0)}$$ asymptotic $m^2_{\perp}$

$$q_x^2 = -m_y^2 + Q_F^2 \qquad \text{asymptotic} \qquad m_x^2$$

The typical order of greatness of the various quantities appearing in Eq.(19) are:

$$\text{22)} \qquad \begin{aligned} &\varepsilon \sim M^{-2} \\ &\alpha \sim x_S \sim \beta \sim x_L \sim M^{-1} \\ &h \sim 1 \\ &Q_F^2 \sim q_x^2 \sim M \end{aligned} \qquad\qquad M = \frac{m_p}{m_e}$$

It will be shown in the next section that Eq.(19) may be solved in closed form when either $\alpha = \beta = 0$ or $\varepsilon = 0$, and with a perturbative treatment in the general case.

We close this section by giving an expression for the energy flux. From Eqs. (7), (9), (11) and (13) we get for the x component of the total flux:

$$\text{23)} \qquad S_x = P_x + T_x = \frac{c}{8\pi} \, \mathrm{Im} \left\{ E_y^* \frac{dE_y}{dx} - i m_y E_y^* E_+ + 2\,\mathrm{Re}(\sigma)\, E_+^* \frac{dE_+}{dx} \right\}$$

It is useful to evaluate this expression in the W.K.B. approximation by replacing d/dx with $i m_x$. From the second of Eqs.(14) we find the polarization:

$$\text{24)} \qquad \frac{E_-}{E_+} = \frac{(m_x - i m_y)^2}{m_x^2 + m_y^2 + 2\,(m_\parallel^2 - R)}$$

and:

$$\text{25)} \qquad \left\{ \begin{aligned} \frac{E_x}{E_+} &= \sqrt{2}\, \frac{m_x^2 + (m_\parallel^2 - R) - i m_x m_y}{m_x^2 + m_y^2 + 2\,(m_\parallel^2 - R)} \\ \frac{E_y}{E_+} &= -i\sqrt{2}\, \frac{(m_\parallel^2 - R) + i m_x m_y + m_y^2}{m_x^2 + m_y^2 + 2\,(m_\parallel^2 - R)} \end{aligned} \right.$$

Directly from Eq.(15) we recover the dispersion relation

26) $$\sigma m_\perp^4 + \left[(m_\parallel^2 - S) + 2\sigma (m_\parallel^2 - R) \right] m_\perp^2 + (m_\parallel^2 - R)(m_\parallel^2 - L) = 0$$

which, provided that:

27) $$\left| \sigma \frac{(m_\parallel^2 - R)}{(m_\parallel^2 - S)} \right| \ll 1$$

may be factorized into the dispersion relations of the fast wave and of the Bernstein wave:

28) $$\begin{cases} m_{\perp,F}^2 = m_{x,F}^2 + m_y^2 \simeq - \dfrac{(m_\parallel^2 - R)(m_\parallel^2 - L)}{(m_\parallel^2 - S)} \\ m_{\perp,B}^2 = m_{x,B}^2 + m_y^2 \simeq - \dfrac{(m_\parallel^2 - S)}{\sigma} \end{cases}$$

Using Eqs.(24)-(25) we get:

29) $$S_x = \frac{c}{8\pi} \, 2 \, Re(m_x) \left\{ \frac{(m_\parallel^2 - R)^2}{| m_x^2 + m_y^2 + 2 (m_\parallel^2 - R)|^2} + Re(\sigma) \right\} |E_+|^2$$

For the fast wave this becomes

30) $$S_x^F = \frac{c}{8\pi} \, 2 Re(m_x^F) \left\{ \frac{|m_\parallel^2 - S|^2}{(m_\parallel^2 - R)^2} + Re(\sigma) \right\} |E_+^F|^2$$

which approaches for $x \to \infty$ to:

31) $$S_x^F = \pm \frac{c}{8\pi} \, 2 q_x \, \frac{(m_\parallel^2 - S_0)^2}{(m_\parallel^2 - R_0)^2} \, |E_+^F|^2$$

while for the Bernstein wave

$$32)\qquad S_x^{B} = \begin{cases} \dfrac{c}{8\pi}\; 2\, n_x^{B}\; \mathrm{Re}(\sigma)\, |E_+^{B}|^2 & \text{high field side} \\ \\ 0 & \text{(evanescent wave) low field side} \end{cases}$$

As expected, the energy is carried mainly by the Poynting vector for the fast wave, and by the kinetic flux for the warm plasma, electrostatic wave. The latter is a backward wave since $\mathrm{Re}(\sigma) < 0$ in the propagating region, so that the direction of the power flow is opposite to the phase velocity.

3 Solutions of the propagation equation

We develop the solutions of Eq.(19) in three distinct cases:

a) $\varepsilon = 0$ $\quad \alpha \neq \beta \neq 0$

b) $\varepsilon \neq 0$ $\quad \alpha = \beta = 0$

c) $\varepsilon \neq 0$ $\quad \alpha \neq \beta \neq 0$

a) $\varepsilon = 0$

This situation is formally identical to the cold plasma limit and occurs whenever we consider a "minority" heating scheme in which the first harmonic of the majority species does not coincide with the fundamental cyclotron frequency of the minority species.

The majority species does not contribute to the warm plasma resonant term, and the latter becomes negligible.In this case the dispension relation shows a true cold plasma resonance near $S = 0$, and the approximation $\varepsilon = 0$ is fully justified.

Examples of this situation are ${}_3He^{++}$ minority in H^+ or D^+ plasmas.

The propagation equation reduces to a 2nd order one:

$$33)\qquad \frac{d^2}{dx^2}\left[\left(1-\frac{\alpha}{x}\right)E_+\right] + q_x^2\left(1-\frac{\beta}{x}\right)E_+ = 0$$

Using the dependent variable:

$$34)\qquad F = \left(1-\frac{\alpha}{x}\right)E_+$$

Eq.(33) transforms into a Budden equation

$$35)\qquad \frac{d^2F}{dx^2} + q_x^2\left(1-\frac{\Delta}{x-\alpha}\right)F = 0$$

$$\Delta = \beta - \alpha$$

Changing in turn the independent variable:

$$36)\qquad \xi = -2iq_x(x-\alpha)$$

$$\kappa = -i\,\frac{\Delta q_x}{2}$$

we get a Whittaker equation

$$37)\qquad \frac{d^2F(\xi)}{d\xi^2} + \left(-\frac{1}{4} + \frac{\kappa}{\xi}\right)F(\xi) = 0$$

whose suitable solutions are the Whittaker functions of 2nd kind |12|

$$38)\qquad F_\pm = W_{\pm\kappa,\frac{1}{2}}(\pm\xi)$$

Using the asymptotic expansion:

$$39)\qquad W_{\kappa,\frac{1}{2}}(\xi) \sim \xi^{\kappa}e^{-\frac{1}{2}\xi} \propto e^{iq_x x} \qquad -\frac{3}{2}\pi < \mathrm{Arg}(\xi) < \frac{3}{2}\pi$$

we see that the solutions corresponding to the cases of a fast wave incident from $+\infty$ (low field side) and from $-\infty$ (high field side) are given by F_- for $x<\alpha$ and by F_+ for $x>\alpha$, respectively. Since the Whittaker functions are defined in the complex ξ-plane cut along the negative real axis, the solution in the opposite region must be the appropriate analytical prolongement.

This is obtained by passing the singularity from above |Appendix 1|.

The result for l.f.s. incidence is:

$$40)\quad \begin{cases} E_+ = A\,\dfrac{x}{x-\alpha}\,W_{-K,\frac{1}{2}}(-\xi) & x<\alpha \\[2ex] E_+ = A\,\dfrac{x}{x-\alpha}\left\{ e^{2\eta_0}\,W_{-K,\frac{1}{2}}(-\xi) - \dfrac{2\pi i\, e^{\eta_0}\, W_{K,\frac{1}{2}}(\xi)}{\Gamma(K)\,\Gamma(1+K)} \right\} & x>\alpha \end{cases}$$

And for h.f.s. incidence:

$$41)\quad E_+ = A\,\frac{x}{x-\alpha}\,W_{K,\frac{1}{2}}(\xi)$$

Here we have defined the optical thickness

$$42)\quad \eta_0 = i\pi K = \frac{\pi}{2}\,\Delta q_x = \frac{\pi}{2}\,(k-1)\,\alpha\, q_x$$

and A is a normalization constant.

The asymptotic behavior of the field results:

$$43)\quad \begin{cases} E_+ \sim e^{-\eta_0}\, e^{-iq_x x} & x \ll \alpha \\[2ex] E_+ \sim e^{-iq_x x} - \dfrac{2\pi i\, e^{-\eta_0}}{\Gamma(K)\,\Gamma(1+K)}\, e^{-iq_x x} & x \gg \alpha \end{cases}$$

for l.f.s. incidence, and:

$$44)\quad \begin{cases} E_+ \sim e^{iq_x x} \\[1ex] E_+ \sim e^{-\eta_0}\, e^{iq_x x} \end{cases}$$

for high field side incidence. The incident amplitude is normalized to 1.

The transmission and reflection coefficient are defined as rates of the respective power fluxes (Eq.31)

$$45)\quad \begin{cases} T = e^{-2\eta_0} \\ R = (1-e^{-2\eta_0})^2 \end{cases} \qquad \text{l.f.s. incidence}$$

$$46)\quad \begin{cases} T = e^{-2\eta_0} \\ R = 0 \end{cases} \qquad \text{h.f.s. incidence}$$

In evaluating (45) we used the identity

$$47)\quad |\Gamma(iy)|^2 = \frac{\pi}{y \,\mathrm{senh}\,(\pi y)} \qquad y \text{ real}$$

Apparently Eqs.(45)-(46) do not show energy conservation since there is a missing energy:

$$48)\quad A = 1 - R - T = \begin{cases} e^{-2\eta_0}(1-e^{-2\eta_0}) & \text{l.f.s. incidence} \\ 1 - e^{-2\eta_0} & \text{h.f.s. incidence} \end{cases}$$

It may be shown |Appendix 2| , that this energy is entirely absorbed at the two ion hybrid "resonance" $x = \alpha$.

b) $\alpha = \beta = 0$

The typical example of this situation is a pure first harmonic heating scheme, with incidence perpendicular to the singular layer $(n_y = 0)$.

The propagation equation becomes

$$49)\quad -\frac{d^3}{dx^3}\left(\frac{\varepsilon}{x}\frac{dE_+}{dx}\right) + \frac{d^2}{dx^2}E_+ - h q_x^2 \frac{d}{dx}\left(\frac{\varepsilon}{x}\frac{dE_+}{dx}\right) + q_x^2 E_+ = 0$$

By setting:

$$F = \frac{1}{x}\frac{dE_+}{dx} \tag{50}$$

and subsequently Laplace-transforming, we get the system

$$\begin{cases} -\varepsilon p\left(p^2 + h q_x^2\right)\tilde{F} + \left(p^2 + q_x^2\right)\tilde{E}_+ = 0 \\ p\,\tilde{E}_+ = -\dfrac{d\tilde{F}}{dp} \end{cases} \tag{51}$$

The result is expressed by an integral representation:

$$E_+(x) = \int \frac{\varepsilon p\left(p^2 + h q_x^2\right)}{p^2 + q_x^2} \exp\left\{ p x - \varepsilon q_x^3 \left[(1-h)\,\mathrm{arctg}\,\frac{p}{q_x} - + (1-h)\frac{p}{q_x} + \frac{1}{3}\left(\frac{p}{q_x}\right)^3 \right] \right\} dp \tag{52}$$

where the integration path must go to infinity in the following directions

$$-\frac{\pi}{6} < \mathrm{Arg}\,p < \frac{\pi}{6}\ ;\quad \frac{\pi}{2} < \mathrm{Arg}\,p < \frac{5}{6}\pi\ ;\quad \frac{7}{6}\pi < \mathrm{Arg}\,p < \frac{3}{2}\pi \tag{53}$$

Eq.(51) differs from that given by Ngan e Swanson |3| in the term before the exponential, whose occurrence assures the correct asymptotic behavior of the Bernstein wave.

Instead, Eq.(49) is essentially the same as given subsequently by Swanson |8|, with minor differences due to the choice of the variable (E_y instead of E_+). Swanson however did not realize that (49) can be explicitly solved. An appropriate choice of four independent integration paths, each corresponding to a well defined asymptotic behavior of the wave, are shown in Fig.(1-a) and in Fig.(1-b) for $x<0$ and $x>0$ respectively.

The asymptotic behavior of Eq.(52) may be evaluated with the saddle point method for the Bernstein wave, and with a method based an the Hanckel contour integral for the Γ function |13| for the fast wave. The latter cannot be evaluated with the saddle point method because the relative saddle points are too close to the essential singularities $\rho = \pm i q_x$.

Defining:

$$54)\quad H(\rho) = \rho x - \varepsilon q_x^3 \left[(1-h)\, \mathrm{arctg}\, \frac{\rho}{q_x} - (1-h)\frac{\rho}{q_x} + \frac{1}{3}\left(\frac{\rho}{q_x}\right)^3 \right]$$

we get:

$$55)\quad H'(\rho) = x - \varepsilon \rho^2 \, \frac{\rho^2 + h q_x^2}{\rho^2 + q_x^2}$$

$$56)\quad H''(\rho) = -\frac{2\varepsilon\rho}{(\rho^2+q_x^2)^2}\left(\rho^4 + 2 q_x^2 \rho^2 + h q_x^2\right)$$

The saddle point condition is equivalent to the dispersion relation:

$$57)\quad 0 = H'(\rho) = x - \frac{\varepsilon\rho^2(\rho^2 + h q_x^2)}{\rho^2 + q_x^2}$$

whose asymptotic solutions are

$$58)\quad \rho_B = \begin{cases} \pm i \left|\frac{x}{\varepsilon}\right|^{1/2} & x<0 \\ \pm \left(\frac{x}{\varepsilon}\right)^{1/2} & x>0 \end{cases} \qquad \text{B-wave}$$

$$59)\quad \rho_F = \pm i q_x \left[1 - \frac{1}{2}(1-h)\frac{\varepsilon q_x^2}{x}\right] \qquad \text{F-wave}$$

And

$$60)\quad H''(\rho_B) = \begin{cases} \mp 2i\,\varepsilon^{1/2}|x|^{1/2} & x<0 \\ \mp 2\,\varepsilon^{1/2} x^{1/2} & x>0 \end{cases}$$

61) $$H''(p_F) = \pm \frac{2i}{(1-\hbar)} \frac{x^2}{\varepsilon q_x^3}$$

In order to expand H(p) around the saddle points, the singularity p_0 must be far enough from the saddle point p_s that:

62) $$| H''(p_s) \ (p_0 - p_s)^2 | \gg 1$$

In the asymptotic region this is true for the Bernstein wave, but not for the fast wave for which the expression (62) is of order unit.

The Stokes' lines relative to p_B are shown in Fig. (1).
We have

63) $$H(p_B) = \begin{cases} \pm \frac{2}{3} \varepsilon^{-1/2} x^{3/2} & x > 0 \\ \mp \eta_p \mp \frac{2}{3} i \varepsilon^{-1/2} |x|^{3/2} & x < 0 \end{cases}$$

where the optical thickness is defined:

64) $$\eta_p = \frac{\pi}{2} (1-\hbar) \varepsilon q_x^3$$

The result of the evaluation of (52) for the B wave is:

65) $$E_+^B \sim e^{\pm i \frac{\pi}{4}} (\varepsilon \pi)^{1/2} \left| \frac{x}{\varepsilon} \right|^{1/4} e^{\mp \eta_p \mp \frac{2}{3} i \varepsilon^{-1/2} |x|^{3/2}}$$

for the paths C_1 and C_4 of Fig. (1-a),

66) $$E_+^B \sim -i (\varepsilon \pi)^{1/2} \left(\frac{x}{\varepsilon} \right)^{1/4} e^{-\frac{2}{3} \varepsilon^{-1/2} x^{3/2}}$$

for the path D_2 of Fig. (1-b).

Let's now evaluate the contributions of the integral C_3 and D_3 (C_2 and C_1 are their complex conjugates).
Changing the variable in Eq. (52)

67) $$p = -i q_x + t$$

we get

$$68)\quad E_+^F = e^{-\frac{3}{2}\eta_P} e^{-iq_x x} \int_{C_3', D_3'} (-t)^{\left(\frac{\eta_P}{\pi i} - 1\right)} e^{tx} \varphi(t)\, dt$$

$$69)\quad \varphi(t) = \frac{-\varepsilon(t - iq_x)\left[(t - iq_x)^2 + h q_x^2\right]}{(t - 2iq_x)} (it + 2q_x)^{-\frac{\eta_P}{\pi i}} \cdot$$

$$\cdot \exp\left\{ -\varepsilon q_x^3 \left[-(1-h)\left(\frac{t}{q_x} - i\right) + \frac{1}{3}\left(\frac{t}{q_x} - i\right)^3 \right] \right\}$$

The integrand in Eq. (68) is dominated by e^{tx} until a distance from the origin of order

$$70)\quad t_0 \sim \left|\frac{x}{\varepsilon}\right|^{1/2}$$

We thus have a good evaluation of(68) by simply replacing $\varphi(t)$ with $\varphi(0)$, provided that:

$$71)\quad |t_0 x| = \varepsilon^{-1/2} |x|^{3/2} \gg 1$$

well satisfied in the W.K.B. region. Then

$$72)\quad E_+^F \sim e^{-\frac{3}{2}\eta_P} \varphi(0) e^{-iq_x x} \int_{C_3'' D_3''} (-t)^{\left(\frac{\eta_P}{\pi i} - 1\right)} e^{tx}\, dt$$

$$73)\quad \varphi(0) = \frac{(1-h)}{2} \varepsilon q_x^2\, e^{i\frac{\eta_P}{\pi} \ln(2q_x)}\, e^{-i\left(\frac{4}{3} - h\right)\varepsilon q_x^3}$$

Here C_3'' and D_3'' must go to infinity in the less restrictive directions $Re(t) > 0$ und $Re(t) < 0$ respectively.
Finally, changing again the variable:

$$74)\quad t = -\frac{z}{x}$$

we get

$$75)\quad E_+^F(x) \sim e^{-iq_x x} e^{-\frac{3}{2}\eta_P} \varphi(0) \left(-\frac{1}{x}\right)^{\frac{\eta_P}{\pi i}} \int_{C_H} (-z)^{\left(\frac{\eta_P}{\pi i} - 1\right)} e^{-z}\, dz$$

where C_H is Hanckel's contour for Γ function integral representation:

76) $$\int_{C_H} (-z)^{-Y} e^{-z}\, dz = -2\pi i\, \Gamma^{-1}(Y)$$

The results, neglecting irrelevant phase factors, are listed below:

77) $$E_+^F \sim \frac{e^{-\frac{3}{2}\eta_p}}{q_x} \frac{2\pi}{\Gamma(i\eta_p/\pi)} e^{-iq_x x} \qquad \text{for } C_3$$

$$E_+^F \sim \frac{e^{-\eta_p/2}}{q_x} \frac{2\pi}{\Gamma(i\eta_p/\pi)} e^{-iq_x x} \qquad \text{for } D_3$$

$$E_+^F \sim \frac{e^{-\frac{3}{2}\eta_p}}{q_x} \frac{2\pi}{\Gamma(i\eta_p/\pi)} e^{iq_x x} \qquad \text{for } C_2$$

$$E_+^F \sim \frac{e^{-\eta_p/2}}{q_x} \frac{2\pi}{\Gamma(-i\eta_p/\pi)} e^{iq_x x} \qquad \text{for } D_1$$

The final step is to connect the solutions of the regions $x \gtrless 0$. This will be done explicitely for the case of low field side incidence, while only the results will be cited for the opposite case.

Incidence from $+\infty$ excludes the paths C_2 and C_3 in $x<0$, which are associated to waves carrying energy from $-\infty$.

Moreover, we must exclude the unphysical growing solution in $x>0$ As shown in Fig. (2-a), the presence of a branch point in $p = -iq_x$ makes contributions from the two rectilinear portions of C_3 to differ by a factor $e^{-2\eta_p}$. So, the contribution of the path C_1 must be multiplied for the weight ($1 - e^{-2\eta_p}$) in order to cancel the growing solution.

The required solution for E_+ can thus be symbolically written as:

78) $$E_+ = (1 - e^{-2\eta_p}) * C_1 + C_3$$

which is expressed by means of the standard integrals in $x<0$ The subsequent modifications of the paths, needed to obtain (78) by means of the integrals in $x>0$ are shown in Fig. (2). The result is

79) $$E_+ = (1 - e^{-2\eta_p})\, D_1 - (1 - e^{-2\eta_p})\, D_2 + D_3$$

Eqs. (78) - (79) together with (65) - (66) and (77) lead to the following connection formula which we have written with the incident amplitude normalized to 1:

$$80)\qquad e^{-iq_x x} + (1-e^{-2\eta_p})e^{iq_x x} + q_x(\varepsilon\pi)^{1/2}e^{\eta_p/2}(1-e^{-2\eta_p})\frac{\Gamma(i\eta_p/\pi)}{2\pi}\left(\frac{x}{\varepsilon}\right)^{1/4}\cdot$$

$$\cdot e^{-\frac{2}{3}\varepsilon^{-1/2}x^{3/2}} \leftrightarrow \qquad (x>0)$$

$$\leftrightarrow\ e^{-\eta_p}e^{-iq_x x} + e^{-\eta_p/2}(1-e^{-2\eta_p})\,q_x(\varepsilon\pi)^{1/2}\frac{\Gamma(i\eta_p/\pi)}{2\pi}\left|\frac{x}{\varepsilon}\right|^{1/4}\cdot$$

$$\cdot e^{-\frac{2}{3}i\varepsilon^{-1/2}|x|^{3/2}} \qquad (x<0)$$

The coefficients of transmission ($T_{F,B}$) and reflection ($R_{F,B}$) on the fast and Bernstein wave are finally obtained as rates of the power fluxes Eqs. (31) - (32).
The results are

$$81)\quad \begin{cases} T_F = e^{-2\eta_p} \\ T_B = e^{-2\eta_p}(1-e^{-2\eta_p}) \\ R_F = (1-e^{-2\eta_p})^2 \\ R_B = 0 \end{cases} \qquad \text{low field side incidence}$$

In a similar manner, one easily obtains

$$82)\quad \begin{cases} T_F = e^{-2\eta_p} \\ T_B = 0 \\ R_F = 0 \\ R_B = 1-e^{-2\eta_p} \end{cases} \qquad \text{high field side incidence}$$

The results are the same as (45) - (46) and (48) except that the energy absorbed now appears converted into the Bernstein wave.

c) General case

The general case includes the "minority heating scheme", when the fundamental frequency of the minority species concides with the first harmonic of the majority species. Typical example is a D^+ plasma with a small concentration of H^+: the warm term σ is now proportional to the majority concentration and is no longer negligible.
The other case which belongs to the general one is the pure first harmonic heating with oblique incidence.

We were not able to obtain an integral representation of the solution like EQ. (55) for the general case. The main difficulty is that substitutions like (34) and (50) lead now to a second order equation in the transformed variables.

However, we can evaluate the coupling coefficients in the limit of small optical thickness, with a technique similar to the Born approximation in scattering problems.

Eq. (19) may be modified as follows

83) $$H_o E_+ = \left(H_o H_1 + H_2 \right) E_+$$

where Ho describes the propagation of "free" fast waves:

84) $$H_o = \frac{d^2}{dx^2} + q_x^2$$

while the r.h.s. describes the "interaction" with the singular layer:

85) $$H_1 = \frac{d}{dx} \frac{\varepsilon}{x} \frac{d}{dx} + \left[\alpha - (1-\ell) \varepsilon q_x^2 \right] \frac{1}{x}$$

86) $$H_2 = -(1-\ell) \varepsilon q_x^2 \frac{d}{dx} \frac{1}{x^2} + q_x^2 \left[(\beta - \alpha) + (1-\ell) \varepsilon q_x^2 \right] \frac{1}{x}$$

Es. (83) may be rewritten by using the appropriate Green function for Ho (The one describing outgoing waves |14|):

87) $$H_0^{-1} \equiv G(x-x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-iK(x-x')}}{-K^2+q_x^2+i r}\,dK = \frac{1}{2 i q_x}\, e^{i q_x |x-x'|}$$
$$(r \to 0^+)$$

88) $$E_+ = E_o + H_1 E_+ + \int_{-\infty}^{+\infty} dx'\, G(x-x')\, H_2' E_+(x')$$

where Eo is any free solution:

89) $$H_o E_o = 0$$

As usually

90) $$G(x-x') = G_+(x-x') + G_-(x-x')$$
$$G_+(x-x') = \begin{cases} G(x-x') & x > x' \\ 0 & x < x' \end{cases}$$
$$G_-(x-x') = \begin{cases} 0 & x > x' \\ G(x-x') & x < x' \end{cases}$$

Then Eq. (83) is transformed into an integral equation

91) $$E_+ = E_o + H_1 E_+ + \int_{-\infty}^{x} G_+(x-x')\, H_2' E_+' dx' + \int_{x}^{+\infty} G_-(x-x')\, H_2' E_+' dx'$$

We now obtain asymptotic formulae for the amplitude of the fast wave.

For l.f.s. incidence we take $E_o = e^{-i q_x x}$. Since asymptotically $H_1 E_+^F$ is negligible with respect to the main term, we get:

92) $$E_+^F \sim e^{-i q_x x} + \int_{-\infty}^{+\infty} G_+(x-x')\, H_2' E_+' dx' =$$
$$= e^{-i q_x x} + e^{i q_x x}\int_{-\infty}^{+\infty} dx' \frac{1}{2 i q_x} e^{-i q_x x'} H_2' E_+' \qquad (x \gg 0)$$

93) $$E_+^F \sim e^{-i q_x x} + \int_{-\infty}^{+\infty} G_-(x-x')\, H_2' E_+' dx' = .$$
$$= e^{-i q_x x}\left[1 + \int_{-\infty}^{+\infty} dx' \frac{1}{2 i q_x} e^{i q_x x'} H_2' E_+'\right] \qquad (x \ll 0)$$

Thus we obtain for the transmitted and reflected amplitude (l.f.s. incidence)

$$94) \quad \begin{cases} A_T = 1 + \int_{-\infty}^{+\infty} dx \, \frac{1}{2iq_x} \, e^{iq_x x} H_2 E_+ \\ A_R = \int_{-\infty}^{+\infty} dx \, \frac{1}{2iq_x} \, e^{-iq_x x} H_2 E_+ \end{cases}$$

and (h.f.s. incidence)

$$95) \quad \begin{cases} A_T = 1 + \int_{-\infty}^{+\infty} dx \, \frac{1}{2iq_x} \, e^{-iq_x x} H_2 E_+ \\ A_R = \int_{-\infty}^{+\infty} dx \, \frac{1}{2iq_x} \, e^{iq_x x} H_2 E_+ \end{cases}$$

Eqs. (94) - (95) are still exact, since they are expressed in terms of the exact solution $E_+(x)$. However, since the latter is not known, we can only obtain a first order evaluation by replacing E_+ with E_o:

$$96) \quad \begin{cases} A_T = 1 + \int_{-\infty}^{+\infty} \frac{dx}{\pi i} \left[(\eta_o + \eta_p) \frac{1}{x} + \frac{i\eta_p}{q_x} \frac{1}{x^2} + \frac{2\eta_p}{q_x^2} \frac{1}{x^3} \right] \\ A_R = \int_{-\infty}^{+\infty} \frac{dx}{\pi i} \, e^{-2iq_x x} \left[(\eta_o + \eta_p) \frac{1}{x} + \frac{i\eta_p}{q_x} \frac{1}{x^2} + \frac{2\eta_p}{q_x^2} \frac{1}{x^3} \right] \end{cases}$$

for l.f.s. incidence and

$$97) \quad \begin{cases} A_T = 1 + \int_{-\infty}^{+\infty} \frac{dx}{\pi i} \left[(\eta_o + \eta_p) \frac{1}{x} - i \frac{\eta_p}{q_x} \frac{1}{x^2} + \frac{2\eta_p}{q_x^2} \frac{1}{x^3} \right] \\ A_R = \int_{-\infty}^{+\infty} \frac{dx}{\pi i} \, e^{2iq_x x} \left[(\eta_o + \eta_p) \frac{1}{x} - \frac{i\eta_p}{q_x} \frac{1}{x^2} + \frac{2\eta_p}{q_x^2} \frac{1}{x^3} \right] \end{cases}$$

for h.f.s. incidence.

Here we have used the optical thicknesses η_o and η_p (Eqs. (42) and (64)).

The pole must be bypassed from above, and the integrals for A_T must be evaluated with the formula (A2).

The results are :

$$98)\quad \begin{cases} A_T = 1 - (\eta_0 + \eta_p) \\ A_R = 2(\eta_p - \eta_0) \end{cases} \qquad \text{l.f.s. incidence}$$

$$99)\quad \begin{cases} A_T = 1 - (\eta_0 + \eta_p) \\ A_R = 0 \end{cases} \qquad \text{h.f.s. incidence}$$

We note that although (94) and (95) are formally very similar, the results are quite different. This is a consequence of the causality requirement, i.e. the displacement of the pole from $x = 0$.

The small optical thickness expansions (98) - (99) suggest the extension to arbitrary η_0 and η_p :

$$100)\quad \begin{cases} |A_T| = e^{-(\eta_0 + \eta_p)} \\ |A_R| = |e^{-2\eta_p} - e^{-2\eta_0}| \end{cases} \qquad \text{l.f.s. incidence}$$

We see that (100) agrees with the correct expressions in the exactly solvable cases. Moreover they satisfy energy conservation:

$$101)\quad T_B \equiv 1 - |A_T|^2 - |A_R|^2 = 1 + e^{-2\eta_0 - 2\eta_p} - e^{-4\eta_0} - e^{-4\eta_p} \geq 0$$

And T_B has to be interpreted as the power converted into the Bernstein wave.

The results may be summarized as follows:

$$102)\quad \begin{cases} T_F = e^{-2(\eta_0 + \eta_p)} \\ R_F = (e^{-2\eta_p} - e^{-2\eta_0})^2 \\ R_B = 0 \\ T_B = 1 + e^{-2\eta_0 - 2\eta_p} - e^{-4\eta_0} - e^{-4\eta_p} \end{cases} \qquad \text{l.f.s. incidence}$$

$$103)\quad \begin{cases} T_F = e^{-2(\eta_0 + \eta_p)} \\ R_F = 0 \\ R_B = 1 - e^{-2(\eta_0 + \eta_p)} \\ T_B = 0 \end{cases} \qquad \text{h.f.s. incidence}$$

Thus, the general case is completely described by two optical thicknesses which we rewrite in terms of physical quantities:

104) $$\eta_0 = \frac{\pi}{2} \frac{(n_{\parallel}^2 - R_0)^2}{(n_{\parallel}^2 - S_0)^2} \frac{1}{q_x} \left(S' + \sigma' n_y^2 \right)$$

$$\eta_p = \frac{\pi}{2} \frac{(n_{\parallel}^2 - R_0)^2}{(n_{\parallel}^2 - S_0)^2} \sigma' q_x$$

An interesting consequence of (105) is that reflection vanishes when $\eta_p = \eta_0$, condition which can be satisfied in the first harmonic heating scheme at 45^0 incidence.

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Appendix 1

The behaviour $\sim 1/x$ of the resonant terms derives from the asymptotic expansion of the Z function:

$$\text{A1)} \quad \frac{1}{x} \rightarrow -\frac{x_0}{R_T} Z\left(\frac{x x_0}{R_T}\right) \simeq -\frac{x_0}{R_T} \operatorname{Re} Z\left(\frac{x x_0}{R_T}\right) - i\sqrt{\pi}\, \frac{x_0}{R_T}\, e^{-\left(\frac{x x_0}{R_T}\right)^2}$$

Performing the limit $m_{\parallel} \rightarrow 0$ $(x_0 \rightarrow \infty)$ in the sense of the distributions we obtain

$$\text{A2)} \quad \lim_{m_{\parallel} \rightarrow 0} \left[-\frac{x_0}{R_T} Z\left(\frac{x x_0}{R_T}\right)\right] = P\frac{1}{x} - i\pi\, \delta(x) =$$

$$= \lim_{\gamma \rightarrow 0^+} \frac{1}{x + i\gamma}$$

So, the causality requirement is taken into account by giving to x a small positive imaginary part.

Appendix 2

The power absorbed in the limit $m_{\parallel} \rightarrow 0$ can be evaluated by replacing x with $x + i\gamma$ and performing the limit $\gamma \rightarrow 0^+$ at the end of the calculation.

The imaginary part of Z must be replaced by:

$$\text{A3)} \quad -\frac{x_0}{R_T} \operatorname{Im} Z\left(\frac{x x_0}{R_T}\right) \rightarrow \operatorname{Im} \frac{1}{x + i\gamma} = \frac{-\gamma}{x^2 + \gamma^2}$$

This may be applied to the cold plasma limit ($\varepsilon = 0$). Considering for example l.f.s. incidence with $\alpha > 0$ and normalizing the field so that the incident flux be unitary we get |15|

$$\text{A4)} \quad \frac{dS_x}{dx} = -e^{-3\eta_0} \frac{(m_{\parallel}^2 - R_0)^2}{(m_{\parallel}^2 - S_0)^2} \frac{S'}{q_x} \frac{\gamma}{x^2 + \gamma^2} \left|\frac{x + i\gamma}{x + i\gamma - \alpha}\right|^2 \left|W_{-\kappa,\frac{1}{2}}(-\xi)\right|^2 =$$

$$= -e^{-3\eta_0} \frac{(m_{\parallel}^2 - R_0)^2}{(m_{\parallel}^2 - S_0)^2} \frac{S'}{q_x} \frac{\gamma}{(x-\alpha)^2 + \gamma^2} \left|W_{-\kappa,\frac{1}{2}}(-\xi)\right|^2$$

The limit $\gamma \to 0^+$ shows that the absorption is localized in $x = \alpha$:

A5) $$\frac{dS_x}{dx} = - \delta(x-\alpha)\; 2\eta_0\, e^{-3\eta_0} \left| W_{-\kappa,\frac{1}{2}}(0) \right|^2$$

and

A6) $$\Delta S_x = \left| \int_{-\infty}^{+\infty} \frac{dS_x}{dx}\, dx \right| = e^{-2\eta_0} \left(1 - e^{-2\eta_0}\right)$$

as expected.

FIGURE CAPTIONS

Fig. 1: Independent integration paths for evaluating the electric field (Eq. 55)

a) Well defined behaviour for $x \to -\infty$

C_1 : Bernstein wave cyrrying energy toward $-\infty$
C_2 : Fast wave carrying energy from $-\infty$
C_3 : Fast wave carrying energy toward $-\infty$
C_4 : Bernstein wave cyrrying energy from $-\infty$

b) Well defined behaviour for $x \to +\infty$

D_1: Fast wave carrying energy toward $+\infty$
D_2: Evanescent Bernstein wave
D_3: Fast wave carrying energy from $+\infty$
D_4: Any other independent path, which includes contributions from the unphysical growing solution

Fig. 2: Subsequent deformations of the paths to find the connection formula in the case of l.f.s. incidence

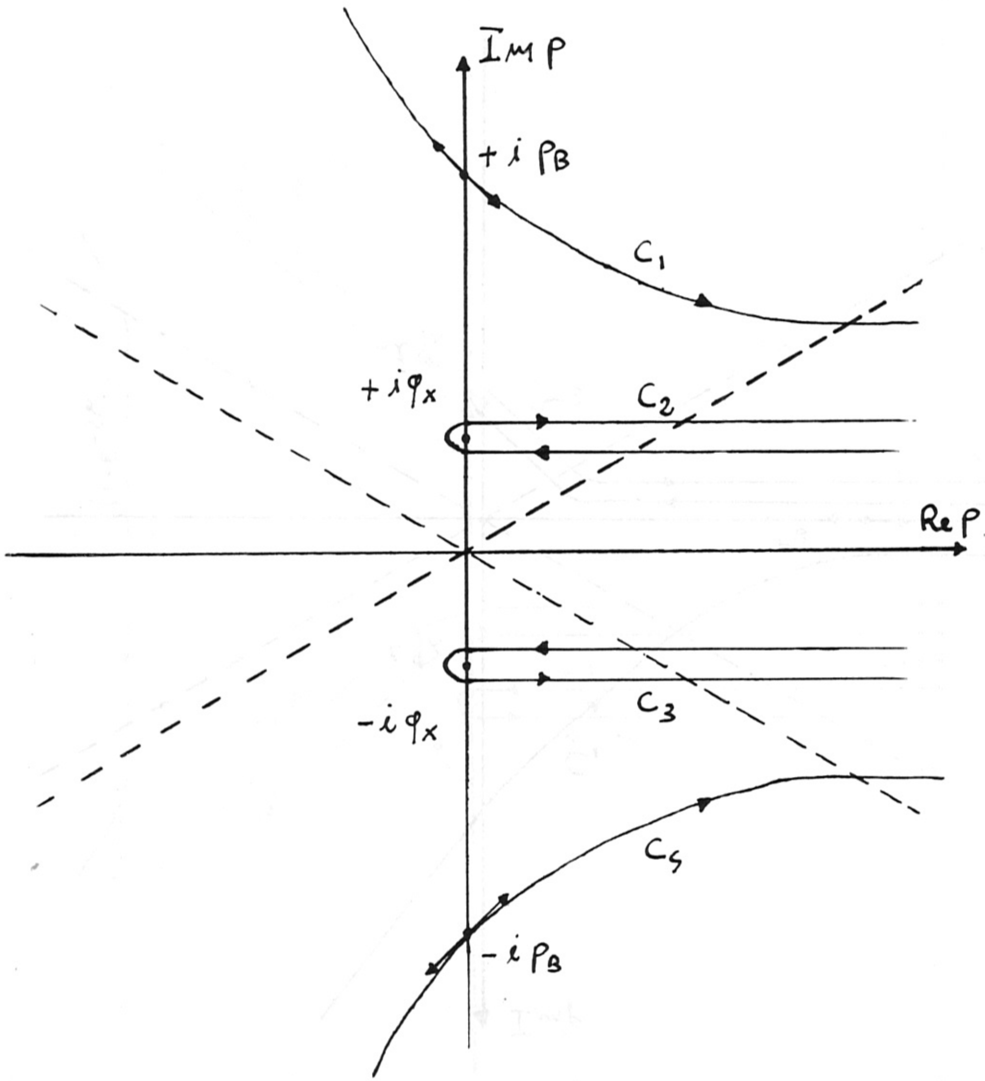


Fig 1 a

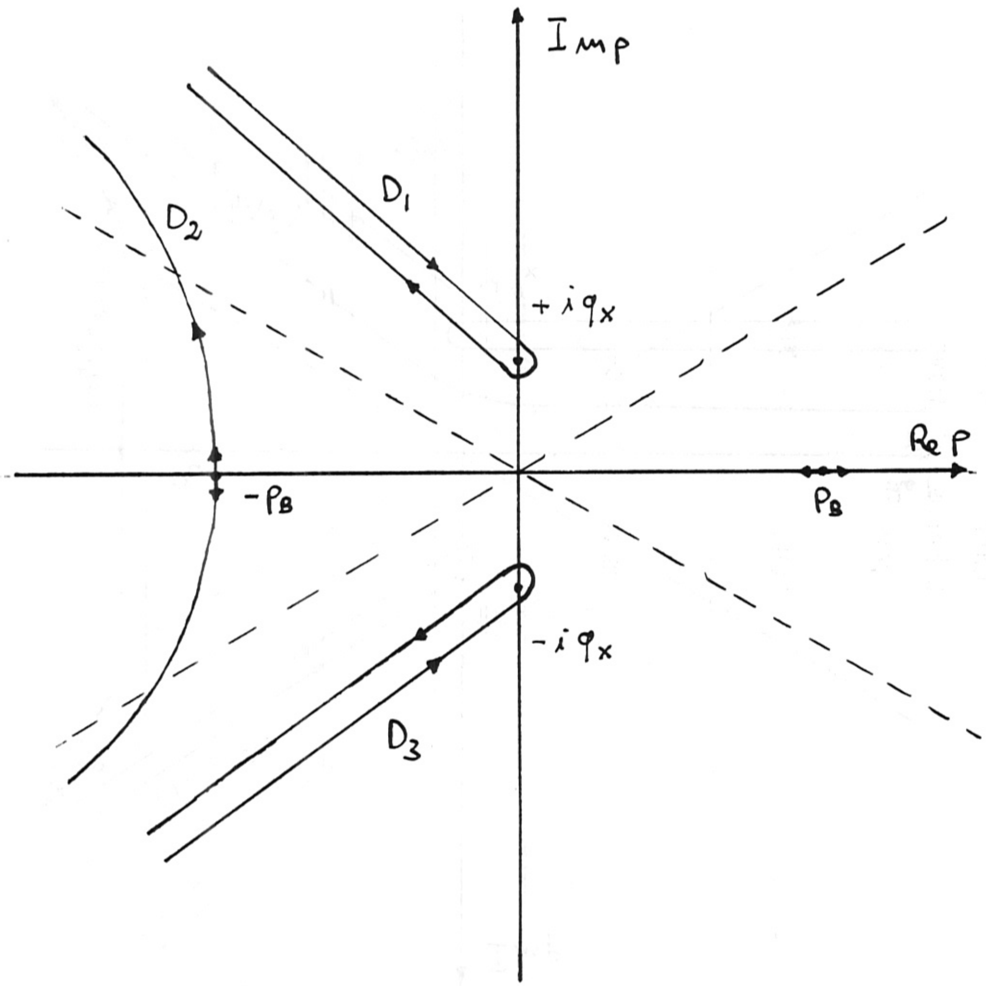


Fig. 1 b

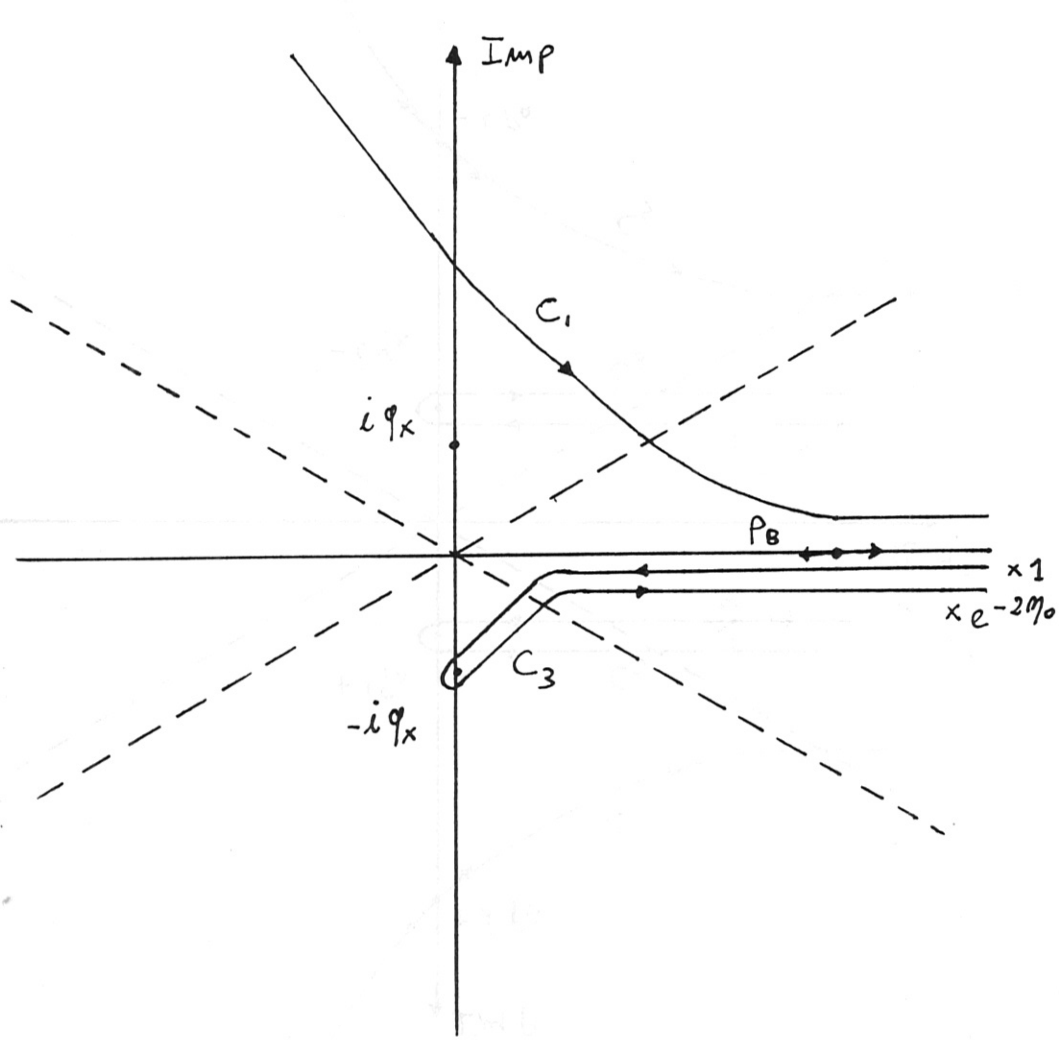

Im p
C_1
iq_x
p_B
$\times 1$
$\times e^{-2\eta_0}$
C_3
$-iq_x$

Fig 2 a

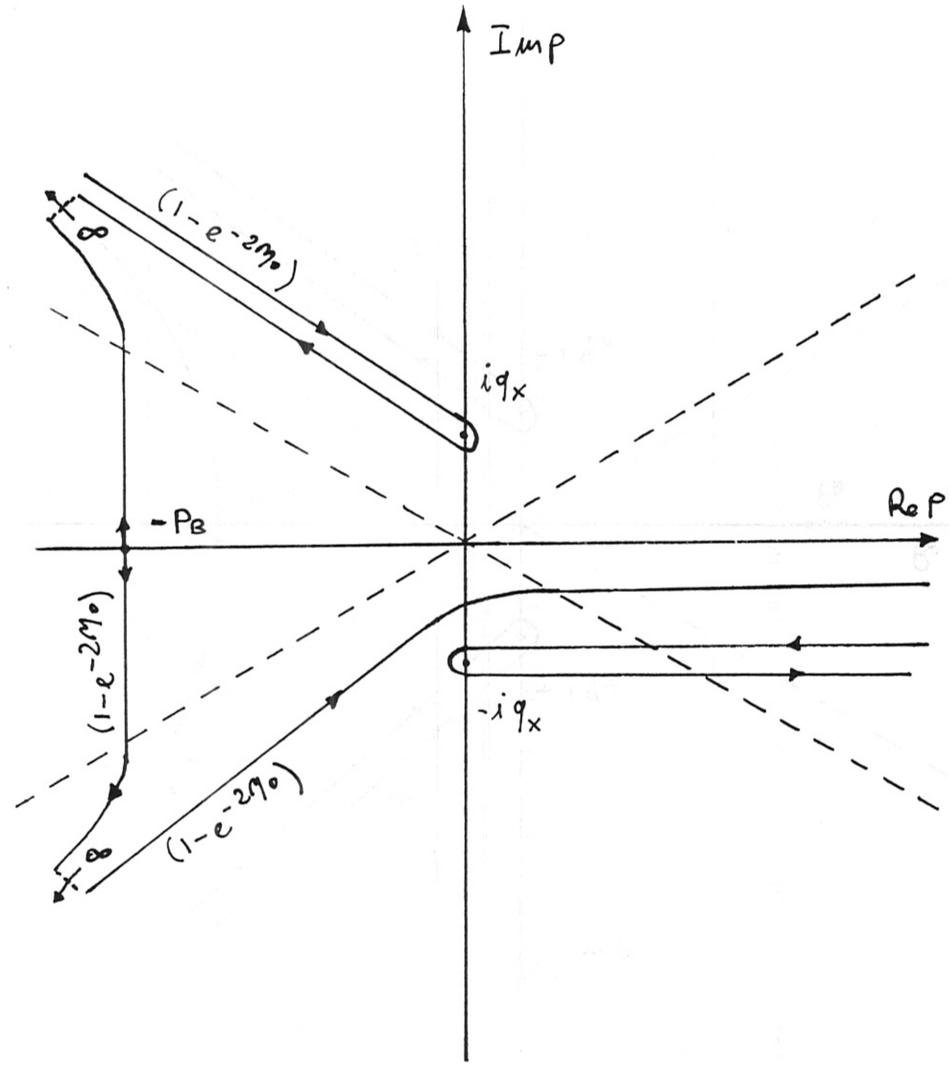
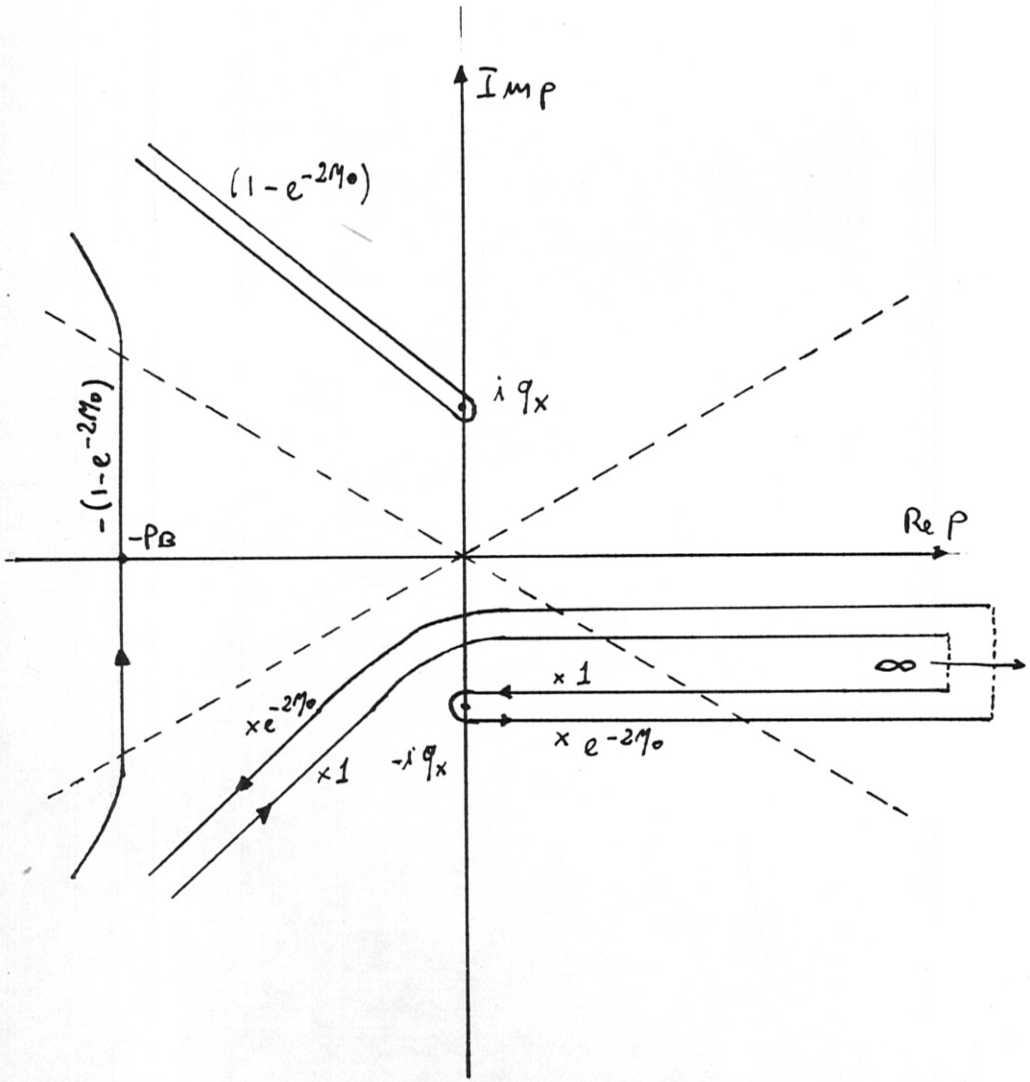
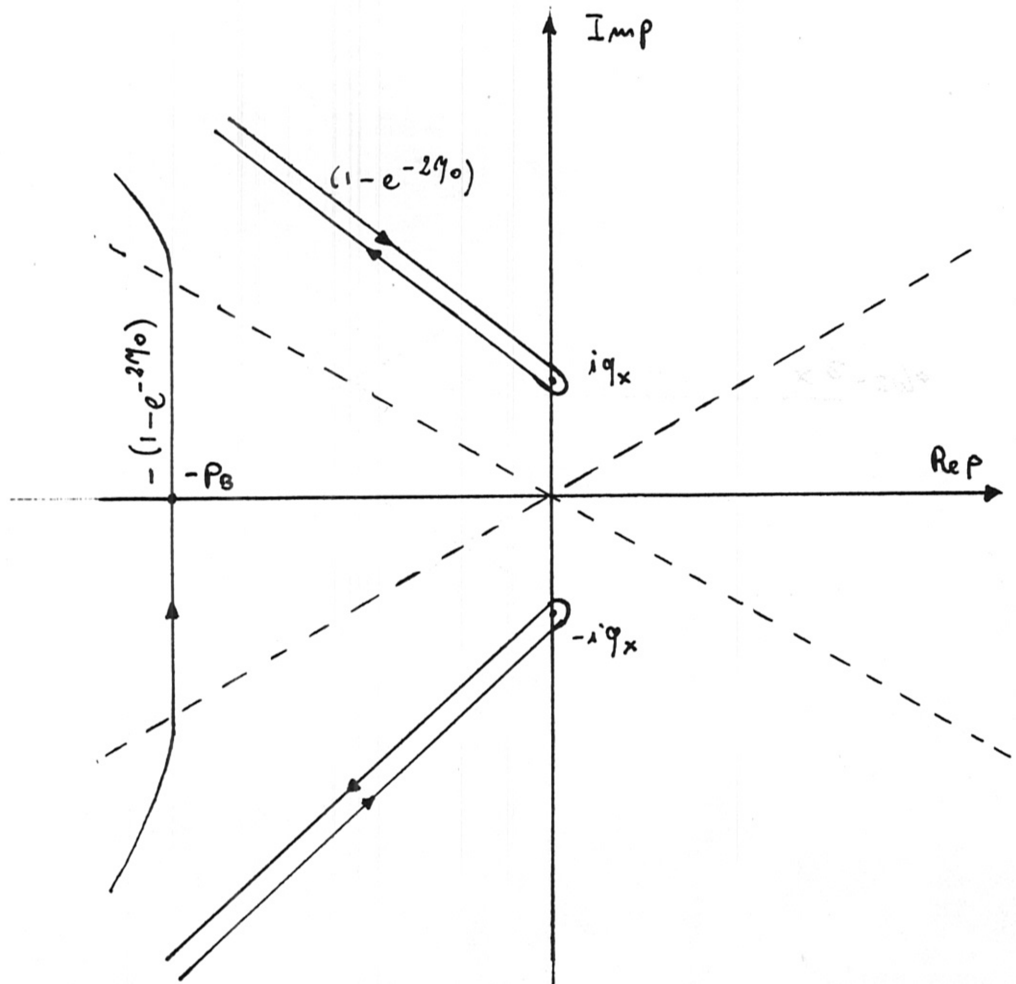

Im p
Re p
iq_x
$-iq_x$
$-p_B$
$(1-e^{-2\eta_0})$
$(1-e^{-2\eta_0})$
$(1-e^{-2\eta_0})$
∞
∞

Fig 2 b



Im p
Re p
$(1-e^{-2\eta_0})$
iq_x
$-(1-e^{-2\eta_0})$
$-p_B$
$\times 1$
∞
$\times e^{-2\eta_0}$
$-iq_x$

Fig 2c



Im p
Re p
$(1-e^{-2\eta_0})$
iq_x
$-(1-e^{-2\eta_0})$
$-p_B$
$-iq_x$

Fig 2d