

(in English)
On Anomalous Plasma Transport as a
Wiener Process

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IPP 6/229

August 1983



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*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem
Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die
Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

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Abstract

A recent theory by MOLVIG et al. on particle transport in turbulent fields with overlapping island structure is critically reviewed. An inconsistency in the derivation of the diffusion coefficient D from the assumption of a Wiener process is pointed out and corrected. In the region $\Delta\Omega \cdot \tau_K \gg 1$ the results if properly interpreted are unchanged. For $\Delta\Omega \cdot \tau_K \ll 1$ the results differ ($\Delta\Omega$ = particle velocity/mode correlation length, τ_K = "randomization" time). Computer experiments are performed to support the critical arguments.

1. Introduction

Experimental evidence /1, 2/ indicates that "anomalous" transport of energy by electrons in plasmas is correlated with a superthermal level of low frequency electromagnetic mode activity. Turbulence theories based on the interaction of linear drift modes, e.g. /3/, the references cited therein, and /5-11/, or on the presence of solitons /12/ have been proposed. Also attempts to evaluate the thermodynamic partition function /13/ have been made.

Recently, Molvig et al. /14/ proposed a mixed Eulerian - Lagrangian transport theory where the turbulent electron motion is considered with the modes frozen in as it were so that the mode spectrum and the transport process can be investigated separately. This theory applies to situations when the modes develop a set of islands for the potential(s) around resonant locations and the islands strongly overlap. In this case the particle trajectories become of the "mixing" type /15/. Trajectories which are initially close to each other diverge exponentially. As a result, after a few Kolmogorov times τ_K ("randomization" times) the orbits may be viewed as a diffusion process with independent random increments with Gaussian distribution (Wiener process). The auto-correlation time τ_{AC} of the Eulerian modes is assumed to be large compared with τ_K . This scheme, called "normal stochastic approximation" (NSA), yields a diffusion coefficient /14/ of the

type

$$D^{NSA} = \sum_{k', \omega'} G^{NSA}_{k', \omega'} \langle |E_{k', \omega'}|^2 \rangle, \tag{1}$$

where $E_{k, \omega}$ is the field strength of a perturbing mode. $G^{NSA}_{k, \omega}$ is the Green's function of the particle motion, essentially a broadened version of $\delta(\omega - kv)$.

The aim of this paper is to point out and correct an inconsistency in the theory /14/ of Molvig et al.. In Sec. 3 it is shown that the assumed diffusion process has not been properly treated. In Sec. 4 this claim is corroborated with numerical experiments. The corrected theory yields a diffusion coefficient which, depending on the parameters, agrees or disagrees with /14/.

Before a short derivation of the relevant equations is given in Sec. 2, it may still be useful to briefly mention some related or contrasting theories.

In 1966 Dupree proposed a turbulence theory /4/ where secularities in the orbits owing to wave particle resonances are avoided by a velocity space diffusion. The diffusion coefficient

$$D = \frac{q^2}{m^2} \int_0^\infty dt \sum_{k, \omega} |E_{k, \omega}|^2 e^{i(kv - \omega)t - \frac{1}{3} k^2 D t^3} \tag{2}$$

has the same structure as (1).

In the same year Rosenbluth et al. /16/ investigated the topology of magnetic surfaces disturbed by modes with a magnetic component. The formation and overlap of islands was discussed. A quasilinear model of field line diffusion was proposed, i.e. a diffusion coefficient analogous to eq.(2) without the t^3 term in the exponent. In 1978 this model was applied /17/ to determine the energy transport of electrons which are tied to the stochastic field lines. In 1979 this quasilinear diffusion coefficient was tested /18/ by numerical solution of model equations which are equivalent to recursion equations as discussed also in /14/ and Sec. 4 below.

1979 Hirshman and Molvig /5/ gave a self-consistent theory of particle diffusion and nonlinear mode evolution for the universal instability in a sheared magnetic field in order to obtain absolute values for the anomalous energy transport. The theory includes orbit diffusion by analogy to the ideas which led to eq. (2). Some of the simplifying assumptions made in /5/ were relaxed in /6/ and /7/.

In addition and in contrast to the work mentioned above, turbulence theory was further developed by Kraichnan and co-workers. The "direct interaction approximation" (DIA), /8,9/ yields (in a slightly simplified version /14/) a frequency and wave number dependent diffusion coefficient which is of the type

$$D_{k,\omega}^{DIA} = \sum_{k',\omega'} G_{k-k',\omega-\omega'}^{DIA} \langle |E_{k',\omega'}|^2 \rangle \quad (3)$$

The physical process underlying eq.(3) is inelastic Compton scattering from mode ω, k to modes ω', k' , while eq.(1) corresponds to primary wave particle resonances $\omega' \approx k'v$. Although the mechanisms are very different, an explicit comparison by Diamond and Rosenbluth /10/ gave no significant discrepancies. The simplified DIA renormalization method used in /10/ had been developed in /11/.

2. Equations of motion

Let us consider a "typical" drift wave situation. Electrons move in a sheared magnetic field $\underline{B} = B_0 (0, x/L_s, 1)$ in the presence of electrostatic waves $\underline{E} = -\nabla\phi$:

$$\phi(\underline{x}, t) = \sum_{m, n=-\infty}^{+\infty} \phi_{m, n}(\underline{x}) e^{i(k_m y + k_n z - \omega_{mn} t)}, \quad (4)$$

where $\underline{x} = (x, y, z)$ and $k_m = 2\pi m/L_y$ etc. In the guiding center approximation the equations of motion are

$$\dot{\underline{x}}_{\perp} = \frac{1}{B} [\underline{E} \times \underline{B}] ; \quad \dot{v}_{\parallel} = \frac{q_e}{m_e} E_{\parallel}, \quad (5)$$

where q_e, m_e are the electron's charge and mass, and $E_{\parallel} = \underline{E} \cdot \underline{B}/B$, etc. If positions, times, velocities and potentials are measured in units of $\rho_s, \omega_*^{-1}, \rho_s \omega_*$ and $(T_e/q_e)(\rho_s/L_n)$ eqs.(5) simplify to

$$\dot{x} = E_y, \quad \dot{y} = -E_x + v_{\parallel} x/L_s \quad (5a)$$

$$\dot{v}_{\parallel} = -\frac{m_i L_n}{m_e \rho_s} E_{\parallel}. \quad (5b)$$

Here $\rho_s = c_s/\Omega_i$, Ω_i = ion gyro frequency, $c_s^2 = kT_e/m_i$, $\omega_* = c_s/L_n$, and $L_n > 0$ is the scale length for the density, varying in the x direction. For small x, $|x|/L_s \ll 1$, it is a good approximation for our purpose to set $v_{\parallel} = \text{const}$ and $z = z_0 + v_{\parallel} t$.

In this case eqs.(5a) may be written as a one-dimensional Hamiltonian system with conjugate variables x, y :

$$\begin{aligned} \dot{x} &= - \frac{\partial H}{\partial y}, & \dot{y} &= \frac{\partial H}{\partial x} \\ H(x, y, t) &= \frac{v_{||}}{L_s} \frac{x^2}{2} + \Phi(x, y, z_0 + v_{||}t, t) \end{aligned} \quad (6)$$

Let $x^0 = x_0$, $y^0 = y_0 + x_0 v_{||}t/L_s$ be the unperturbed solution for $\Phi = 0$. Transformation from x, y to $\delta x = x - x^0$, $\delta y = y - y^0$ yields

$$\begin{aligned} \delta \dot{x} &= - \sum_{m, n=-\infty}^{+\infty} im \Phi_{m, n}(x_0 + \delta x) e^{i [k_m (y_0 + \delta y) + \\ &+ (k_m v_{||} x_0 / L_s + k_n v_{||} - \omega_{mn}) t + k_n z_0]} \end{aligned} \quad (7)$$

$$\delta \dot{y} = \frac{v_{||}}{L_s} \delta x + \sum_{m, n} \frac{\partial \Phi_{m, n}(x_0 + \delta x)}{\partial x_0} e^{i [\dots]}.$$

For simplicity we shall neglect the x dependence of the modes, as done in /14, 18/.

Resonances occur for $k_{||}(x_0)v_{||} - \omega_{mn} = 0$, where driving terms are only weakly time dependent. Consistently with the assumption $v_{||} = \text{const}$, it is a good approximation for drift waves to neglect ω_{mn} compared with $k_{||}v_{||}$, because it shifts the resonant positions x_0 only slightly. Around each resonant position x_0 , now at

$k_{||} = k_m x_o / L_s + k_n = 0$, there is an island in which the particles are trapped if the islands are well separated.

With these approximations and a further change of units: $x' = L_z x / (L_s L_y)$, $y' = 2\pi y / L_y$, $t' = 2\pi v_{||} t / L_z$ and $\phi' = L_z^2 \phi / (v_{||} L_s L_y^2)$ one finally obtains (omitting the slashes)

$$\dot{\delta x} = \sum_{m,n=-\infty}^{+\infty} |\phi_{m,n}| m \sin [m(y_o + \delta y) + (m x_o + n) t + \phi_{mn}] \quad (8)$$

$$\dot{\delta y} = \delta x ,$$

where $\phi_{m,n} e^{i k_n z_o} = |\phi_{m,n}| e^{i \phi_{mn}}$. Initial conditions at an arbitrarily fixed time $t_o = 0$ are $\delta x = \delta y = 0$.

For the purpose of numerical investigation further simplification of eqs.(8) is achieved by assuming that the mode strength $\phi_{m,n}$ is the same for all n , $-\infty \leq n \leq \infty$. With

$$\sum_{n=-\infty}^{+\infty} e^{i n t} = 2\pi \sum_{l=-\infty}^{+\infty} \delta(t - 2\pi l) , \quad (9)$$

and $z_o = 0$ without loss of generality, the differential equations (8) are replaced by the recursions

$$\delta y_1 = \delta y_{1-1} + 2\pi \delta x_{1-1} , \quad (10)$$

$$\delta x_1 = \delta x_{1-1} + 2\pi \sum_{m>0} A_m m \sin [m(y_o + \delta y_1) + 2\pi l m x_o + \phi_m]$$

where $A_m = 2|\phi_m|$. Initial conditions are $\delta x_0 = \delta y_0 = 0$.

Actually, eqs.(10) are not the most general discretized version of eqs.(8). For arbitrarily chosen $t_0 \neq 0$ there is a time lag ΔT , $0 < \Delta T < 2\pi$ between the first δ -function pulse before t_0 and t_0 itself. $\Delta T \rightarrow 0$ was chosen in eqs.(10). At the opposite end, $\Delta T \rightarrow 2\pi$, the order of the recursions is reversed, with δy_1 and δx_{1-1} on the r.h. sides replaced by δy_{1-1} and δx_1 . For the sake of comparison with theoretical models it is advantageous to consider both limiting cases, see Sec. 4.

Equations (8) were solved numerically in /14/ while eqs.(10) were solved both in /14/ and /18/ in order to obtain diffusion coefficients. In Sec. 4 we solve eqs.(10) again, but for the purpose of obtaining correlation functions like $\langle [\delta y(t) - \delta y(t')]^2 \rangle$.

3. Diffusion process

In 1981 Molvig et al. /14/ proposed the "normal stochastic approximation" to describe turbulent plasma response. As stated in the introduction, the theory is based on the chaotic "mixing" type behaviour of the particle motion in strongly overlapping islands of the potential. Any small segment of starting values y_0 , for example, is mapped onto the whole available range 2π after a time τ_K , called Kolmogorov time. As a consequence, the Lagrangian perturbations $\delta x(t)$, $\delta y(t)$ develop much finer spatial scales than the Eulerian correlation length of the modes $\Phi(\underline{x}, t)$. This disparity of scales together with the assumption that the "mixing" time τ_K is small compared with the autocorrelation time τ_{AC} of the modes is used in NSA to perform an average over the micro-scale fluctuations first, with the modes Φ frozen in. The averaging is performed with δx and δy as normally distributed random variables, i.e. a Gaussian distribution, because this is what the central limit theorem predicts for the cumulative effect of many independent increments $\Delta\delta x$, $\Delta\delta y$ which together yield the orbits δx , δy .

To be more precise, a diffusion or Wiener process was assumed to describe the "radial" motion $\delta x(t)$. In a symmetric Wiener process /19/ the probability density to find a particle at position x at time t when at $t = 0$ it was at $x = 0$ is given by

$$P(x(t)) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}, \quad t > 0. \quad (11)$$

The probability density to find the particle at time t' at position x' and at a later time t at x is the product of two independent probability densities:

$$P(x(t), x'(t')) = \frac{1}{4\pi D} \frac{1}{\sqrt{t'(t-t')}} e^{-\frac{x'^2}{4Dt'}} e^{-\frac{(x-x')^2}{4D(t-t')}} \quad (12)$$

$$t > t' > 0,$$

etc., where D is a diffusion constant.

Let Wiener averages be denoted by $\langle \rangle_W$. For a symmetric Wiener process x owing to its Gaussian nature it holds /20/ that

$$\langle e^{i a x} \rangle_W = e^{-\frac{a^2}{2} \langle x^2 \rangle_W}, \quad (13)$$

$$\langle e^{i(ax + bx')} \rangle_W = e^{-\frac{1}{2} \langle (ax + bx')^2 \rangle_W}.$$

From eq.(11) it follows that

$$\langle x^2 \rangle_W = 2 D t, \quad t > 0 \quad (14)$$

and, for $y(t) = \int_0^t x(t') dt'$, (see Appendix A)

$$\langle y^2(t) \rangle_W = \frac{2}{3} D t^3, \quad t > 0. \quad (15)$$

In Appendix A it is shown that for $t > t' > 0$

$$\begin{aligned} \langle (y - y')^2 \rangle_W &= \frac{2}{3} D (t-t')^2 \left[(t-t') + 3t' \right] \\ &= \frac{2}{3} D (t-t')^2 \left[3t - 2(t-t') \right] \end{aligned} \quad (16)$$

The fact that this is not a pure function of $t-t'$ is important in the following.

From eqs.(8) and (14) it is straight-forward to derive an equation for the diffusion coefficient D:

$$\begin{aligned} D &= \frac{1}{2t} \langle [\delta x(t)]^2 \rangle_W \\ &= \sum_{m,n} \sum_{m',n'} \langle \phi_{m,n} | \phi_{m',n'} \rangle \cdot \frac{-1}{2t} \int_0^t dt_1 \int_0^t dt_2 \\ &\quad \cdot \langle e^{i \Psi_{m,n}(t_1) + i \Psi_{m',n'}(t_2)} \rangle_W \end{aligned} \quad (17)$$

where

$$\Psi_{m,n}(t) = m \left[y_0 + \delta y(t) \right] + (mx_0 + n)t + \phi_{mn} .$$

With the symmetries involved, and transforming to $\tau = t_1 - t_2$

one obtains

$$D = \sum_{m,n} \sum_{m',n'} |m m'| |\phi_{m,n} \phi_{m',n'}| \cdot \frac{1}{t} \int_0^t dt_1 \int_0^{t_1} d\tau$$

$$\cdot \exp i\{(m+m')y_0 + [(m+m')x_0 + (n+n')]t_1 - (m'x_0 + n')\tau + \phi_{m,n}$$

$$+ \phi_{m',n'}\} \cdot \exp \left\{ \frac{-1}{2} \langle [m\delta y(t_1) + m'\delta y(t_1-\tau)]^2 \rangle_W \right\}. \quad (18)$$

For large times the rapid t_1 oscillations reduce the double sums to $m + m' = 0$, $n + n' = 0$, (see below) and with equ.(16) one obtains

$$D = \sum_{m,n} |m\phi_{m,n}|^2 \frac{1}{t} \int_0^t dt_1 \int_0^{t_1} d\tau \exp \left\{ i(mx_0 + n)\tau - \frac{m^2 D}{3} \tau^2 (3t_1 - 2\tau) \right\}. \quad (19)$$

An upper bound to D is obtained by taking the absolute values of the terms in the sum, by replacing 2τ by its maximum $2t_1$, and by extending the τ integration to infinity. The result is

$$D \leq \sum_{m,n} |m| |\phi_{m,n}|^2 \frac{\sqrt{3\pi}}{2\sqrt{D}} \frac{1}{t} \int_0^t dt_1 \frac{1}{\sqrt{t_1}}.$$

As a consequence, D does not tend to a constant value for large times, but goes to zero instead for $t \rightarrow \infty$:

$$D^{3/2} \leq \sqrt{3\pi} \sum_{m,n} |m| |\phi_{m,n}|^2 \frac{1}{\sqrt{t}}. \quad (20)$$

In Appendix B it is shown that the same result holds if the terms $m + m' \neq 0$, $n + n' \neq 0$ are retained.

We have thus reached the preliminary surprising result: Provided that $\sum_{m,n} |m| |\phi_{m,n}|^2 < \infty$, the diffusion of particles (in the case of strong island overlap) cannot be understood as a simple diffusive Wiener process with a time-independent diffusion constant.

The opposite conclusion, reached by Molvig et al. /14/, is based on the following error: On page 330 the correlation function $\langle [\delta y(t_1) - \delta y(t_1 - \tau)]^2 \rangle_W$ is mistakenly replaced by $\frac{2}{3} D \tau^3$. As a consequence the t_1 integral is trivially done and cancels the factor $1/t$, and $D = D_M$ is determined by the inconsistently obtained relation

$$D_M = \sum_{m,n} |m \phi_{m,n}|^2 \int_0^\infty d\tau e^{i(mx_0 + n)\tau - \frac{m^2 D_M}{3} \tau^3}, \quad (21)$$

the structure of which is analogous to eq.(2).

In /14/ the authors nevertheless present computational evidence that diffusion can be described with a finite diffusion coefficient in agreement with eqs.(14), (15) and (21). In the next section this discrepancy is analyzed and a final conclusion is reached on the diffusive nature of the particle motion.

4. Resolution of discrepancy

In /14/ numerical solutions are presented for both the discretized equations of motion (10) and the original equations (8). For the discretized case our proviso, $\sum_{m,n=-\infty}^{+\infty} |m| |\phi_{m,n}|^2$ finite, is violated since $\phi_{m,n} = \phi_m = \text{const}(n)$. For this case the derivation of D has to be reconsidered. This is done next, and, supported by numerical experiments very good agreement with a Wiener process with finite D is obtained. Since particle motion in the physically realistic case $\sum_{m,n} |m| |\phi_{m,n}|^2 < \infty$ should not be totally different from the degenerate case $\phi_{m,n} = \phi_m$, it is even more important to resolve the discrepancy with our previous result $D \rightarrow 0$ for large times. This is done at the end of this section.

For $\phi_{m,n} = \phi_m$ one obtains from eq.(19) with the help of eq.(9)

$$D = \pi \sum_m |m \phi_m|^2 + 2\pi \sum_m |m \phi_m|^2 \sum_{l>0} \frac{1}{l} \int_0^t dt_1 e^{\left[i m x_0 \tau - \frac{m^2 D \tau^2}{3} (3t_1 - 2\tau) \right]} \Big|_{\tau=2\pi l} \quad (22)$$

The first term comes from $l = 0$. (A factor 0.5 results from properly treating the τ integration at $\tau = 0$.) The $l > 0$ terms are easily seen to converge to zero for $t \gg \tau_K = (m^2 D)^{-1/3}$, so that for later times $D = D_0$, where

$$D_0 = \pi \sum_{m=-\infty}^{+\infty} m^2 |\phi_m|^2 \quad (23)$$

In /14/ the erroneously derived D_M was further evaluated for $\Delta\Omega \cdot \tau_K \gg 1$, where $\Delta\Omega$ ($=\Delta k_{||} v_{||}$) is the effective width of $\Omega = m\omega_0 + n$ as function of m and n . In this case the complex exponential function in eq.(21) is rapidly oscillating and may be replaced by a δ - function. With $\phi_{m,n} = \phi_m$ and the summation over n treated as an integral

$$D_M = \pi \sum_{m,n} m^2 |\phi_{m,n}|^2 \delta(m\omega_0 + n) \quad (24)$$

is found to agree with D_0 . For $\Delta\Omega \cdot \tau_K \ll 1$ see below.

In order to test the Wiener hypothesis we solved the recursions (10) numerically for N equally spaced initial values y_0 , $N \gg 1$, with phases ϕ_m chosen at random. Apart from the moments

$$R_0(t) = \langle [\delta y(t)]^2 \rangle, \quad S_0(t) = \langle \exp iM\delta y(t) \rangle, \quad (25)$$

as in /14/, where $\langle \rangle$ is the numerically obtained ensemble average, we also investigated the correlations

$$R_r = R(t, t'_r) = \langle [\delta y(t) - \delta y(t'_r)]^2 \rangle \quad (26)$$

$$S_r = S(t, t'_r) = \langle \exp iM [\delta y(t) - \delta y(t'_r)] \rangle$$

for $r = 1$ and 2 , where $\tau_r = t - t'_r = r \cdot \Delta t$. M is a typical mode number m , and $\Delta t = 2\pi$ is the interval between two δ - function pulses. The results are shown in Figures 1 - 3.

In Figs. 1 and 2 the numerically obtained values of R_0 , R_1 , R_2 are plotted as functions of t , for $t - t'$ fixed, as series of crosses. Each cross corresponds to one iteration. The solid curves are the analytic formulae (15), (16) for the same correlation functions in a Wiener process, with D given by eq.(23). The agreement is quite good. (It becomes even better if the arbitrary initial phase shift ΔT , mentioned in Sec. 2, is properly chosen. With $\Delta T = 2\pi$ the crosses are distributed about the same distance to the other side of the analytic curves). The essential result of the figures is that indeed $\langle [\delta y(t) - \delta y(t')]^2 \rangle$ is in fact not a function of $t - t'$ alone, contrary to assumptions in the literature /14/.

In Fig. 1 the time scale τ_K for randomization of the particle motions, $\tau_K^{-3} = M^2 D$, is large compared with the interval $\Delta t = 2\pi$ between two pulses: $\tau_K = 5.3 \Delta t$. In Fig. 2 it is the opposite: $\tau_K = 0.0029 \Delta t$. Obviously, the agreement between theory and numerical results does not depend much on $\tau_K / \Delta t$. The parameters in Fig. 1 were chosen as follows: $m_{\min} = 200$, $m_{\max} = 550$, $A = 2 |\phi_m| = 2 \cdot 10^{-9} = \text{const}$ for $m_{\min} \leq m \leq m_{\max}$, $\phi_m = 0$ otherwise, and 50 iterations. In Fig. 2 there are $m_{\min} = 365$, $m_{\max} = 385$, $A = 2 \cdot 10^{-5}$, and 100 iterations. For both sets of parameters the island overlap criterion /14/ is well satisfied. $N = 6000$ and $x_0 = 1 + 1.2/\pi$ in all figures.

In Fig. 3 the crosses are S_0 and $S_r = \langle \exp iM(\delta y - \delta y') \rangle$,

$r = 1$ and 2 , again for fixed $\tau_r = t - t'_r$, as functions of t , while the solid curves correspond to

$$\begin{aligned} \langle \exp iM \delta y \rangle_W &= \exp \left[-\frac{M^2}{3} D t^3 \right] \\ \langle \exp iM (\delta y - \delta y') \rangle_W &= \exp \left[-\frac{M^2}{3} D \tau_r^2 (3t - 2\tau_r) \right]. \end{aligned} \quad (27)$$

In addition, the dashed lines for comparison show the functions $\exp \left[-\frac{M^2}{3} D \tau_r^3 \right]$ as used in /14/. Again, the correlations are seen to be functions of both t and $\tau = t - t'$.

Figure 3 is a good test of the Gaussian nature of the particle diffusion. Even if the second order moments $\langle (\delta y - \delta y')^2 \rangle$ are in agreement with a Wiener process, the higher order cumulants /20/ could change eqs.(13) (with $x = \delta y$) so that the exponents on the right-hand sides would have, in addition, contributions from $\langle (\delta y)^n \rangle$, $\langle (\delta y - \delta y')^n \rangle$, $n > 2$. A Gaussian probability distribution is the only one for which these higher moments disappear from the exponents. The good agreement between the numerical results and the analytic Wiener curves in Fig. 3 therefore supports the Gaussian nature of the diffusion process.

The parameters in Fig. 3 are the same as in Fig. 1. For $\tau_K \ll \Delta t$ no corresponding figure is shown because already after the first iterations the numerical results are determined by noise, which becomes dominant after a few τ_K . Noise is due to the finite number of start positions. Round-off errors, which on the CRAY computer may be artificially enhanced, do not play a role

in Figs. 1 and 3. In Fig. 2, however, round-off errors accumulate somewhat erratically up to about 4 %.

With the foregoing results it is obvious that a Wiener type diffusion process is a valid model for the degenerate case $\phi_{m,n} = \phi_m$. It remains to understand why the numerical results /14/ for a mode spectrum with finite energy are also in agreement with the diffusion coefficient (24) when theory predicts that they should not.

From eq.(19) it is possible to obtain both an upper and a lower bound, D_+ and D_- , for D , with D_+ more detailed than in relation (20). The bounds follow from taking the maximum or the minimum of τ in the correlation term $3t_1 - 2\tau$ in eq.(19). With $c_+^2 = 1/3$, $c_-^2 = 1$, $\tau_K^{-3} = m^2 D_{\pm}$ there is, after a transformation of variables

$$D_{\pm} = \sum_{m,n} m^2 |\phi_{m,n}|^2 \tau_K \frac{1}{c_{\pm}^2 \tilde{t}} \int_0^t dt_1 \frac{1}{\sqrt{t_1}} \int_0^{t_1^{3/2} \frac{i\Omega \tau_K}{c_{\pm} \sqrt{t_1}} \tau} d\tau e^{-\tau^2} \quad (28)$$

where $\tilde{t} = t/\tau_K$ and $\Omega = m x_0 + n$. Let us define

$$t_c = \tau_K (\Delta\Omega \cdot \tau_K)^2. \quad (29)$$

Consider first the case $\Delta\Omega \cdot \tau_K \gg 1$. For $t \gg t_c$ one obtains approximately

$$D_{\pm} = \frac{1}{c_{\pm}} \sum_{m,n} m^2 |\phi_{m,n}|^2 \frac{\tau_K}{t} \int_0^t dt_1 \frac{1}{\sqrt{t_1}} \int_0^{\infty} d\tau e^{-\tau} \quad (30)$$

which is proportional to $t^{-0.5}$. This is the decline, as predicted in Sec. 3. For $\tau_K \ll t \ll t_c$, however, the complex exponential function in eq. (28) is rapidly oscillating and may be replaced by a δ -function in $\Omega\tau_K/c_{\pm}\sqrt{t_1}$, so that

$$D = D_{\pm} = \pi \sum_{m,n} m^2 |\phi_{m,n}|^2 \delta(m x_0 + n). \quad (31)$$

Thus, up to times of order $t_c \gg \tau_K$ the details of the correlation function $\langle [\delta y(t) - \delta y(t')]^2 \rangle$ do not matter: there is a finite time-independent diffusion constant (31), and it agrees with the erroneously derived one, eq.(24).

The interpretation of t_c is straight forward. For fixed m and a finite interval Δn of n values the region Δx where resonances are possible is given by $\Omega(\Delta x) = 0$, i.e. $\Delta x = \Delta n/m = \Delta\Omega/m$. According to eq.(14) the time $t_{\Delta x}$ for a particle to diffusively cross this region is determined by $(\Delta x)^2 = (\Delta\Omega/m)^2 = 2Dt_{\Delta x}$. With $\tau_K^{-3} = m^2 D$ it follows that $t_c \cong t_{\Delta x}$. Hence, for $t > t_c$ the particles have left the resonance layer of overlapping islands, and the assumed process of diffusion cannot take place any more. The numerical results in /14/ ($\Delta\Omega \cdot \tau_K \gg 1$ was satisfied) were obviously obtained for times that did not exceed t_c .

For $\Delta\Omega \cdot \tau_K \ll 1$ and $t \gtrsim \tau_K$ the $t^{-0.5}$ decline of eq.(30) is

again recovered. Owing to the strong diffusion the particles leave the resonant zone even before their motion has become random. No steady state diffusion exists. In contrast, eq.(21) yields a well defined but erroneous value $D_m \sim \left(\sum_{m,n} |m| |\phi_{m,n}|^2 \right)^{3/4}$.

We are thus led to the following final conclusions: Provided the particles stay long enough in the region of resonant island overlap the diffusion may be considered as a Wiener process, i.e. a process with Gaussian probability distribution where present events are independent of previous ones. Numerically obtained correlation functions support this conclusion. The diffusion coefficient itself is not a sensitive function of the details of the correlation functions.

Acknowledgement

The author is grateful to P. Gräff for helpful discussions concerning stochastic processes.

Appendix A

Let $x(t)$ be a symmetrical Wiener process as described by eqs.(11), (12). In order to obtain correlation functions for the integrated Wiener process $y(t) = \int_0^t dt' x(t')$ it is useful to determine $\langle x(t)x(t') \rangle_W$ first. Let $x = x(t)$, $x' = x(t')$ and $t > t' > 0$. It follows that

$$\begin{aligned} \langle xx' \rangle_W &= \langle x'(x - x') + x'^2 \rangle_W = \langle x'^2 \rangle_W \\ &= \frac{1}{\sqrt{4\pi Dt'}} \int_{-\infty}^{+\infty} dx' x'^2 e^{-\frac{x'^2}{4Dt'}} = 2Dt' \end{aligned}$$

where the first term did not contribute because both factors are independent and odd. For general t and t' we have

$$\langle x(t)x(t') \rangle_W = 2D \cdot \text{minimum}(t, t') \quad (\text{A.1})$$

From this we get

$$\begin{aligned} \langle y(t)y(t' \leq t) \rangle_W &= \int_0^t dt_1 \int_0^{t'} dt_2 \langle x(t_1)x(t_2) \rangle_W \\ &= 2D \int_0^{t'} dt_2 \int_0^t dt_1 \min(t_1, t_2) \quad (\text{A.2}) \\ &= 2D \int_0^{t'} dt_2 \left[\int_0^{t_2} dt_1 t_1 + \int_{t_2}^t dt_1 t_2 \right] = D \int_0^{t'} dt_2 (-t_2^2 + 2t_2 t) \\ &= \frac{1}{3} D t'^2 (3t - t') \end{aligned}$$

For $t' = t$ one obtains eq.(15) for $\langle y^2 \rangle_W$. Also, repeated application of eq.(A.2) yields

$$\begin{aligned} \langle [my(t) + m'y(t' \leq t)]^2 \rangle_W &= \\ &= \frac{2}{3} D [(m+m')^2 t^3 + m'(m+m')\tau(\tau^2 - 3t^2) + 3m'^2 \tau^2 t - 2m'^2 \tau^3] \\ &= \frac{2}{3} D [(m+m')^2 t'^3 + 3m(m+m')t'^2 \tau + 3m^2 t' \tau^2 + m^2 \tau^3] \end{aligned} \quad (A.3)$$

with $\tau = t - t'$. For $m + m' = 0$ one obtains eqs.(16). The last expression (A.3) is particularly useful for application in Appendix B.

Appendix B

In order to give an upper bound on D , for general m, m' and n, n' one needs a lower bound on $K = 2DK/3 = \langle [m\delta y(t_1) + m'\delta y(t_2 \leq t_1)]^2 \rangle_W$. We minimize K with respect to $\mu = m + m'$ at fixed m . According to eq.(A.3) K is of the form

$$K(\mu) = a\mu^2 + b\mu + c$$

with $a = t_2^3$, $b = 3mt_2^2 \tau$, $c = 3m^2 t_2 \tau^2 + m^2 \tau^3$, $\tau = t_1 - t_2 \geq 0$.

The minimum occurs at $\mu_0 = -0.5b/a$ and its value is

$$K(\mu_0) = -\frac{b^2}{4a} + c = \frac{m^2 \tau^2}{4} (3t_1 + \tau) .$$

Hence, a lower bound is given by

$$K \geq K(\mu_0) \geq \frac{3}{4} m^2 \tau^2 t_1 .$$

The same consideration that led from eq.(19) to eq.(20) applies again, so that $D \rightarrow 0$ for $t \rightarrow \infty$ results for general m, n and m', n' .

Figure captions

Fig. 1: Time evolution of moment R_0 and correlation functions R_1, R_2 , eqs.(25), (26). Crosses: numerical solution, solid curves: analytic Wiener process. Kolmogorov time $\tau_K \gg$ iteration period.

Fig. 2: Same as Fig. 1 but $\tau_K \ll$ iteration period.

Fig. 3: Time evolution of moment S_0 and correlation functions S_1, S_2 , eqs.(25), (26). Dashed lines: S_1, S_2 according to Ref. /14/. Parameters as in Fig. 1.

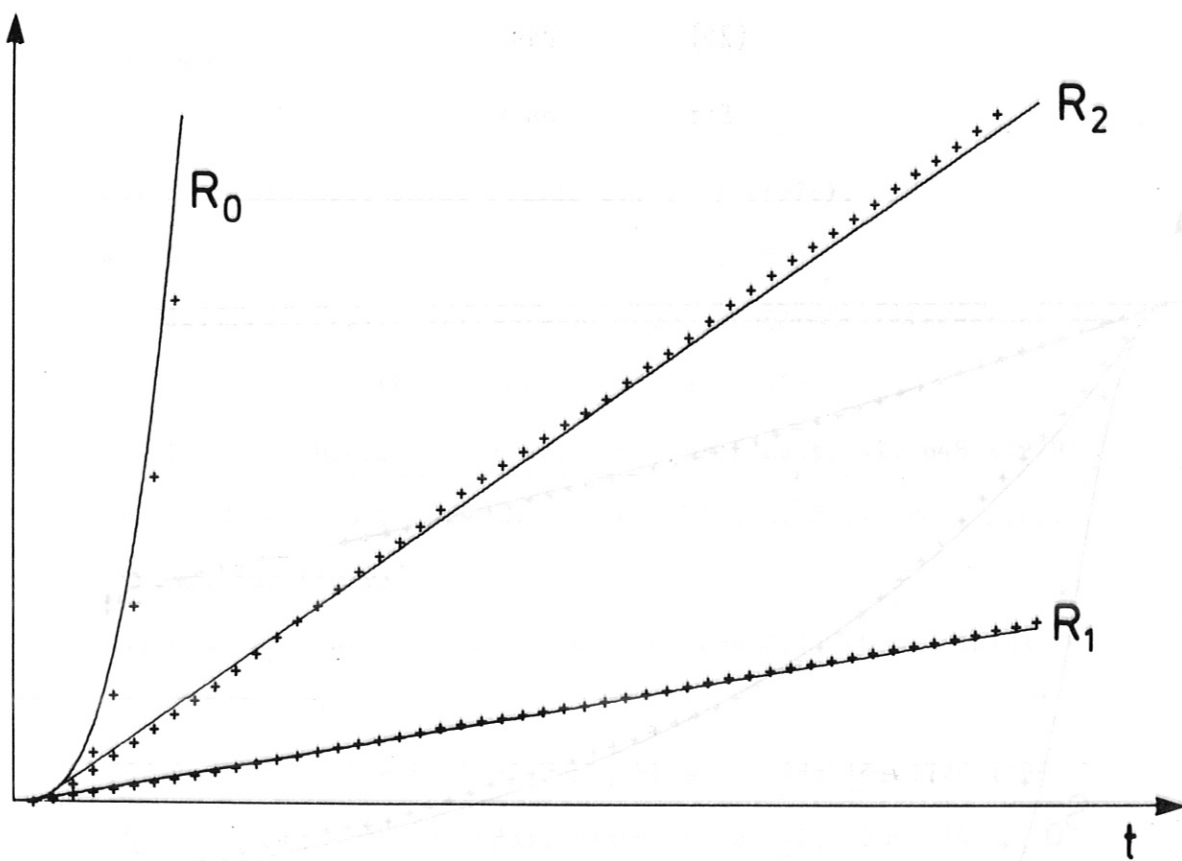


Figure 1

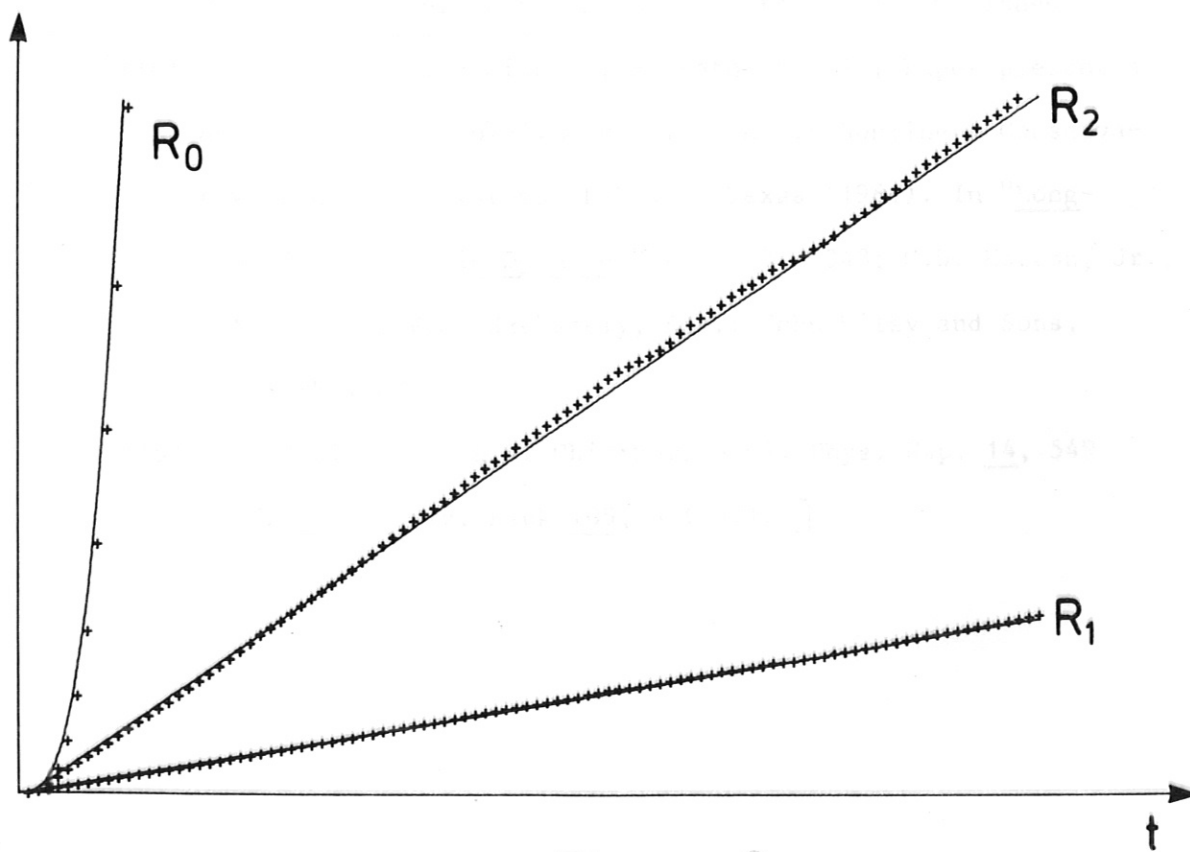


Figure 2

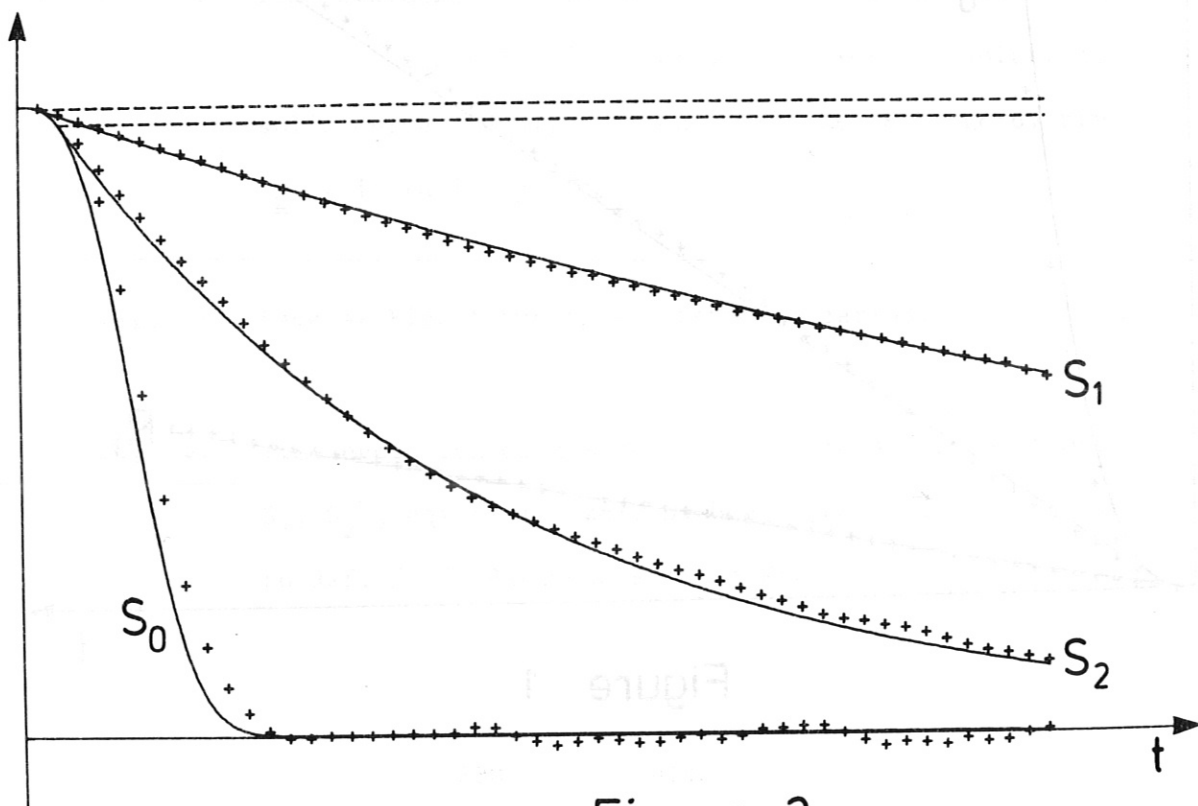


Figure 3

References

- /1/ E. Mazzucato, Phys. Fluids 21, 1063 (1978).
- /2/ TFR - Group, Plasma Phys. 25, 641 (1983)
- /3/ A. Rogister, G. Hasselberg, Phys. Fluids 26, 1467 (1983)
- /4/ T.H. Dupree, Phys. Fluids 9, 1773 (1966)
- /5/ S.P. Hirshman, K. Molvig, Phys. Rev. Lett. 42, 648 (1979)
- /6/ K. Molvig, S.P. Hirshman, J.C. Whitson, Phys. Rev. Lett. 43, 582 (1979)
- /7/ C.O. Beasley, Jr., K. Molvig, W. van Rij, Phys. Fluids 26, 678 (1983)
- /8/ S.A. Orszag, R.H. Kraichnan, Phys. Fluids 10, 1720 (1967)
- /9/ D.F. DuBois, M. Espedal, Plasma Phys. 20, 1209 (1978)
- /10/ P.H. Diamond, M.N. Rosenbluth, Phys. Fluids 24, 1641 (1981)
- /11/ T.H. Dupree, D.J. Tetreault, Phys. Fluids 21, 425 (1978)
- /12/ J.D. Meiss, W. Horton, Phys. Fluids 25, 1838 (1982)
- /13/ H. Tasso, K. Lerbinger, Phys. Letters, to be published
- /14/ K. Molvig, J.P. Freidberg, R. Potok et al., Paper presented at Workshop on Long-Time Predictions in Nonlinear Conservative Dynamical Systems, Lakeway, Texas (1981). In "Long-Time Prediction in Dynamics" pp. 319 - 343; C.W. Horton, Jr., L.E. Reichl, V.G. Szebehely, Eds.; John Wiley and Sons, New York 1983.
- /15/ G.M. Zaslavskii, B.V. Chirikov, Sovj. Phys. Usp. 14, 549 (1972) [Usp. Fiz. Nauk 105, 3 (1971)]

- /16/ M.N. Rosenbluth, R.Z. Sagdeev, J.B. Taylor, G.M. Zaslavskii,
Nucl. Fusion 6, 297 (1966)
- /17/ A.B. Rechester, M.N. Rosenbluth, Phys. Rev. Lett. 40, 38
(1978)
- /18/ A.B. Rechester, M.N. Rosenbluth, R.B. White, Phys. Rev. Lett.
42, 1247 (1979)
- /19/ I.I. Gihman, A.V. Skorohod, "Stochastic Differential Equa-
tions", Springer Verlag, Berlin 1972
- /20/ E. Parzen, "Modern Probability Theory and its Applications",
J. Wiley and Sons, Inc., New York 1967