NEW VARIATIONAL FORMULATION OF MAXWELL - VLASOV AND GUIDING CENTER THEORIES

LOCAL CHARGE AND ENERGY CONSERVATION LAWS

D. Pfirsch

IPP 6/226 July 1983

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*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

Abstract:

A new variational formulation of Maxwell-Vlasov and related theories is given in terms of a common Lagrangian density for both the "Vlasov particles" and the Maxwell fields. This formulation is used to derive in a consistent way, on the one hand, correct charge and current densities and, on the other, corresponding energy and energy flux densities. All of these densities generally show in addition to particle like contributions electric polarization and magnetization terms. By some limiting procedure collisionless guiding center theories with polarization drifts included are also treated. In this way local energy conservation laws are formulated for such theories, which has not been possible up to now.

## INTRODUCTION

Systems allowing a Lagrangian formulation of the equations describing them often show such properties as conservation laws which are not present in other systems. Moreover, the derivation of, say, such conservation laws is often facilitated quite a bit when a variational formulation is used instead of the explicit form of the underlying equations. For the usual Vlasov-Maxwell theory F.E. Low once gave a variational formulation [1]. His Lagrangian for the Vlasov part was based on the particle equations of motion in the usual Lagrange picture, i.e. with the position $\underline{x}$ of a particle being a dependent variable $\underline{x}$ (t; $\underline{x}_1$, $\underline{v}_1$), where $\underline{x}_1$, $\underline{v}_1$ represent initial values of $\underline{x}$ and $\dot{\underline{x}}$. On the other hand, the Maxwellian part is described in a Eulerian picture with $\underline{x}$ being an independent variable. These two different pictures cause some difficulties in applying Low's Lagrangian formulation. In this paper the Vlasov part will also be based on a Eulerian picture of the particle motion which is provided by the Hamilton-Jacobi theory. This will be done for rather general Vlasov-like theories, which also allows collisionless kinetic guiding center theories to be treated. The latter will be based on a Lagrangian for guiding center motions in a Lagrange picture given by H.K. Wimmel [2] from which Littlejohn's guiding center mechanics, with the polarization drift included [3], can be rederived. In order to allow the use of a Hamilton-Jacobi theory for this case, a limiting procedure has to be applied, starting with a somewhat modified Lagrangian allowing a Hamiltonian.

The new formulation will be used in order to obtain correct expressions for charge and current densities as well as for corresponding energy and momentum densities and their flux densities for the stated general class of Vlasov-like theories. All these expressions will eventually be given in terms of usual quantities such as velocities, magnetic moments etc. They generally also show in addition to particle like contributions electric polarization and magnetization terms. The latter two are, of course, identical to zero for the usual particle Vlasov theory. They are, however, essential for kinetic guiding center theories, and one important result of this paper is that for these theories correct charge and current densities are derived, leading in a consistent way to local energy conservation laws, a result which had not yet been obtained in the case with polarization drifts /2/ . It will also be shown that these charge and current densities fulfill certain criteria resulting from an exact Vlasov theory, such as a vanishing current density in time-independent but inhomogeneous magnetic fields for "isotropic" guiding center distribution functions. Drift motions are thus exactly compensated by magnetization effects in this case.

The paper is organized as follows: In Sec. I the Hamilton-Jacobi theory is reviewed in a way suited to its later application. In Sec. II Lagrangians for Vlasov like theories are introduced. These are specialized in Sec. III to ones that allow coupling to the electromagnetic field. First expressions for charge and current densities are obtained there. In Sec. IV energy and energy flux densities are derived. All the densities are expressed by conventional variables in Sec. V. Section VI presents an application to the usual Vlasov-Maxwell theory yielding the already known expressions, and in Sec. VII kinetic guiding center Maxwell theories are treated for which self-consistent expressions for the various densities are given for the first time.

## I. Hamilton-Jacobi theory

In order to allow the inclusion of systems more general than the usual Vlasov theory, especially kinetic guiding center theories, we consider a more general space than the normal $\underline{x}$ - space with position vectors

$$\underline{y} = (y_1, \ldots, y_n) = (\underline{y}_1 , \underline{y}_2) \quad . \tag{1}$$

$$\underline{y}_1 = (y_1, y_2, y_3) = \underline{x} \tag{2}$$

is the position vector in normal $\underline{x}$ - space. $\underline{y}_2$ is assumed to be of dimension $n_2$. Corresponding to $\underline{y}$ we need variables

$$\underline{\alpha} = (\alpha_1, \ldots., \alpha_n) \tag{3}$$

such that we can define a function

$$S = S(\underline{y}, \underline{\alpha}, t) \tag{4}$$

for each particle species to be a complete solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(\underline{y}, \frac{\partial S}{\partial \underline{y}}, t) = 0 , \tag{5}$$

where $H(\underline{y}, \underline{P}, t)$ is the Hamiltonian for the "particles" as a function of the canonical variables $\underline{y}$ and $\underline{P}$.

The following relations hold:

$$\frac{\partial S}{\partial \underline{y}} = \underline{P}(\underline{y}, \underline{\alpha}, t) , \quad \frac{\partial H}{\partial \frac{\partial S}{\partial \underline{y}}} = \underline{V}(\underline{y}, \underline{\alpha}, t) = \frac{d}{dt}\underline{y} ,$$

$$(\frac{\partial}{\partial t} + \underline{V} \cdot \frac{\partial}{\partial \underline{y}}) \frac{\partial S}{\partial \underline{\alpha}} = 0 , \tag{6}$$

where $\underline{V}$ is the velocity of "particles" in $\underline{y}$ space and $\partial S/\partial \underline{\alpha}$ appears to be a constant of the motion. A general constant of the motion is a function of $\underline{\alpha}$ and $\frac{\partial S}{\partial \underline{\alpha}}$. Furthermore, one can form the Van Vleck determinant

$$w = \left\| \frac{\partial^2 S}{\partial \alpha_i \, \partial y_k} \right\| = w\,(\underline{y},\ \underline{\alpha},\ t), \tag{7}$$

which can be shown to satisfy the continuity equation /4/

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \underline{y}} \cdot (w\,\underline{V}) = 0\ . \tag{8}$$

## II. Lagrangians for Vlasov-like equations

If used as a Lagrangian, the following functional of $S(\underline{y},\underline{\alpha},t)$ and an additional function $\phi(\underline{y},\underline{\alpha},t)$ has the property of yielding Vlasov-like equations:

$$L_V\ (t) = - \int d^n y\ d^n \alpha\ (\frac{\partial S}{\partial t} + H\ (\underline{y},\ \frac{\partial S}{\partial \underline{y}},\ t))\ \phi\ (\underline{y},\ \underline{\alpha},\ t). \tag{9}$$

Hamilton's principle

$$\delta \int_{t_1}^{t_2} L_V\ (t)\ dt = 0 \quad \text{with} \quad \delta S = \delta\phi = 0 \text{ at } t_1,\ t_2 \tag{10}$$

together with the assumption that certain partial integrations over $\underline{y}$ can be performed with vanishing contributions from boundaries gives by variation of $\phi$

$$\frac{\partial S}{\partial t} + H\ (\underline{y},\ \frac{\partial S}{\partial \underline{y}},\ t) = 0 \tag{5}$$

and by variation of S

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial \underline{y}} \cdot (\underline{V}\phi) = 0 \quad . \tag{11}$$

If we define

$$\tilde{f}\,(\underline{y}, \underline{\alpha}, t) \equiv \frac{\phi}{w} \, , \tag{12}$$

with $w$ given by eq.(7), then from eq. (8)

$$\frac{\partial \tilde{f}}{\partial t} + \underline{V} \cdot \frac{\partial \tilde{f}}{\partial \underline{y}} = 0 \; . \tag{13}$$

Thus $\tilde{f}$ is a constant of the motion:

$$\tilde{f}\,(\underline{y}, \underline{\alpha}, t) = \hat{f}\,(\underline{\alpha}, \frac{\partial S}{\partial \underline{\alpha}})$$

and therefore

$$\phi = w\, \hat{f}(\underline{\alpha}, \frac{\partial S}{\partial \underline{\alpha}}) \quad . \tag{14}$$

Since $\hat{f}$ is a constant of the motion, it satisfies the "Vlasov" equation when $\underline{\alpha}$ is replaced by $\underline{P}$ via $\underline{P} = \partial S/\partial \underline{y}$ , and it will in fact turn out to be the distribution function for the particle species considered or at least to be closely related to it in the case of guiding center theories. In the case of H being an N particle Hamiltonian $\hat{f}$ is a solution of Liouville's equation. This case will, however, not be considered here.

## III. Coupling to the electromagnetic field

In the following $\nu$ denotes the particle species with charge $e_\nu$ and mass $m_\nu$. For the coupling to the electromagnetic field we have to restrict the "particle" Hamiltonians to correspond to gauge invariant theories, i.e.

$$H_\nu = e_\nu \, \Phi \, (\underline{x}, t) + \hat{H}_\nu \, (\underline{p} - \frac{e_\nu}{c} \underline{A} \, (\underline{x},t) \, , \underline{P}_2, \underline{E}, \underline{B}) \, , \tag{15}$$

where

$$\begin{aligned} &\underline{x} = \underline{y}_1 \quad , \quad \underline{p} = \underline{P}_1 = \text{ canonical conjugate to } \underline{x} \\ &\underline{P}_2 = \text{ canonical conjugate to } \underline{y}_2 \, . \end{aligned} \tag{16}$$

Furthermore, we have

$$\underline{v}_\nu = \frac{\partial H_\nu}{\partial \underline{p}} = \frac{\partial H_\nu}{\partial \underline{P}_1} = \underline{V}_1 \, , \, \underline{V}_2 = \frac{\partial H_\nu}{\partial \underline{P}_2} \, . \tag{17}$$

In the usual Vlasov theory $\hat{H}_\nu$ does not depend on $\underline{P}_2$, $\underline{E}$, $\underline{B}$. $\underline{E}$ and $\underline{B}$ follow from $\Phi$ and $\underline{A}$:

$$\underline{E} = - \frac{1}{c} \frac{\partial \underline{A}}{\partial t} - \nabla \Phi, \; \underline{B} = \text{curl } \underline{A}. \tag{18}$$

The Lagrange density for the vacuum fields is

$$\mathcal{L}_M = \frac{1}{8\pi} (\underline{E}^2 - \underline{B}^2), \tag{19}$$

and we take as the Lagrangian for the coupled Maxwell-"Vlasov" system

$$L(t) = L_V + L_M =$$

$$= - \sum_\nu \int d^n\alpha \int d^n y \left(\frac{\partial S_\nu}{\partial t} + e_\nu \Phi + \hat{H}_\nu \left( \frac{\partial S_\nu}{\partial \underline{x}} - \frac{e_\nu}{c} \underline{A}, \frac{\partial S_\nu}{\partial \underline{y}_2}, \underline{E}, \underline{B}\right)\right) \phi_\nu$$

$$+ \int d^3x \frac{1}{8\pi} (\underline{E}^2 - \underline{B}^2) . \tag{20}$$

Using this function in Hamilton's principle, we obtain by variation of

$$\phi_\nu: \quad \frac{\partial S_\nu}{\partial t} + e_\nu \Phi + \hat{H}_\nu = 0 , \tag{21}$$

$$S_\nu: \quad \phi_\nu = w_\nu \hat{f}_\nu \left(\underline{\alpha}, \frac{\partial S_\nu}{\partial \underline{\alpha}}\right), \tag{22}$$

$$\Phi : \quad \frac{1}{4\pi} \operatorname{div} \underline{E} = \sum_\nu \int d^n\alpha \, d^{n_2} y_2 \, w_\nu \hat{f}_\nu e_\nu$$

$$+ \operatorname{div} \left[\sum_\nu \int d^n\alpha \, d^{n_2} y_2 \, w_\nu \hat{f}_\nu \frac{\partial \hat{H}_\nu}{\partial \underline{E}}\right] \tag{23}$$

$$= \rho = \text{charge density} ,$$

$$\underline{A}: \frac{c}{4\pi}\left(\text{curl } \underline{B} - \frac{1\partial}{c\partial t}\underline{E}\right) =$$

$$\sum_\nu \int d^n\alpha \, d^n 2y_2 \, w_\nu \, \hat{f}_\nu \, e_\nu \, \underline{v}_\nu$$

$$- \frac{\partial}{\partial t} \sum_\nu \int d^n\alpha \, d^n 2y_2 \, w_\nu \, \hat{f}_\nu \, \frac{\partial \hat{H}_\nu}{\partial \underline{E}} \tag{24}$$

$$- c \text{ curl} \sum_\nu \int d^n\alpha \, d^n 2y_2 \, w_\nu \, \hat{f}_\nu \, \frac{\partial \hat{H}_\nu}{\partial \underline{B}}$$

$$= \underline{j} = \text{current density} \quad .$$

Equations (23) and (24) are the inhomogeneous Maxwellian equations with expressions for the charge and current densities on the r.h.s. The current density - and correspondingly the charge density - shows three different contributions: "particle", electric polarization and magnetization contributions (the latter without a counterpart in the charge density). The "particle" contribution obeys by itself a conservation law which is a consequence of the gauge invariance of the theory expressed by the combinations $\frac{\partial S_\nu}{\partial t} + e_\nu \Phi$ and $\frac{\partial S_\nu}{\partial \underline{x}} - \frac{e_\nu}{c} \underline{A}$. The electric polarization and magnetization contributions also separately obey conservation laws. There exists therefore some ambiguity in defining such densities. The fact that unique expressions are obtained for the charge and current densities is a consequence of deriving them from the variational principle. It is only these expressions which can be consistent with a local conservation law for the energy which will be derived in the next section. A further simplification of eqs. (23) and (24) is obtained in Sec. V.

## IV. Energy and energy flux densities

In order to derive expressions for these quantities, we first introduce

$$z \equiv (t,\ \underline{y}) = (z_o,\ z_1,\ \ldots.,\ z_n),\ \underline{z_a} = (t,\underline{x})\ ; \tag{25}$$

$$\beta = (\nu,\ \underline{\alpha})\ ,\quad \sum_{\beta} \ldots. \equiv \sum_{\nu} \int d^n\alpha\ \ldots.,$$

$$\psi_\beta^{\ 1}(\underline{z}) = \phi_\nu\ (\underline{y},\ \underline{\alpha},\ t),$$

$$\psi_\beta^{\ 2}(\underline{z}) = S_\nu\ (\underline{y},\ \underline{\alpha},\ t), \tag{26}$$

$$\Phi_\mu(\underline{z_a}) = (\Phi,\ \underline{A}) \qquad , \qquad \mu = 0,1,2,3\ .$$

We then define with

$$\mathcal{L}_\beta = -\ \phi_\nu\ \left(\frac{\partial S_\nu}{\partial t} + e_\nu \Phi + \hat{H}_\nu\ \left(\frac{\partial S_\nu}{\partial \underline{x}} - \frac{e_\nu}{c}\underline{A},\ \frac{\partial S_\nu}{\partial \underline{y}_2},\ \underline{E},\ \underline{B}\right)\right)$$

$$= \quad {}_\beta\ \left(\psi_\beta^{\ i}\ ,\ \frac{\partial \psi \beta^i}{\partial \underline{z}}\ ,\ i = 1,2;\ \Phi_\mu,\ \frac{\partial \Phi_\mu}{\partial \underline{z}_a},\ \mu = 0,1,2,3\right)$$

$$\mathcal{L}_M = \frac{1}{8\pi}\ (\underline{E}^2 - \underline{B}^2) = \mathcal{L}_M\ \left(\frac{\partial \Phi_\mu}{\partial \underline{z}_a}\ ,\ \mu = 0,1,2,3\right) \tag{27}$$

$$\mathcal{L} = \sum_\beta \int \mathcal{L}_\beta\ d^{n_2} y_2 + \mathcal{L}_M$$

a canonical tensor

$$\Theta_{kl} = \sum_{\mu=0}^{3} \frac{\partial \Phi_\mu}{\partial z_{ak}} \frac{\partial \mathcal{L}}{\partial \frac{\partial \Phi_\mu}{\partial z_{al}}} + \sum_\beta \sum_{i=1}^{2} \int d^{n_2} y_2 \frac{\partial \psi_\beta^{\ i}}{\partial z_k} \frac{\partial \mathcal{L}_\beta}{\partial \frac{\partial \psi_\beta^i}{\partial z_l}} \tag{28}$$

$$- \delta_{lk} \mathcal{L} ; \quad l,k = 0,1,2,3 \ .$$

With the summation convention applied we have

$$\frac{\partial \Theta_{kl}}{\partial z_{al}} = \sum_{\mu=0}^{3} \frac{\partial}{\partial z_{al}} \left( \frac{\partial \Phi_\mu}{\partial z_{ak}} \frac{\partial \mathcal{L}}{\partial \frac{\partial \Phi_\mu}{\partial z_{al}}} \right) + \sum_\beta \sum_{i=1}^{2} \int d^{n_2} y_2 \frac{\partial}{\partial z_\lambda} \left( \frac{\partial \psi_\beta^{\ i}}{\partial z_k} \frac{\partial \mathcal{L}_\beta}{\partial \frac{\partial \psi_\beta^{\ i}}{\partial z_\lambda}} \right)$$

$$- \frac{\partial \mathcal{L}}{\partial z_{ak}} \ . \tag{29}$$

Because of the integration over $\underline{y}_2$ in the second term of the r.h.s. the summation over $\lambda$ could be extended to go from 0 to n instead of just from 0 to 3.

The Euler-Lagrange equations are

$$\frac{\partial}{\partial z_\lambda} \frac{\partial \mathcal{L}_\beta}{\partial \frac{\partial \psi_\beta^{\ i}}{\partial z_\lambda}} - \frac{\partial \mathcal{L}_\beta}{\partial \psi_\beta^{\ i}} = 0 \quad , \quad \text{all } \beta \ , \ i = 1,2$$

$$\frac{\partial}{\partial z_{ak}} \frac{\partial \mathcal{L}}{\partial \frac{\partial \Phi_\mu}{\partial z_{ak}}} - \frac{\partial \mathcal{L}}{\partial \Phi_\mu} = 0 \quad , \quad \mu = 0,1,2,3 \ . \tag{30}$$

Furthermore, we have

$$\frac{\partial \mathcal{L}}{\partial z_{ak}} = \sum_{\mu=0}^{3} \left( \frac{\partial \Phi_\mu}{\partial z_{ak}} \frac{\partial \mathcal{L}}{\partial \Phi_\mu} + \frac{\partial^2 \Phi_\mu}{\partial z_{ak}\, \partial z_{al}} \frac{\partial \mathcal{L}}{\partial \frac{\partial \Phi_\mu}{\partial z_{al}}} \right) \tag{31}$$

$$+ \sum_\beta \sum_{i=1}^{2} \int d^{n_2} y_2 \left( \frac{\partial \psi_\beta{}^i}{\partial z_k} \frac{\partial \mathcal{L}_\beta}{\partial \psi_\beta{}^i} + \frac{\partial^2 \psi_\beta{}^i}{\partial z_k\, \partial z_\lambda} \frac{\partial \mathcal{L}_\beta}{\partial \frac{\partial \psi_\beta{}^i}{\partial z_\lambda}} \right) .$$

Using eqs. (30) and (31) in eq. (29), we obtain in a manner similar to that usual in field theory

$$\frac{\partial \Theta_{kl}}{\partial z_{al}} = \frac{\partial \Theta_{ko}}{\partial t} + \frac{\partial \Theta_{ki}}{\partial x_i} = 0 \tag{32}$$

This - alsmost - constitutes the local conservation law for energy and momentum; almost, because $\Theta_{kl}$ is not symmetric.

In the following only energy conservation which is related to $\Theta_{01}$, will be considered. The "Vlasov" contributions to $\Theta_{00}$ are

$$\Theta_{00}^{V} = \sum_\beta \sum_{i=1}^{2} \int d^{n_2} y_2 \frac{\partial \psi_\beta{}^i}{\partial t} \frac{\partial \mathcal{L}_\beta}{\partial \frac{\partial \psi_\beta{}^i}{\partial t}} - \sum_\beta \int d^{n_2} y_2 \mathcal{L}_\beta$$

$$= \sum_\nu \int d^n\alpha\, d^{n_2} y_2 \frac{\partial S_\nu}{\partial t} \frac{\partial \mathcal{L}_\beta}{\partial \frac{\partial S_\nu}{\partial t}} - \sum_\nu \int d^n\alpha\, d^{n_2} y_2 \mathcal{L}_\beta$$

$$= \sum_\nu \int d^n\alpha\, d^{n_2} y_2\, H_\nu\, \phi_\nu$$

$$= \sum_\nu \int d^n\alpha\, d^{n_2} y_2\, w_\nu\, \hat{f}_\nu\, H_\nu \quad . \tag{33}$$

In a similar way, one obtains

$$(\Theta^{V}_{0i}) = \sum_{\nu} \int d^n\alpha \int d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; H_\nu \; \mathbf{v}_\nu \; . \tag{34}$$

The full expressions for $\Theta_{00}$, $\Theta_{0i}$ are rather complicated:

$$\begin{aligned}
\Theta_{00} = & \sum_{\nu} \int d^n\alpha \int d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; (e_\nu \; \Phi + \hat{H}_\nu - \mathbf{E} \cdot \frac{\partial \hat{H}_\nu}{\partial \mathbf{E}}) \\
& + \Phi \; \mathrm{div} \left[ \sum_{\nu} \int d^n\alpha \int d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; \frac{\partial H_\nu}{\partial \mathbf{E}} \right] \\
& - \rho \; \Phi + \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \\
& + \mathrm{div} \left[ \Phi \left( \frac{1}{4\pi} \mathbf{E} - \sum_{\nu} \int d^n\alpha \; d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; \frac{\partial \hat{H}_\nu}{\partial \mathbf{E}} \right) \right],
\end{aligned} \tag{35}$$

$$\begin{aligned}
(\Theta_{0i}) = & \sum_{\nu} \int d^n\alpha \; d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; (e_\nu \; \Phi + \hat{H}_\nu) \; \mathbf{v}_\nu \\
& + \frac{\partial \Phi}{\partial t} \left[ \sum_{\nu} \int d^n\alpha \; d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; \frac{\partial \hat{H}_\nu}{\partial \mathbf{E}} - \frac{1}{4\pi} \mathbf{E} \right] \\
& - c \sum_{\nu} \int d^n\alpha \; d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; \frac{\partial \hat{H}_\nu}{\partial \mathbf{B}} \times \mathbf{E} \\
& - c \sum_{\nu} \int d^n\alpha \; d^{n2}y_2 \; w_\nu \; \hat{f}_\nu \; \frac{\partial \hat{H}_\nu}{\partial \mathbf{B}} \times \nabla\Phi \\
& + \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} - \frac{c}{4\pi} \mathbf{B} \times \nabla\Phi \; .
\end{aligned} \tag{36}$$

These expressions have the property that all non-gauge-invariant terms cancel in the continuity equation

$$\frac{\partial \Theta_{00}}{\partial t} + \frac{\partial \Theta_{0i}}{\partial x_i} = 0$$

and can therefore be disregarded. What is left can then indeed be interpreted as energy density $\varepsilon$ and energy flux density $\underline{\eta}$:

$$\varepsilon = \sum_\nu \int d^n\alpha\, d^{n2}y_2\, w_\nu\, \hat{f}_\nu\, (\hat{H}_\nu - \underline{E} \cdot \frac{\partial \hat{H}_\nu}{\partial \underline{E}}) + \frac{1}{8\pi} (\underline{E}^2 + \underline{B}^2) , \qquad (37)$$

$$\underline{\eta} = \sum_\nu \int d^n\alpha\, d^{n2}y_2\, w_\nu\, \hat{f}_\nu\, (\hat{H}_\nu \underline{v}_\nu + c\, \underline{E} \times \frac{\partial \hat{H}_\nu}{\partial \underline{B}}) + \frac{c}{4\pi} \underline{E} \times \underline{B}. \qquad (38)$$

Like charge and current densities, these expressions show "particle" electric polarization and magnetization contributions. In the following we get a further simplification of these expressions as well as of the expressions for the charge and current densities.

## V. Integration over the additional coordinates $\underline{y}_2$

From eq. (15) we have

$$\frac{\partial H_\nu}{\partial \underline{y}_2} = 0 \qquad (39)$$

and therefore

$$\underline{P}_2 = \text{const.} = \underline{\alpha}_2 \ . \tag{40}$$

We can thus write

$$\underline{\alpha} = (\underline{\alpha}_1, \underline{\alpha}_2) = (\underline{\alpha}_1, \underline{P}_2) \ . \tag{41}$$

The function $S_\nu$ separates into

$$S_\nu = S_{\nu 1} + S_{\nu 2} \ , \tag{42}$$

with

$$S_{\nu 2} = \underline{\alpha}_2 \cdot \underline{y}_2 \ , \quad \frac{\partial S_{\nu 1}}{\partial \underline{y}_2} = 0. \tag{43}$$

From this it follows that

$$\frac{\partial^2 S_\nu}{\partial \alpha_i \ \partial y_{2k}} = \delta_{ik} \tag{44}$$

and therefore

$$w_\nu = \left\| \frac{\partial^2 S_{\nu 1}}{\partial \alpha_{1i} \ \partial x_k} \right\| , \tag{45}$$

in which i,k only assume the values 1,2,3. A consequence of eq. (45)

is that

$$\frac{\partial w_\nu}{\partial \underline{y}_2} = 0. \tag{46}$$

The integrands of $\rho$, $\underline{j}$, $\varepsilon$, $\underline{\eta}$ therefore only depend on $y_2$ via $\hat{f}_\nu$ $(\underline{\alpha}, \partial S_\nu/\partial\underline{\alpha})$ through $\partial S_\nu/\partial\underline{\alpha}_2 = \underline{y}_2$ + vector independent of $\underline{y}_2$. The result of the $\underline{y}_2$ integrations is therefore

$$\int d^{n2}y_2\ \hat{f}_\nu\ (\underline{\alpha}, \frac{\partial S_\nu}{\partial \underline{\alpha}}) = \overline{f}_\nu\ (\underline{\alpha}, \frac{\partial S_{\nu 1}}{\partial \underline{\alpha}}) , \tag{47}$$

which is a constant of motion in $\underline{x}$-space. We can now replace the $\underline{\alpha}_1$ integration by a $\underline{p}$ integration by using $\underline{p} = \partial S_{\nu 1}/\partial \underline{x}$. This gives

$$d^3p = \left\| \frac{\partial^2 S_{\nu 1}}{\partial \alpha_{1k}\ \partial x_i} \right\| d^3\alpha_1 = w_\nu\ d^3\alpha_1 \tag{48}$$

and

$$\overline{f}_\nu\ (\underline{\alpha}, \frac{\partial S_{\nu 1}}{\partial \underline{\alpha}}) = f_\nu(\underline{x}, \underline{p}, t; \underline{\alpha}_2) , \tag{49}$$

which is now a solution of the "Vlasov" equation in x, $\underline{p}$ - space:

$$\frac{\partial f_\nu}{\partial t} + [H_\nu, f_\nu] = 0 . \tag{50}$$

where the brackets are Poisson brackets.

With all this we can now replace on the expressions for $\rho$, $\underline{j}$, $\varepsilon$ $\underline{\eta}$

$$\int d^3\alpha_1 \, d^{n2}y_2 \, w_\nu \, \hat{f}_\nu \, .... \text{ by } \int d^3p \, f_\nu \, ......, \tag{51}$$

from which the interpretation of $f_\nu$ as distribution function also becomes evident. This is the final general result of this paper. In the next sections these results are applied to the usual Vlasov case and to the case of kinetic guiding center theories.

## VI. The usual Vlasov-Maxwell theory

Here we have $\underline{y} = \underline{x}$, $\underline{\alpha} = \underline{\alpha}_1$ ,

$$\int d^{n2}\alpha_2 ... \rightarrow 1 \, .... \, , \tag{52}$$

$$\frac{\partial H_\nu}{\partial \underline{E}} = \frac{\partial H_\nu}{\partial \underline{B}} = 0. \tag{53}$$

Because of eq. (53) only particle contributions remain, and we find the well-known expressions

$$\rho = \sum_\nu \int d^3p \, f_\nu \, (\underline{x}, \underline{p}, t) \, e_\nu \, , \tag{54}$$

$$\underline{j} = \sum_\nu \int d^3p \, f_\nu \, (\underline{x}, \underline{p}, t) \, e_\nu \, \underline{v_\nu} \, (\underline{x}, \underline{p}, t) \, ,$$

$$\varepsilon = \sum_\nu \int d^3p \, f_\nu \, \hat{H}_\nu + \frac{1}{8\pi} (\underline{E}^2 + \underline{B}^2) \, , \tag{55}$$

$$\underline{\eta} = \sum_\nu \int d^3p \, f_\nu \, \hat{H}_\nu \, \underline{v_\nu} + \frac{c}{4\pi} \, \underline{E} \times \underline{B} \, .$$

## VII. Kinetic guiding center - Maxwell theory

According to Wimmel [2] Littlejohn's guiding center motions including polarization drifts [3] can be obtained from a Lagrangian

$$L_W = \frac{e}{c}\, \dot{\underline{x}} \cdot \left(\underline{A}(\underline{x}, t) + \frac{mc}{e}(v_{\shortparallel}\, \underline{b} + \underline{v_E})\right) - e\,\Phi$$

$$- \mu B - \frac{m}{2}\left(v_{\shortparallel}^2 + \underline{v}_E^2\right) , \tag{56}$$

$$\underline{b} = \underline{B}/B \; , \; \underline{v}_E = c\,(\underline{E} \times \underline{B})/B^2 \; ,$$

$$\mu = \text{magnetic moment,}$$

if in $\delta \int_{t_1}^{t_2} L_W\, dt = 0$ $\underline{x}\,(t)$, $v_{\shortparallel}(t)$ are varied independently while $\mu$ is kept constant. $\mu$ therefore only appears as a parameter. This Lagrangian, however, does not allow to construct a Hamiltonian as a function of canonical variables. In order to obtain such a Hamiltonian, we have to modify the Lagrangian and also introduce an additional coordinate. The following choice is a possible one:

$$L_g = \frac{1}{2}\,\varepsilon\, \dot{\underline{x}}^2 + L_W \; , \quad \varepsilon > 0, \; \varepsilon \to 0, \; \varepsilon|\dot{\underline{x}}|^2 \to 0,$$

$$\underline{y} = (\underline{x}, \zeta) \; , \quad \dot{\underline{y}} = (\dot{\underline{x}}, \dot{\zeta} = v_{\shortparallel}) \; . \tag{57}$$

From this we find canonical momenta

$$\underline{P} = (\underline{P_1}, P_2) = (\underline{p}, p_\zeta) ,$$

$$\underline{p} = \varepsilon \underline{\dot{x}} + \frac{e}{c} \underline{A} + m (v_{\shortparallel} \underline{b} + \underline{v_E}) , \tag{58}$$

$$p_\zeta = m (\underline{b} \cdot \underline{\dot{x}} - v_{\shortparallel}) .$$

Since $L_g$ does not depend on $\zeta$ , the corresponding Hamiltonian does not either and therefore $p_\zeta$ = const. To be in agreement with Littlejohn and Wimmel, this constant has to be taken as zero:

$$p_\zeta = m (\underline{b} \cdot \underline{\dot{x}} - v_{\shortparallel}) = 0, \tag{59}$$

i.e. $v_{\shortparallel} = \underline{b} \cdot \underline{\dot{x}}$ .

In this case one finds

$$\underline{\dot{x}} = \frac{1}{\varepsilon} (\underline{p} - \frac{e}{c} \underline{A}) - \frac{m}{\varepsilon(m+\varepsilon)} \underline{b}\, \underline{b} \cdot (\underline{p} - \frac{e}{c} \underline{A})$$

$$- \frac{m}{\varepsilon} \underline{v_E} , \tag{60}$$

$$v_{\shortparallel} = \frac{1}{m+\varepsilon} \underline{b} \cdot (\underline{p} - \frac{e}{c} \underline{A}) .$$

Using this in

$$H_g = \underline{\dot{x}} \cdot \frac{\partial L_g}{\partial \underline{\dot{x}}} + v_{\shortparallel} \frac{\partial L_g}{\partial v_{\shortparallel}} - L_g$$

$$= \frac{1}{2} \varepsilon \underline{\dot{x}}^2 + m v_{\shortparallel} (\underline{b} \cdot \underline{\dot{x}} - v_{\shortparallel}) + \frac{m}{2} (v_{\shortparallel}^2 + \underline{v}_E^2)$$

$$+ \mu B + e \Phi ,$$

one obtains the Hamiltonian

$$H_g = \frac{1}{2\varepsilon} (\underline{p} - \frac{e}{c} \underline{A})^2 - \frac{1}{2\varepsilon} \frac{m}{m+\varepsilon} \left[ (\underline{p} - \frac{e}{c} \underline{A}) \cdot \underline{b} \right]^2 \tag{61}$$

$$- \frac{m}{\varepsilon} \underline{v_E} \cdot (\underline{p} - \frac{e}{c} \underline{A}) + \frac{m}{2} (1 + \frac{m}{\varepsilon}) \underline{v_E}^2 + \mu B + e \Phi .$$

This function of the canonical variables is to be used in order to get $\partial H_g/\partial \underline{E}$ and $\partial H_g/\partial \underline{B}$. After insertion of $\underline{p}$ again from eq. (58) this results in

$$\frac{\partial H_g}{\partial \underline{E}} = \frac{mc}{B^2} (\underline{\dot{x}} - \underline{v_E}) \times \underline{B} ,$$

$$\frac{\partial H_g}{\partial \underline{B}} = - \frac{m v_{\shortparallel}}{B} (\underline{\dot{x}_\perp} - \underline{v_E}) - \frac{mc}{B^2} (\underline{\dot{x}_\perp} - \underline{v}_E) \times \underline{E} \tag{62}$$

$$+ \frac{2m}{B} \underline{v_E} \cdot (\underline{\dot{x}} - \underline{v_E}) \underline{b} + \mu \underline{b} ,$$

$$\underline{\dot{x}}_{\perp} = \underline{b} \times (\underline{\dot{x}} \times \underline{b}) \quad . \tag{63}$$

These relations are correct for all $\varepsilon$. $\varepsilon$ only occurs in the original expressions with the momenta as variables instead of $\underline{\dot{x}}$. We now replace the integration over $\underline{p}$ by an integration over $\underline{\dot{x}}$. Because of eq. (58) we have

$$d^3p = \varepsilon^2 \; m \; d^3\dot{x} \tag{64}$$

and we therefore define

$$F(\underline{x}, \underline{\dot{x}}, t) \equiv \varepsilon^2 m \; f \; (\underline{x}, \underline{p}, t) \; . \tag{65}$$

The integration over $\underline{\dot{x}}$ is simplified when going to the limit $\varepsilon \to 0$. For this purpose we first write down the exact equations of motion:

$$\varepsilon \; \underline{\ddot{x}} = e \; \underline{\hat{E}} + \frac{e}{c} \; \dot{x} \times \underline{\hat{B}} \quad , \tag{66}$$

$$\underline{\hat{E}} = - \frac{1}{c} \frac{\partial \underline{\hat{A}}}{\partial t} - \nabla \hat{\Phi} \quad , \quad \underline{\hat{B}} = \text{curl}/\underline{\hat{A}} \quad ,$$

$$\underline{\hat{A}} = \underline{A} + \frac{mc}{e} (v_{\parallel} \underline{b} + \underline{v_E}) \quad , \qquad (67)$$

$$\hat{\Phi} = \Phi + \frac{\mu}{e} B + \frac{m}{2e} \underline{v_E}^2 \quad ,$$

$$v_{\parallel} = \underline{b} \cdot \underline{\dot{x}} \; .$$

With

$$\underline{\hat{E}} = \underline{E}^* - \frac{m}{e} \dot{v}_{\parallel} \underline{b} \qquad (68)$$

eq. (66) can be decomposed into

$$\varepsilon \, \underline{\ddot{x}} \times \underline{b} = e \, \underline{E}^* \times \underline{b} + \frac{e}{c} (\underline{\hat{B}_{\perp}} v_{\parallel} - \underline{\dot{x}_{\perp}} \hat{B}_{\parallel}) \; , \qquad (69)$$

$$\varepsilon \, \underline{\ddot{x}} \cdot \underline{b} = e \, E^*_{\parallel} - m\dot{v}_{\parallel} + \frac{e}{c} (\underline{\dot{x}_{\perp}} \times \underline{\hat{B}_{\perp}}) \cdot \underline{b}, \qquad (70)$$

where

$$\underline{\hat{B}_{\perp}} = \underline{b} \times (\underline{\hat{B}} \times \underline{b}) \; , \; \hat{B}_{\parallel} = \underline{b} \cdot \underline{\hat{B}} \; . \qquad (71)$$

Equation (69) implies a "gyromotion" with a frequency of order $1/\varepsilon$, thus

$$\varepsilon\, \underline{\ddot{x}} \times \underline{b} = \varepsilon \frac{d}{dt} (\underline{\dot{x}} \times \underline{b}) + \text{terms vanishing for } \varepsilon \to 0 \ , \qquad (72)$$

$$\varepsilon\, \underline{\ddot{x}} \cdot \underline{b} = \varepsilon \frac{d}{dt} (\underline{\dot{x}} \cdot \underline{b}) + \text{terms vanishing for } \varepsilon \to 0 \ , \qquad (73)$$

$$= 0 \text{ for } \varepsilon \to 0 \ .$$

Equation (69) therefore yields the following result, being exact for $\varepsilon \to 0$: The motion consists of a drift

$$\underline{\dot{x}_\perp}^{0} = \underline{v_D} = \frac{1}{\hat{B}_{\parallel}} (v_{\parallel}\, \underline{\hat{B}_\perp} + c\, \underline{E}^{*} \times \underline{b}) \qquad (74)$$

on which is superimposed a gyromotion with frequency

$$\omega_{g\varepsilon} = \frac{e\hat{B}_{\parallel}}{\varepsilon c} \qquad (75)$$

and a gyroradius

$$r_{g\varepsilon} = \frac{\varepsilon |\underline{\dot{x}_\perp} - \underline{v_D}| c}{e\hat{B}_{\parallel}} \ . \qquad (76)$$

The motion possesses an adiabatic invariant

$$\mu_\varepsilon = \frac{\frac{1}{2}\,\varepsilon\,|\underline{\dot{x}}_\perp - \underline{v}_D|^2}{\hat{B}_\parallel} \tag{77}$$

which in the limit $\varepsilon \to 0$ becomes an exact constant of the motion. Since for $\varepsilon \to 0$ $\omega_{g\varepsilon} \to \infty$ , $r_{g\varepsilon} \to 0$ , the integration over the phase of the "gyromotion" yields $2\pi$ for the remainder, and we can replace

$$\begin{aligned} &d^3\dot{x} \text{ by } 2\pi\, d\mu_\varepsilon \frac{\hat{B}_\parallel}{\varepsilon}\, dv_\parallel\ , \\ &\underline{\dot{x}} \text{ by } \underline{v}_D\ . \end{aligned} \tag{78}$$

We now write

$$F\ (\underline{x},\ \underline{\dot{x}},\ t) = \frac{\varepsilon}{m}\, f_g\ (\underline{x},\ v_\parallel,\ \mu,\ t)\ \delta\ (\mu_\varepsilon)\ , \tag{79}$$

where $\delta\ (\mu_\varepsilon)$ is to be understood in such a way that integration over $\mu_\varepsilon$ only gives a contribution for $\mu_\varepsilon \to 0$ for $\varepsilon \to 0$. Furthermore, a dependence on $\mu$ is added, which is possible because $\mu$ only had the character of a parameter. We can also sum our expressions over $\mu$, i.e. integrate, which is analogous to summing over different particle species. We therefore arrive at the replacement rule

$$\int d^3\dot{x}\ F\ (\underline{x},\ \underline{\dot{x}},\ t)\ \ldots \to 2\pi \int \frac{\hat{B}_\parallel}{m}\, d\mu\, dv_\parallel\, f_g\ (\underline{x},\ v_\parallel,\ \mu,\ t)\ \ldots \tag{80}$$

With eqs. (80), (65), (64), (63), (62), (51) the general expressions for charge and current densities (23), (24) and for energy and energy flux densities (37), (38) yield

$$\rho = \sum_\nu \frac{2\pi}{m_\nu} \int \hat{B}_{\nu\shortparallel}\, d\mu\, dv_\shortparallel\, f_{g\nu}\, e_\nu \tag{81}$$

$$+ \operatorname{div} \sum_\nu \frac{2\pi}{m_\nu} \int \hat{B}_{\nu\shortparallel}\, d\mu\, dv_\shortparallel\, f_{g\nu} \frac{m_\nu c}{B} (\underline{v}_{D\nu} - \underline{v}_E) \times \underline{b},$$

$$\underline{j} = \sum_\nu \frac{2\pi}{m_\nu} \int \hat{B}_{\nu\shortparallel}\, d\mu\, dv_\shortparallel\, f_{g\nu}\, e_\nu\, (\underline{v}_{D\nu} + v_\shortparallel \underline{b})$$

$$- \frac{\partial}{\partial t} \sum_\nu \frac{2\pi}{m_\nu} \int \hat{B}_{\nu\shortparallel}\, d\mu\, dv_\shortparallel\, f_{g\nu} \frac{m_\nu c}{B} (\underline{v}_{D\nu} - \underline{v}_E) \times \underline{b} \tag{82}$$

$$- c \operatorname{curl} \sum_\nu \frac{2\pi}{m_\nu} \int \hat{B}_{\nu\shortparallel}\, d\mu\, dv_\shortparallel\, f_{g\nu} \Big[\mu \underline{b} - \frac{m_\nu c}{B^2} (\underline{v}_{D\nu} - \underline{v}_E) \times \underline{E}$$

$$- \frac{2m_\nu}{B} \underline{v}_E \cdot (\underline{v}_E - \underline{v}_{D\nu}) \underline{b} - \frac{m_\nu v_\shortparallel}{B} (\underline{v}_{D\nu} - \underline{v}_E)\Big],$$

$$\varepsilon = \sum_\nu \frac{2\pi}{m_\nu} \int \hat{B}_{\nu\shortparallel}\, d\mu\, dv_\shortparallel\, f_{g\nu} \,\cdot$$

$$\cdot \Big[\frac{m}{2} (v_\shortparallel^2 + v_{D\nu}^2 - (\underline{v}_{D\nu} - \underline{v}_E)^2) + \mu B\Big] \tag{83}$$

$$+ \frac{1}{8\pi} (\underline{E}^2 + \underline{B}^2),$$

$$\underline{\eta} = \sum_\nu \frac{2\pi}{m_\nu} \int \hat{B}_{\nu\shortparallel}\, d\mu\, dv_\shortparallel\, f_{g\nu} \,\cdot$$

$$\Big[\Big(\frac{m_\nu}{2} (v_\shortparallel^2 + \underline{v}_E^2) + \mu B\Big) (v_{D\nu} + v_\shortparallel \underline{b})$$

$$+ (\mu B + 2m_\nu v_E \cdot (v_{D\nu} - \underline{v}_E))\, \underline{v}_E$$

$$- \frac{m_\nu c^2}{B^2} \; \underline{E} \times ((\underline{v}_{D\nu} - \underline{v}_E) \times \underline{E}) \qquad (84)$$

$$- \frac{m_\nu v_\parallel c}{B} \; \underline{E} \times (\underline{v}_{D\nu} - \underline{v}_E)]$$

$$+ \frac{c}{4\pi} \; \underline{E} \times \underline{B} \; ,$$

where $\hat{B}_{\nu\parallel} = \underline{b} \cdot \hat{\underline{B}}_\nu$ , $\hat{\underline{B}}_\nu$ being defined in eq. (67), $\underline{v}_{D\nu}$ is given by eqs. (74), (68), (67), $f_{g\nu}$ is a solution of the drift kinetic equation

$$\frac{\partial f_{g\nu}}{\partial t} + \underline{v}_{D\nu} \cdot \frac{\partial f_{g\nu}}{\partial \underline{x}} + \dot{v}_\parallel \frac{\partial f_{g\nu}}{\partial v_\parallel} = 0 \quad , \qquad (85)$$

with $\dot{v}_\parallel$ following from eqs. (70), (73), (68), (67):

$$m_\nu \, \dot{v}_\parallel = e_\nu E^*_{\nu\parallel} + \frac{e_\nu}{c} \, (\underline{v}_D \times \hat{\underline{B}}_\nu) \cdot \underline{b} \quad . \qquad (86)$$

It has thus been established that the drift kinetic equation (85) together with Maxwell's equations with charge and current densities (81), (82) form a self-consistent system obeying a local energy conservation law $\dot{\varepsilon}$ + div $\underline{\eta} = 0$, with $\varepsilon$ and $\underline{\eta}$ given by eqs. (83), (84). Moreover, the current density (82) has the property of vanishing for $f_{g\nu} = f_{g\nu} (\frac{m_\nu}{2} v_\parallel^2 + \mu B)$, which corresponds to a stationary solution of (85) for $\dot{\underline{A}} = \Phi = 0$ but $\underline{A}$ $(\underline{x})$ arbitrary. One thus gets a compensation of drift and magnetization currents as with solutions $f_\nu$ (v) of the exact Vlasov equation.

## Summary

By using a Eulerian description of point mechanics in the form of a Hamilton - Jacobi equation a new variational formulation of Maxwell-Vlasov and related theories has been given within a Eulerian picture for both the electromagnetic and the Vlasov parts. This formulation allowed us to derive correct charge and current densities as well as corresponding energy and energy flux densities consisting of particle like, electric polarization and magnetization contributions. By some limiting procedure collisionless guiding center theories with polarization drifts included were also treated. In this way it was possible for the first time to formulate consistently local energy conservation laws for such theories. Also, Liouville's equation could be formulated variationally likewise.

## Acknowledgement

The author would like to thank P. Gräff and H.K. Wimmel for valuable discussions.

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