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Application to Drift Wave Turbulence

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(in English)

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The k-spectrum of Korteweg-de Vries turbulence is computed exactly. It consists of a superposition of Lorentzians in k. It is determined by both the steepening and the dispersive term. Application to drift waves in a tokamak shows qualitative agreement with measurements.

Exact k-spectrum for the KdV Equation: Application to Drift Wave Turbulence

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Abstract: The k-spectrum of Korteweg-de Vries turbulence is computed exactly. It consists of a superposition of Lorentzians in k . It is determined by both the steepening and the dispersive term. Application to drift waves in a tokamak shows qualitative agreement with measurements.

A method of studying KdV (Korteweg-de Vries) and drift wave turbulence was given in a previous note [1]. This approach based on thermodynamics of continua and borrowed from solid-state physics is pursued here. It allows one to find the k-spectrum explicitly and to compare it with experimental measurements.

As in [1], the starting point is the KdV equation, but written here with coefficients C_1 and C_2 which characterize the physical system of which the KdV equation is supposed to be an approximation or a model.

If φ is the physical variable, we have

$$\frac{\partial \varphi}{\partial t} - C_1 \varphi \frac{\partial \varphi}{\partial x} + C_2 \frac{\partial^3 \varphi}{\partial x^3} = 0 \quad (1)$$

where $C_1 \varphi$ and $\frac{C_2}{\text{length}^2}$ have the dimension of a velocity. Equation (1) can be written in canonical form:

$$\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial x} \frac{\delta H}{\delta \varphi} \quad (2)$$

with

$$H = \int \frac{1}{6} C_1 \varphi^3 dx + \int \frac{1}{2} C_2 \varphi_x^2 dx \quad (3)$$

As mentioned in [1], the Hamiltonian (3) cannot be interpreted as an energy because of the cubic term. Instead a non-canonical Hamiltonian can be used:

$$H_E = \frac{1}{2} \int \varphi^2 dx \quad (4)$$

To obtain a canonical variable with this Hamiltonian we transform as in [1]:

$$\varphi = A v^2 + B v_x \quad (5)$$

where v is a new dimensionsless variable and A and B are determined below. After substituting eq. (5) in eq. (1) one obtains

$$(2Av + B \frac{\partial}{\partial x})(v_t - C_1 A v^2 v_x + C_2 v_{xxx}) - (C_1 B^2 - C_2 6A) v_x v_{xx} = 0. \quad (6)$$

If

$$C_1 B^2 = 6C_2 A \quad (7)$$

then

$$v_t = C_1 A v^2 v_x - C_2 v_{xxx} \quad (8)$$

is a solution of eq. (6).

Inserting eq. (5) into eq. (4) the new Hamiltonian is

$$H_E = \frac{1}{2} \int dx (A^2 v^4 + B^2 v_x^2) \quad (9)$$

A is now defined such that Hamiltonian (9) produces eq. (8) in the form

$$v_t = \frac{\partial}{\partial x} \frac{\delta H_E}{\delta v} \quad (10)$$

We find $A = \frac{C_1}{6}$ and $B = \sqrt{C_2}$ from eq. (7) and

$$H_E(v) = \frac{1}{2} \int (\frac{C_1^2}{36} v^4 + C_2 v_x^2) dx \quad (11)$$

or for later convenience

$$H_E(v) = \int \frac{dx}{\xi} (bv^4 + cv_x^2) \quad (12)$$

with $\frac{b}{\xi} = \frac{C_1^2}{72}$, $\frac{c}{\xi} = \frac{C_2}{2}$ and the length ξ

will be defined later.

The partition function is meaningfully defined as a functional integral over the field phase variable $v(x)$ as

$$Z = \int \mathcal{D}(v) e^{-\beta H_E(v)} \quad (13)$$

The method of integration is given in [2], where v is supposed to have periodic boundary conditions on a distance L . Z is first written in discretized form:

$$Z = \lim_{N \rightarrow \infty} \frac{1}{\mathcal{D}^N} \int \prod_{i=1}^N dv_i \exp \left[-\beta \frac{\Delta x}{\xi} \left(b v_{i+1}^4 + c \left(\frac{v_{i+1} - v_i}{\Delta x} \right)^2 \right) \right] \quad (14)$$

with $L = N \Delta x$

The integration can be done successively using the integral transfer operator

$$\begin{aligned} \frac{1}{\mathcal{D}} \int dv_i \exp \left[-\beta \frac{\Delta x}{\xi} \left(b v_{i+1}^4 + c \left(\frac{v_{i+1} - v_i}{\Delta x} \right)^2 \right) \right] \phi_n(v_i) = \\ = \exp \left[-\beta \frac{\Delta x}{\xi} E_n \right] \phi_n(v_{i+1}) \end{aligned} \quad (15)$$

and yields

$$Z = \sum_{n=0}^{\infty} e^{-\beta \frac{L}{\xi} E_n} \quad (16)$$

In the limit of $\Delta x \rightarrow 0$ eq. (15) can be written as a Schrödinger-like equation:

$$\left[-\frac{1}{4} \frac{\partial^2}{\partial v^2} + v^4 \right] \phi_n(v) = E_n \beta^{4/3} \beta_0^{-1/3} \phi_n(v) \equiv E_n \phi_n(v) \quad (17)$$

where $\beta_0^{-1} = b$ and ξ has been chosen equal to $\frac{c}{b}$. From eq. (17)

one can obtain the eigenvalues E_n and eigenfunctions $\phi_n(v)$. It has been solved numerically using finite elements, which allows the lowest

eigenvalues E_n and corresponding eigenfunctions to be determined with high accuracy. The partition function is hence completely known.

To obtain the k -spectrum of the turbulence, we compute first the space correlation function

$$C(x) = \langle \delta\varphi(x) \delta\varphi(0) \rangle \quad (18)$$

where $\delta\varphi(x) = \varphi(x) - \langle \varphi(x) \rangle$ and the brackets mean averages over the canonical distribution. $C(x)$ is hence a functional integral of the type

$$\int \mathcal{D}(v) \delta\varphi(x) \delta\varphi(0) \frac{e^{-\beta H\epsilon}}{Z} \quad (19)$$

which can be computed in a similar manner to Z by using the transfer integral operator technique. Using the very accurate approximation

$$Z \simeq e^{-\beta \frac{L}{3} \epsilon_0}, \quad \text{one obtains}$$

$$C(x) = \frac{C_1^2}{36} \sum_{n=1}^{\infty} \left\{ \langle \phi_n | v^2 | \phi_0 \rangle - \langle \phi_n | v | \phi_0 \rangle \times \right. \\ \left. \times (E_n - E_0)^2 \left(\frac{\beta_0}{\beta} \right)^{2/3} \right\} \exp \left[-\frac{x}{3} \left(\frac{\beta_0}{\beta} \right)^{1/3} (E_n - E_0) \right] \quad (20)$$

where the brackets mean integration in Hilbert space of the $\phi(v)$.

Finally, the spectrum is given by

$$S(k) = \int dx e^{ikx} C(x) = 2 \sum_{n=1}^{\infty} \frac{q_n}{p_n} \frac{1}{1 + k^2/p_n^2} \quad (21)$$

where

$$p_n = \frac{1}{\xi} \left(\frac{\beta_0}{\beta} \right)^{1/3} (E_n - E_0) \quad (22)$$

and

$$Q_n = \frac{C_1^2}{36} \left[\langle \Phi_n | v^2 | \Phi_0 \rangle - \langle \Phi_n | v | \Phi_0 \rangle (p_n \xi)^2 \right] \quad (23)$$

Application to Drift Waves Turbulence

As discussed in [1], we take as simplest model for the nonlinear drift wave equation the KdV equation where the steepening term is taken from [3] and the dispersive term is representative of finite ion gyroradius effect, although this term is not exactly derived (an exact derivation would lead to 2-dimensional equations and would exceed the scope of this contribution). This leads us to take

$$C_1 = \frac{C}{B} \frac{T'}{T} \quad C_2 = \left(\frac{kT_c}{eB} \right)^2 \frac{1}{\omega_{ci}} \frac{n'}{n} \quad (24)$$

where primes denote derivatives with respect to the radial direction and x will be identified with the poloidal direction (which is unusual). This means that

$$\xi = \frac{6\sqrt{C_2}}{C_1}, \quad \beta_0 = \frac{12}{C_1\sqrt{C_2}} \quad (25)$$

If we assume a fluctuation level $\frac{e\phi}{kT} \approx 1\%$, then β^{-1} can be taken of the order of $\int \frac{\phi^2}{2} dx$. For a typical tokamak discharge we may assume $B \approx 10^4$ Gauss. $T_i \approx T_e \approx 10^7$ degrees,

$L \approx 2\pi R \approx 120$ cm, $\frac{n}{n'} \approx 2$ cm, $\frac{T}{T'} \approx 2$ cm. Then all quantities in the spectrum can be computed, yielding the curve $S(k)$ versus k/p_2 -

see figure - whose width is essentially, but not completely, determined by the inverse of the ion gyroradius.

This is qualitatively (and quantitatively up to a reasonable factor) in agreement with the measurements [4] done on Alcator. A better agreement cannot be expected because the theory is incomplete in the physics and in the number of dimensions. It shows, however, the importance of the temperature gradient leading to the steepening term and the finite ion gyroradii.

It would be interesting to apply the spectrum found here (eq. (21)) to other areas in physics to which the KdV equation applies and where turbulence is observed.

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(see Fig. 7)

