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Kinetic Guiding-center Equations for the Theory
of Drift Instabilities and Anomalous Transport

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Abstract

Littlejohn's guiding-center mechanics, with the polarization drift included, is rederived from a Lagrangian. The respective collisionless kinetic guiding-center theory is then established. Even though single-guiding-center energy is exactly conserved here in time-independent fields, the total energy of the resulting guiding-center plasma and its self-consistent fields is only approximately conserved (to leading order in $\varepsilon \equiv r_g / L$). This is unlike in the case without polarization drift, where exact conservation of total energy obtains. Nevertheless, the present formalism seems to be important for the theory of drift waves and anomalous transport.

1. Introduction

A kinetic guiding-center theory that includes the polarization drift and provides for exact conservation of the total energy of the guiding-center plasma and its self-consistent electromagnetic fields is desirable for treating drift instabilities and anomalous transport. Littlejohn's guiding-center mechanics [1, 2] seems particularly attractive as a basis because it guarantees energy conservation for single guiding-centers in time-independent fields and validity of Liouville's theorem for an appropriate phase space volume element. In the case of drift scaling [3], i.e. for

$$\frac{cE}{v_{th} B} = O(\varepsilon) , \qquad \varepsilon \equiv r_g / L , \tag{1.1}$$

where the polarization drift and other higher-order effects are omitted, a collisionless kinetic guiding-center theory and the respective moment equations were in fact formulated for this guiding-center mechanics, with energy-conserving self-consistent coupling to Maxwell's equations [3]. By defining on "effective current density" of the guiding-center plasma it was possible there to conserve the total energy of the guiding-center plasma and its self-consistent fields [3].

In this paper the case of "guiding-center scaling", i.e.

$$\frac{cE_{\perp}}{v_{th} B} = O(1) \quad ; \quad \frac{cE_{\parallel}}{v_{th} B} = O(\varepsilon) , \tag{1.2}$$

with allowance for the polarization drift is considered. In Sect. 2 the equations of motion of the guiding centers are derived from an appropriate Lagrangian [2]. In Sect. 3 energy conservation for single guiding centers in time-independent fields and Liouville's theorem are proved. In Sect. 4 the respective collisionless kinetic equation for the guiding centers is derived, and moment equations for guiding-center density and energy density are established. In Sect. 5 it is shown that conservation of the total energy of the guiding-center plasma and its self-consistent electromagnetic fields is only approximate, i.e. it holds to leading order in $\varepsilon \equiv r_g / L$ (r_g = gyro-radius, L = characteristic macroscopic length), but not exactly. The consequences of this approximate energy conservation must be considered when applying this mathematical formalism to the theory of drift instabilities and anomalous transport. On the other hand, single-guiding-center energy conservation and validity of Liouville's theorem make this kinetic guiding-center theory particularly attractive. In Sect. 6 the conclusions are given, and Appendix A verifies the results of Sect. 2 by comparing them with Littlejohn's Hamiltonian theory [1].

2. Consistent Guiding-center Mechanics with Polarization Drift

In agreement with earlier work [3, 4, 5], we call a guiding-center mechanics "consistent" when it conserves single-guiding-center energy in time-independent fields and satisfies a Liouville's theorem. The following is an extension of Littlejohn's Lagrangian formalism [2] which switches from "drift scaling" to "guiding-center scaling" (see Sect. 1). We start with the guiding-center Lagrangian

$$L \equiv \frac{e}{c} \underset{\sim}{A}^{*} \cdot \underset{\sim}{v} - e\phi - W_K , \tag{2.1}$$

with L depending on the guiding-center variables $t, \underset{\sim}{x}, \underset{\sim}{v} \equiv \dot{\underset{\sim}{x}}, v_{\parallel}$, and the parameters μ, m, e, c. Here $\underset{\sim}{A}^{*}$ is a modified vector potential [1], viz.

$$\underset{\sim}{A}^{*} \equiv \underset{\sim}{A} + \frac{mc}{e} \left\{ v_{\parallel} \hat{\underset{\sim}{b}} + \underset{\sim}{v}_E \right\} , \tag{2.2}$$

$\phi(t, \underset{\sim}{x})$ is the scalar potential, and W_K is the kinetic energy in the form

$$W_K \equiv \mu B + \frac{m}{2} \left(v_{\parallel}^2 + v_E^2 \right) . \tag{2.3}$$

The other quantities are: $\underset{\sim}{B}(t, \underset{\sim}{x})$ the magnetic field, $\hat{\underset{\sim}{b}} \equiv \underset{\sim}{B}/B$ the unit vector in the field direction, $\underset{\sim}{v}_E$ the usual $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, viz.

$$\underset{\sim}{v}_E \equiv \frac{c}{B} \underset{\sim}{E} \times \hat{\underset{\sim}{b}} , \tag{2.4}$$

with $\underset{\sim}{E}(t, \underset{\sim}{x})$ the electric field; the (scalar) magnetic moment μ is an adiabatic invariant, i.e. $\dot{\mu} = 0$, and $v_{\parallel}$ is the "parallel velocity", i.e. the guiding-center velocity component parallel to $\underset{\sim}{B}$; but the relation $v_{\parallel} = (\underset{\sim}{v} \cdot \underset{\sim}{b})$ is not yet implied here; it will follow from one of the Lagrangian equations.

In what follows the "modified fields" $\underset{\sim}{B}^*$ and $\underset{\sim}{E}^*$ [1, 3] will be needed. They are defined by

$$\underset{\sim}{B}^* \equiv \mathrm{curl}\ \underset{\sim}{A}^* , \tag{2.5}$$

$$\underset{\sim}{E}^* \equiv -\nabla\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}^*}{\partial t} \tag{2.6}$$

so that they satisfy the modified homogeneous Maxwell equations, viz.

$$\mathrm{div}\ \underset{\sim}{B}^* = 0 \tag{2.7}$$

and

$$\frac{\partial \underset{\sim}{B}^*}{\partial t} = -c\ \mathrm{curl}\ \underset{\sim}{E}^* . \tag{2.8}$$

Explicit expressions of $\underset{\sim}{B}^*$ and $\underset{\sim}{E}^*$ are

$$\underset{\sim}{B}^* \equiv \underset{\sim}{B} + \frac{mc}{e}\left\{ v_{\parallel}\ \mathrm{curl}\ \hat{\underset{\sim}{b}} + \mathrm{curl}\ \underset{\sim}{v}_E \right\} , \tag{2.9}$$

$$\underset{\sim}{E}^* \equiv \underset{\sim}{E} - \frac{m}{e}\left\{ v_{\parallel}\frac{\partial \hat{\underset{\sim}{b}}}{\partial t} + \frac{\partial \underset{\sim}{v}_E}{\partial t} \right\} . \tag{2.10}$$

It should be noted that the gauge used for $\underset{\sim}{E}^{*}$ is different from that of Littlejohn [1] (see Appendix A). An important quantity is the "parallel" component of $\underset{\sim}{B}^{*}$, viz.

$$B_{\parallel}^{*} \equiv B + \frac{mc}{e} \hat{\underset{\sim}{b}} \cdot \left\{ v_{\parallel} \, \mathrm{curl} \, \hat{\underset{\sim}{b}} + \mathrm{curl} \, \underset{\sim}{v}_{E} \right\}. \tag{2.11}$$

All time and space derivatives are performed with $v_{\parallel}$ and μ kept constant.

Equations (2.1) through (2.4) are a plausible ansatz for including the polarization drift and obtaining a "consistent" theory. The Lagrangian equations of motion [6] that follow can be proved to be correct by comparing them with Littlejohn's earlier results [1] (see Appendix A below). Firstly, the equation

$$\frac{\partial L}{\partial v_{\parallel}} = 0 \tag{2.12}$$

yields the relation

$$\underset{\sim}{v} \cdot \hat{\underset{\sim}{b}} = v_{\parallel} \quad ; \quad \underset{\sim}{v}_{\parallel} \equiv v_{\parallel} \hat{\underset{\sim}{b}} . \tag{2.13}$$

The other equations of motion are obtained from

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \underset{\sim}{v}} \right) = \frac{\partial L}{\partial \underset{\sim}{x}} \equiv \nabla L , \tag{2.14}$$

which reads

$$\frac{e}{c} \frac{d\underset{\sim}{A}^{*}}{dt} = \frac{e}{c} \nabla \left(\underset{\sim}{A}^{*} \cdot \underset{\sim}{v} \right) - e \nabla \phi - \nabla W_{K} . \tag{2.15}$$

By using

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} + \dot{v}_{\parallel} \frac{\partial}{\partial v_{\parallel}} \tag{2.16}$$

and several of the above relations this can be transformed to yield

$$m \dot{v}_{\parallel} \hat{\underset{\sim}{b}} = \left(e \underset{\sim}{E}^{*} - \nabla W_{K} \right) + \frac{e}{c} \underset{\sim}{v} \times \underset{\sim}{B}^{*} \; . \tag{2.17}$$

This equation can be decomposed into an equation for the guiding--center velocity $\underset{\sim}{v}$ and another one for the "parallel acceleration" $\dot{v}_{\parallel}$, viz.

$$\underset{\sim}{v} = v_{\parallel} \frac{\underset{\sim}{B}^{*}}{B_{\parallel}^{*}} + \frac{c}{e B_{\parallel}^{*}} \left(e \underset{\sim}{E}^{*} - \nabla W_{K} \right) \times \hat{\underset{\sim}{b}} \tag{2.18}$$

and

$$\dot{v}_{\parallel} = \frac{1}{m B_{\parallel}^{*}} \underset{\sim}{B}^{*} \cdot \left(e \underset{\sim}{E}^{*} - \nabla W_{K} \right) \tag{2.19}$$

$$= \frac{1}{m v_{\parallel}} \underset{\sim}{v} \cdot \left(e \underset{\sim}{E}^{*} - \nabla W_{K} \right) . \tag{2.19a}$$

By substituting the modified fields and W_K more explicit expressions are obtained, viz.

$$\underset{\sim}{v} = v_{\parallel} \hat{\underset{\sim}{b}} + \frac{c}{B_{\parallel}^{*}} \underset{\sim}{E} \times \hat{\underset{\sim}{b}} + \frac{\mu}{m \Omega^{*}} \hat{\underset{\sim}{b}} \times \nabla B$$
$$+ \frac{v_{\parallel}^{2}}{\Omega^{*}} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} + \frac{v_{\parallel}}{\Omega^{*}} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial t}$$

$$+ \frac{1}{2\Omega^*}\,\hat{\underset{\sim}{b}} \times \nabla\left(v_E^2\right) + \frac{1}{\Omega^*}\,\hat{\underset{\sim}{b}} \times \frac{\partial \underset{\sim}{v}_E}{\partial t}$$

$$+ \frac{v_\parallel}{\Omega^*}\,\mathrm{curl}_\perp\,\underset{\sim}{v}_E\,, \tag{2.20}$$

with

$$\Omega^* = \frac{e\,B_\parallel^*}{mc} = \Omega + \hat{\underset{\sim}{b}}\cdot\left\{v_\parallel\,\mathrm{curl}\,\hat{\underset{\sim}{b}} + \mathrm{curl}\,\underset{\sim}{v}_E\right\}, \tag{2.21}$$

and

$$\dot{v}_\parallel = \frac{e}{m}E_\parallel - \frac{\mu}{m}\frac{\partial B}{\partial s} - \frac{1}{2}\frac{\partial}{\partial s}\left(v_E^2\right)$$

$$+ \underset{\sim}{v}\cdot\left\{v_\parallel\frac{\partial \hat{\underset{\sim}{b}}}{\partial s} - \hat{\underset{\sim}{b}}\times\mathrm{curl}\,\underset{\sim}{v}_E\right\} - \hat{\underset{\sim}{b}}\cdot\frac{\partial \underset{\sim}{v}_E}{\partial t}\,, \tag{2.22}$$

with the usual definition $\partial/\partial s \equiv \hat{\underset{\sim}{b}}\cdot\nabla$.

The energy equation following from the above equations reads

$$\frac{dW_K}{dt} = \frac{\partial W_K}{\partial t} + e\,\underset{\sim}{E}^*\cdot\underset{\sim}{v} \tag{2.23}$$

or, more explicitly,

$$\frac{dW_K}{dt} = e\,\underset{\sim}{E}\cdot\underset{\sim}{v} + \mu\frac{\partial B}{\partial t} + \frac{m}{2}\frac{\partial}{\partial t}\left(v_E^2\right)$$

$$- m\,\underset{\sim}{v}\cdot\left\{v_\parallel\frac{\partial \hat{\underset{\sim}{b}}}{\partial t} + \frac{\partial \underset{\sim}{v}_E}{\partial t}\right\} \tag{2.24}$$

This can be given the form

$$\frac{dW_K}{dt} = e\underset{\sim}{E} \cdot \underset{\sim}{v} - \underset{\sim}{\mu} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} - m(\underset{\sim}{v}_\perp - \underset{\sim}{v}_E) \cdot \frac{\partial \underset{\sim}{v}_E}{\partial t} \,, \tag{2.25}$$

where the last term is now a correction of order ε , and the vectorial magnetic moment $\underset{\sim}{\mu}$ is defined by

$$\underset{\sim}{\mu} \equiv -\mu \hat{\underset{\sim}{b}} + \frac{m v_\parallel}{B} (\underset{\sim}{v}_\perp - \underset{\sim}{v}_E) \,. \tag{2.26}$$

Equation (2.25) is useful when dealing with the kinetic theory (see Sect. 4). The above theory and the work of Littlejohn [1] are compared in Appendix A. It will follow that the above equations are in agreement with the results Littlejohn [1] obtained from a Hamiltonian formalism. Since Littlejohn [1] derives his results direct from particle dynamics, this agreement proves the Lagrangian of eq. (2.1) to be correct.

3. Energy Theorem and Liouville's Theorem

From eq. (2.23) or (2.24) conservation of single-guiding-center energy [1] follows in time-independent fields, viz.

$$\frac{d}{dt}\left(W_K + e\phi\right) = 0. \tag{3.1}$$

Liouville's theorem will be proved for the phase space volume element [1, 3]

$$d\tau \equiv \frac{2\pi}{m} B_\parallel^* \, d^3x \, dv_\parallel \, d\mu, \tag{3.2}$$

i.e. one proves, for a co-moving $d\tau$,

$$d\dot{\tau} \equiv \frac{d}{dt}(d\tau) = 0 \tag{3.3}$$

or, equivalently [1, 3],

$$\delta \equiv \frac{\partial B_\parallel^*}{\partial t} + \operatorname{div}\left(B_\parallel^* \underset{\sim}{v}\right) + \frac{\partial}{\partial v_\parallel}\left(B_\parallel^* \dot{v}_\parallel\right) = 0, \tag{3.3a}$$

where $\underset{\sim}{v}$ must be taken from eq. (2.18) and $\dot{v}_\parallel$ from eq. (2.19). The scalar magnetic moment μ has now been included among the phase space coordinates $\{\alpha_i\} \equiv \{\underset{\sim}{x}, v_\parallel, \mu\}$, which makes no difference in the verification of eqs. (3.3), (3.3a) since $\dot{\mu} = 0$.

<u>To prove</u> eq. (3.3a), we list

$$\frac{\partial B_\parallel^*}{\partial t} = \underset{\sim}{B}^* \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} + \hat{\underset{\sim}{b}} \cdot \frac{\partial \underset{\sim}{B}^*}{\partial t}, \tag{3.4}$$

$$\mathrm{div}\,(B_{\parallel}^{*}\,\underset{\sim}{v}) = c\,\hat{\underset{\sim}{b}}\cdot\mathrm{curl}\,\underset{\sim}{E}^{*} - c\,\underset{\sim}{E}^{*}\cdot\mathrm{curl}\,\hat{\underset{\sim}{b}}$$

$$+\ \frac{c}{e}\,\nabla W_{K}\cdot\mathrm{curl}\,\hat{\underset{\sim}{b}}\ , \tag{3.5}$$

$$\frac{\partial}{\partial v_{\parallel}}\left(B_{\parallel}^{*}\,\dot{v}_{\parallel}\right) = \frac{1}{m}\,\frac{\partial}{\partial v_{\parallel}}\left\{\underset{\sim}{B}^{*}\cdot\left(e\underset{\sim}{E}^{*} - \nabla W_{K}\right)\right\}, \tag{3.6}$$

to obtain

$$\delta = \frac{e}{m}\,\underset{\sim}{B}^{*}\cdot\left\{\frac{\partial \underset{\sim}{E}^{*}}{\partial v_{\parallel}} + \frac{m}{e}\,\frac{\partial \hat{\underset{\sim}{b}}}{\partial t}\right\}$$

$$+\ \frac{1}{m}\left(e\underset{\sim}{E}^{*} - \nabla W_{K}\right)\cdot\left\{\frac{\partial \underset{\sim}{B}^{*}}{\partial v_{\parallel}} - \frac{mc}{e}\,\mathrm{curl}\,\hat{\underset{\sim}{b}}\right\}. \tag{3.7}$$

The modified homogeneous Maxwell equations [eqs. (2.7) and (2.8)] were used to derive eq. (3.7). Inspection of eqs. (2.9) and (2.10) for the "modified fields" shows that both sets of curly brackets in eq. (3.7) vanish. This proves the validity of eq. (3.3a) and hence of eq. (3.3), i.e. Liouville's theorem for the $d\tau$ of eq. (3.2). The results of this section agree with Littlejohn's earlier results [1] ; Littlejohn [1] used a Hamiltonian formalism rather than the Lagrangian approach he proposed later [2] for the case of "drift scaling".

4. Collisionless Kinetic Equation for the Guiding Centers

A collisionless kinetic theory is given for the guiding-center mechanics with polarization drift as presented in Sects. 2 and 3. The phase space is 5-dimensional, with coordinates $\{\alpha_i\} \equiv \{\underset{\sim}{x}, v_\parallel, \mu\}$, i = 1 to 5. The volume element in phase space, $d\tau$, as given by eq. (3.2), is Liouvillian, i.e. $d\dot{\tau} = 0$ for a $d\tau$ that moves with the guiding centers in phase space. The guiding center distribution function f is defined by

$$dN \equiv f \, d\tau , \tag{4.1}$$

with $f = f(t, \underset{\sim}{x}, v_\parallel, \mu)$, N being the number of guiding centers in phase space. The collisionless kinetic equation expresses conservation of dN in a volume element $d\tau$ that moves with the guiding centers, i.e.

$$\frac{d}{dt}(dN) \equiv \frac{d}{dt}(f \, d\tau) = 0. \tag{4.2}$$

This equation can be reformulated [3] to read

$$\frac{\partial}{\partial t}\left(B_\parallel^* f\right) + \operatorname{div}\left(B_\parallel^* f \underset{\sim}{v}\right) + \frac{\partial}{\partial v_\parallel}\left(B_\parallel^* f \dot{v}_\parallel\right) = 0. \tag{4.3}$$

This form of the kinetic equation holds independently of whether $d\tau$ is Liouvillian or not. Since our $d\tau$ <u>is</u> in fact Liouvillian, eq. (4.3) can be simplified to read

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \dot{v}_\parallel \frac{\partial f}{\partial v_\parallel} = 0. \tag{4.4}$$

Here $\underset{\sim}{v}$ must be taken from eq. (2.18) or (2.20) and $\dot{v}_\parallel$ from eq. (2.19) or (2.22).

When moment equations are to be derived from the kinetic equation the use of eq. (4.3) is to be preferred. An equation of continuity and an energy equation will be derived. Integration of eq. (4.3) over $dv_\parallel \, d\mu$ for $-\infty < v_\parallel < +\infty$ and $0 \leq \mu < \infty$ yields the equation of continuity, viz.

$$\frac{\partial n_1}{\partial t} + \operatorname{div} \underset{\sim}{\Gamma}_1 = 0 \, , \tag{4.5}$$

where

$$n_1 \equiv \int d\tau_v \, f \tag{4.6}$$

is the guiding-center density in position space,

$$\underset{\sim}{\Gamma}_1 \equiv \int d\tau_v \, f \, \underset{\sim}{v} \tag{4.6a}$$

is the guiding-center flux density, and

$$d\tau_v \equiv \frac{2\pi}{m} B_\parallel^* \, dv_\parallel \, d\mu \tag{4.7}$$

is the volume element in guiding-center velocity space. To obtain eq. (4.5), it is assumed that $(B_\parallel^* f \dot{v}_\parallel) \to 0$ for $|v_\parallel| \to \infty$.

In order to formulate the energy equation, let us define the kinetic energy density, viz.

$$D_1 \equiv \int d\tau_v \, f \, W_K \, , \tag{4.8}$$

the kinetic energy flux density, viz.

$$\underset{\sim}{F}_1 \equiv \int d\tau_v \, f \, W_K \, \underset{\sim}{v} \, , \tag{4.9}$$

and the vectorial magnetic moment density, viz.

$$\underset{\sim}{M} \equiv \int d\tau_v \, f \, \underset{\sim}{\mu} \, . \tag{4.9a}$$

Here W_K is to be taken from eq. (2.3), $\underset{\sim}{v}$ from eq. (2.18) or (2.20), and $\underset{\sim}{\mu}$ from eq. (2.26). By multiplying eq. (4.3) by $W_K \, dv_\parallel \, d\mu$ and integrating over $(v_\parallel , \mu)$ space one obtains the energy equation in the form

$$\frac{\partial D_1}{\partial t} + \operatorname{div} \underset{\sim}{F}_1 = \int d\tau_v \, f \, \frac{dW_K}{dt} \tag{4.10}$$

or [eqs. (2.25) and (2.26)]

$$\frac{\partial D_1}{\partial t} + \operatorname{div} \underset{\sim}{F}_1 = e \, \underset{\sim}{E} \cdot \underset{\sim}{\Gamma}_1 - \underset{\sim}{M} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} - m \, \frac{\partial \underset{\sim}{v}_E}{\partial t} \cdot \left(\underset{\sim}{\Gamma}_{1\perp} - \underset{\sim}{\Gamma}_E \right) . \tag{4.11}$$

Here $\underset{\sim}{\Gamma}_{1\perp} \equiv \underset{\sim}{\Gamma}_1 - (\hat{\underset{\sim}{b}} \cdot \underset{\sim}{\Gamma}_1) \hat{\underset{\sim}{b}}$ is the perpendicular part of $\underset{\sim}{\Gamma}_1$, and

$$\underset{\sim}{\Gamma}_E \equiv n_1 \underset{\sim}{v}_E \, . \tag{4.12}$$

It would be desirable to transform eq. (4.11) so that the new right-hand side acquires the form $\underset{\sim}{E} \cdot \underset{\sim}{j}_{eff}$, $\underset{\sim}{j}_{eff}$ being the "effective guiding-center current density" of one guiding-center component [3, 4]. This would allow exact energetic coupling of the guiding-center theory with Maxwell's equations. As a first step the following form of the energy equation is obtained:

$$\frac{\partial D_1}{\partial t} + \operatorname{div} \underset{\sim}{F}_2 = \underset{\sim}{E} \cdot \left(e \underset{\sim}{\Gamma}_1 + c \operatorname{curl} \underset{\sim}{M} \right) - m \frac{\partial \underset{\sim}{v}_E}{\partial t} \cdot \left(\underset{\sim}{\Gamma}_{1\perp} - \underset{\sim}{\Gamma}_E \right) , \tag{4.13}$$

with the modified energy flux density

$$\underset{\sim}{F}_2 \equiv \underset{\sim}{F}_1 + c \underset{\sim}{M} \times \underset{\sim}{E} . \tag{4.14}$$

The induction law, i.e.

$$\frac{\partial \underset{\sim}{B}}{\partial t} = - c \operatorname{curl} \underset{\sim}{E} , \tag{4.15}$$

was used here. Further transformations will be considered in Sect. 5.

5. Energetic Coupling of Kinetic Guiding-center Theory and Maxwell's Equations

In order to provide exact conservation of the total energy of the guiding-center plasma and the fields the guiding-center energy equation, viz. eq. (4.13), would have to be further transformed so that the right-hand side of the transformed equation would acquire the form $\underset{\sim}{E} \cdot \underset{\sim}{j}_{eff}$, with the identity

$$\mathrm{div}\, \underset{\sim}{j}_{eff} = - \frac{\partial}{\partial t} \rho_{eff} \,, \tag{5.1}$$

where ρ_{eff} and $\underset{\sim}{j}_{eff}$ refer to a single guiding-center component and are allowed to differ from en_1 and $e\underset{\sim}{\Gamma}_1$, respectively, by higher-order corrections in $\varepsilon \equiv r_g / L$. The form of the last term of the r.h.s. of eq. (4.13) suggests, however, that such a transformation is generally not possible.

To make this plausible, we shall try two different transformations and identify the mathematical obstacles. As a first attempt, let us try partial differentiation with respect to time of the troubling term, viz.

$$\gamma \equiv - m \frac{\partial \underset{\sim}{v}_E}{\partial t} \cdot \left(\underset{\sim}{\Gamma}_{1\perp} - \underset{\sim}{\Gamma}_E \right) . \tag{5.2}$$

After some manipulation this yields a new form of the energy equation, viz.

$$\frac{\partial D_2}{\partial t} + \mathrm{div}\, \underset{\sim}{F}_2 = \underset{\sim}{E} \cdot \left\{ e \underset{\sim}{\Gamma}_1 + e \underset{\sim}{\Gamma}_2 + c\, \mathrm{curl}\, \underset{\sim}{M} \right\}, \tag{5.3}$$

with the definitions

$$D_2 \equiv D_1 + m\, \underset{\sim}{v}_E \cdot \left(\underset{\sim}{\Gamma}_{1\perp} - \underset{\sim}{\Gamma}_E \right) \tag{5.4}$$

and

$$\underset{\sim}{\Gamma}_2 \equiv \frac{1}{\Omega} \underset{\sim}{\hat{b}} \times \frac{\partial}{\partial t} \left(\underset{\sim}{\Gamma}_{1\perp} - \underset{\sim}{\Gamma}_E \right). \tag{5.5}$$

For general field configurations it appears to be impossible to find an explicit expression for a density n_2 such that

$$\mathrm{div}\, \underset{\sim}{\Gamma}_2 + \frac{\partial n_2}{\partial t} = 0 \tag{5.6}$$

is identically satisfied. It follows that

$$\underset{\sim}{j}_{\mathrm{eff}} \equiv e \underset{\sim}{\Gamma}_1 + e \underset{\sim}{\Gamma}_2 + c\, \mathrm{curl}\, \underset{\sim}{M} \tag{5.7}$$

does not identically satisfy eq. (5.1). Since this is in contradiction to Maxwell's equations, where current density and charge density do obey an equation of continuity, exact energetic coupling of guiding-center theory and Maxwell's equations cannot be provided by eqs. (5.3) to (5.5).

As a second attempt, one observes that in

$$\gamma \equiv -m \frac{\partial \underset{\sim}{v}_E}{\partial t} \cdot \left(\underset{\sim}{\Gamma}_{1\perp} - \underset{\sim}{\Gamma}_E \right) \tag{5.8}$$

one may write

$$\underset{\sim}{\Gamma}_{1\perp} - \underset{\sim}{\Gamma}_E \equiv \hat{\underset{\sim}{b}} \times \underset{\sim}{K} \, , \tag{5.9}$$

with $\underset{\sim}{K} \cdot \hat{\underset{\sim}{b}} = 0$ [see eqs. (2.20), (4.6), and (4.12)]. Hence one has

$$\gamma = -m \left(\frac{\partial \underset{\sim}{v}_E}{\partial t} \times \hat{\underset{\sim}{b}} \right) \cdot \underset{\sim}{K} \, . \tag{5.10}$$

On the other hand, the following identity holds:

$$\frac{\partial \underset{\sim}{v}_E}{\partial t} \times \hat{\underset{\sim}{b}} = -\frac{1}{B} \underset{\sim}{v}_E \times \frac{\partial \underset{\sim}{B}}{\partial t} - \frac{c}{B} \frac{\partial \underset{\sim}{E}_\perp}{\partial t} \, . \tag{5.11}$$

Insertion in eq. (5.10) yields

$$\gamma = -\frac{m}{B} \left(\underset{\sim}{v}_E \times \underset{\sim}{K} \right) \cdot \frac{\partial \underset{\sim}{B}}{\partial t} + \frac{mc}{B} \underset{\sim}{K} \cdot \frac{\partial \underset{\sim}{E}_\perp}{\partial t} \, . \tag{5.12}$$

The first term of the r.h.s. can be transformed in the desired way by

$$\begin{aligned} -\underset{\sim}{M}_1 \cdot \frac{\partial \underset{\sim}{B}}{\partial t} &= c \, \underset{\sim}{M}_1 \cdot \operatorname{curl} \underset{\sim}{E} \\ &= c \, \underset{\sim}{E} \cdot \operatorname{curl} \underset{\sim}{M}_1 - c \operatorname{div} \left(\underset{\sim}{M}_1 \times \underset{\sim}{E} \right) , \end{aligned} \tag{5.13}$$

which is similar to the transformation that led from eq. (4.11) to (4.13). However, no transformation appears to be available that would identically yield

$$\frac{mc}{B}\,\underset{\sim}{K}\cdot\frac{\partial \underset{\sim}{E}_{\perp}}{\partial t} = e\,\underset{\sim}{E}\cdot\underset{\sim}{\Gamma}_3 - \frac{\partial D_3}{\partial t} - \operatorname{div}\underset{\sim}{F}_3 \,, \tag{5.14}$$

with

$$\operatorname{div}\underset{\sim}{\Gamma}_3 + \frac{\partial n_3}{\partial t} = 0 . \tag{5.15}$$

It therefore appears that our second attempt does not provide exact energetic coupling of guiding-center theory and Maxwell's equations either. It should be mentioned that preliminary investigations convey the impression that this problem will persist for guiding-center theories with polarization drift other than Littlejohn's, e.g. theories with exact single-guiding-center energy conservation, but without a Liouville's theorem.

Even though exact conservation of total energy cannot be established, total energy is in fact conserved to leading order in $\varepsilon \equiv r_g/L$. This can be seen by considering eq. (4.13) and defining

$$\underset{\sim}{j}_{eff} \equiv e\,\underset{\sim}{\Gamma}_1 + c\,\operatorname{curl}\underset{\sim}{M} . \tag{5.16}$$

The equation of continuity [eq. (5.1)] is then satisfied with

$$\rho_{eff} \equiv e n_1 \,, \tag{5.17}$$

where n_1 has been defined in eq. (4.6). Furthermore, if ρ_{eff} and $\underset{\sim}{j}_{eff}$ are inserted on the right-hand sides of Maxwell's inhomogeneous equations, then one has $(\alpha = i, e)$

$$\frac{\partial}{\partial t}\left[\frac{1}{8\pi}\left(E^2 + B^2\right)\right] + \operatorname{div}\left[\frac{c}{4\pi}\,\underset{\sim}{E}\times\underset{\sim}{B}\right]$$

$$+ \frac{\partial}{\partial t}\sum_\alpha D_{1\alpha} + \operatorname{div}\sum_\alpha \underset{\sim}{F}_{2\alpha}$$

$$= -\sum_\alpha m_\alpha \frac{\partial \underset{\sim}{v}_E}{\partial t}\cdot\left(\underset{\sim}{\Gamma}_{1\perp\alpha} - \underset{\sim}{\Gamma}_{E\alpha}\right), \tag{5.18}$$

the energy source term on the r.h.s. being of higher order in ε, viz. of $O(\varepsilon)$. This shows that the total energy is approximately conserved.

6. Conclusion

Non-relativistic guiding-center equations of motion that include the polarization drift [1, 2] were rederived from an appropriate Lagrangian. They conserve single-guiding-center energy in time-independent fields, obey a Liouville's theorem for an appropriate phase space volume element, and are identical with the results of Littlejohn's Hamiltonian theory [1] when higher-order terms in $\varepsilon \equiv r_g / L$ are dropped (see Appendix A). From this guiding-center mechanics, collisionless kinetic guiding-center equations and moment equations - for guiding-center density and energy density - were deduced. Unlike in the case without polarization drift [3], it appears here that the kinetic guiding-center theory <u>with</u> polarization drift, when coupled with Maxwell's equations, does not exactly conserve the total energy of the guiding-center plasma and its self-consistent fields. Rather an artificial energy source of higher order in $\varepsilon \equiv r_g / L$, viz. $O(\varepsilon)$, exists. Preliminary investigations indicate that this property will persist for other guiding-center theories with polarization drift whenever exact energy conservation for single guiding centers holds. This property must be taken into account when the theory is applied, e.g. to problems of drift instabilities and anomalous transport. Still, employing Littlejohn's mechanics [1, 2] appears to be the best possible basis for a rational kinetic guiding-center theory. In addition, Littlejohn's guiding-center mechanics [1, 2, 3] continues to be superior to earlier guiding-center theories as far as single guiding centers in given external fields are concerned. Moreover, in the case of "drift scaling" where the polarization drift and other higher-order corrections are omitted, conservation of the total energy of the guiding-center plasma and its self-consistent fields _is_ in fact compatible with conservation of single-guiding-center energy (and with validity of Liouville's theorem) [3, 4].

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Appendix A. Verification of the Results of Sect. 2

It is shown here that the results of Sect. 2 agree with the results that Littlejohn derived with a Hamiltonian formalism [1]. Unlike in this paper, Littlejohn [1] uses the gauge

$$\underset{\sim}{E}{}_L^* \equiv \underset{\sim}{E} - \frac{m}{e}\left\{ v_\parallel \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} + \frac{\partial \underset{\sim}{v}_E}{\partial t} \right\}$$

$$- \frac{m}{2} \nabla\left(v_E^2\right) + O\left(\varepsilon^2\right) \tag{A.1}$$

and, in effect,

$$\left(W_K\right)_L \equiv \frac{m}{2} v_\parallel^2 + \mu B + O\left(\varepsilon^2\right). \tag{A.2}$$

This is equivalent to replacing Φ by

$$\Phi_L^* \equiv \Phi + \frac{m}{2e} v_E^2 + O\left(\varepsilon^2\right) \tag{A.3}$$

and leaves the expressions L, $\left(e\Phi + W_K\right)$, and $\left(e\underset{\sim}{E}^* - \nabla W_K\right)$ invariant. Littlejohn's [1] definitions of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are the same as in this paper, except for terms of $O\left(\varepsilon^2\right)$. It should be noted, however, that Littlejohn [1] uses the convention m = c = e = sign (e) = 1, while in this paper normal Gaussian units and sign (e) = $\pm$ 1 are employed.

When the differences in gauge, in units, and in notation are taken into account, it can easily be seen that eq. (2.18) above for $\underset{\sim}{v}$ is identical with eq. (D 9) of Littlejohn [1] . Equally, eq. (2.19) above for $\dot{v}_{\parallel}$ is identical with eq. (D 10) of Littlejohn [1] . This proves that our choice of the Lagrangian L [eqs. (2.1) through (2.4)] is correct. In addition, Littlejohn [1] shows that his (and hence our) guiding-center equations of motion agree to leading orders in ε with the usual ones, viz.

$$\underset{\sim}{v} = v_{\parallel} \hat{\underset{\sim}{b}} + \underset{\sim}{v}_E + \underset{\sim}{v}_{\nabla B} + \frac{1}{\Omega} \hat{\underset{\sim}{b}} \times \left\{ \frac{\partial}{\partial t} + v_{\parallel} \frac{\partial}{\partial s} + \underset{\sim}{v}_E \cdot \nabla \right\} \left(v_{\parallel} \hat{\underset{\sim}{b}} + \underset{\sim}{v}_E \right) + O(\varepsilon^2) , \tag{A.4}$$

with

$$\underset{\sim}{v}_{\nabla B} \equiv \frac{\mu}{m\Omega} \hat{\underset{\sim}{b}} \times \nabla B , \tag{A.5}$$

and

$$\dot{v}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m} \frac{\partial B}{\partial s} + \underset{\sim}{v}_E \cdot \left\{ \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} + v_{\parallel} \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} + \underset{\sim}{v}_E \cdot \nabla \hat{\underset{\sim}{b}} \right\} + O(\varepsilon). \tag{A.6}$$

Here, eqs. (A.4) and (A.6) are transscriptions of Littlejohn's [1] eqs. (D 11) and (D 12).

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