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Lagrangian Formulation of a Consistent
Relativistic Guiding Center Theory

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Abstract

A new relativistic guiding center mechanics is presented that conserves energy (in time-independent fields) and satisfies a Liouville's theorem. The theory reduces to Littlejohn's theory in the non-relativistic limit and agrees to leading orders in $\varepsilon \equiv r_g / L$ with the relativistic theory by Morozov and Solov'ev (which generally lacks a Liouville's theorem). The new theory is developed from an appropriate Lagrangian and is supplemented by a collisionless relativistic kinetic equation for the guiding centers. Moment equations for guiding center density and energy density are also derived.

1. Introduction

An energy-conserving relativistic guiding center mechanics exists [1]. However, this theory by Morozov and Solov'ev does not generally obey a Liouville's theorem. A Liouville's theorem is desirable for several reasons ([2], [3]), particularly in order to formulate a rational kinetic theory. This paper presents a new relativistic guiding center mechanics that satisfies both an energy theorem and a Liouville's theorem. In agreement with earlier work [3] we term such a guiding center theory "consistent". The starting point is a guiding center Lagrangian which is a relativistic generalization of a non-relativistic Lagrangian given by Littlejohn [4]. It also provides for conservation of canonical momenta in cases of spatial symmetry. The theory agrees to leading orders in $\varepsilon \equiv r_g / L$ (r_g = gyro-radius, L = macroscopic length scale) with the relativistic guiding center theory of Morozov and Solov'ev [1]. Relativistic drift scaling (see Appendix C) is used throughout. From the new guiding center mechanics a collisionless guiding center kinetic theory is derived with the methods of ref.[3]. Moment equations (continuity, energy) are obtained from the kinetic equation, and an effective guiding center current is defined. Our theory reduces to the one given by Littlejohn and the present author ([2], [3], [4]) in the non-relativistic limit.

It will be seen that the new, consistent, relativistic guiding center mechanics is not Lorentz invariant. Similarly, Littlejohn's non-relativistic theory ([2], [3], [4]) is not Galilei invariant. Strong arguments exist to the effect that this lack of Lorentz invariance (or Galilei invariance, respectively) is unavoidable in consistent guiding center theories, because conservation of energy and Lorentz invariance (or Galilei invariance) appear to be incompatible in guiding center theories [6]. On the other hand, non-invariant theories are frequent in plasma physics; for instance, all existing guiding center theories cited in [3] and in this paper are non-invariant. A Galilei invariant, non-relativistic guiding center theory has been formulated, but it lacks an energy theorem [6]. Concerning the principle of relativity this physical law remains, of course, effective. However, any non-invariant theory is of necessity non-unique, as is explained in ref.[6].

The paper is organized as follows. The relativistic guiding center mechanics is given in Sec.2, while the kinetic theory is presented in Sec.3. Section 4 presents the conclusions. For reference, Appendix A collects the main results of Morozov and Solov'ev [1], Appendix B those by Littlejohn [4], and Appendix C explains the drift scaling in the relativistic case.

2. Consistent Relativistic Guiding Center Mechanics

In agreement with earlier work [3] we call a guiding center theory "consistent" if it satisfies an energy theorem (in time-independent fields) and a Liouville's theorem. We start with the guiding center Lagrangian

$$L \equiv \frac{e}{c} \underset{\sim}{A}^{*} \cdot \underset{\sim}{v} - e\phi - W_K , \tag{2.1}$$

with L depending on the variables t, $\underset{\sim}{x}$, $\underset{\sim}{v} \equiv \dot{\underset{\sim}{x}}$, $u_{\parallel}$ and the parameters $J_{\perp}$, e, m_0, and c. Here $\underset{\sim}{A}^{*}$ is a modified vector potential, viz.

$$\underset{\sim}{A}^{*} \equiv \underset{\sim}{A}(t, \underset{\sim}{x}) + \frac{m_0 c}{e} u_{\parallel} \hat{\underset{\sim}{b}}(t, \underset{\sim}{x}) , \tag{2.2}$$

$\phi(t, \underset{\sim}{x})$ is the scalar potential, and W_K is the kinetic energy (including the rest energy) in the form

$$W_K \equiv m_0 c^2 \left(1 + \frac{u_{\parallel}^2}{c^2} + \frac{J_{\perp} B}{c^2}\right)^{\frac{1}{2}} , \tag{2.3}$$

with $B(t, \underset{\sim}{x})$ the magnetic field strength, $\underset{\sim}{B}(t, \underset{\sim}{x})$ the magnetic field, and $\hat{\underset{\sim}{b}} \equiv \underset{\sim}{B}/B$ the unit vector in the direction of $\underset{\sim}{B}$. The parameter $J_{\perp}$ is the perpendicular adiabatic invariant [1] and satisfies the relations $\dot{J}_{\perp} \equiv dJ_{\perp}/dt = 0$ and

$$J_{\perp} B = \gamma^2 w_{\perp}^2 = \frac{2\gamma\mu B}{m_0} , \tag{2.4}$$

with μ the (scalar) magnetic moment, $\underset{\sim}{w}_{\perp}$ the gyration velocity (without the guiding center drift), and

$$\gamma \equiv \frac{W_K}{m_0 c^2} = \left(1 + \frac{u_\parallel^2}{c^2} + \frac{J_\perp B}{c^2}\right)^{\frac{1}{2}}. \tag{2.5}$$

The abbreviation

$$v_\parallel \equiv u_\parallel / \gamma \tag{2.6}$$

will also be used. However, the interpretation of $v_\parallel$ as the "parallel velocity" of the guiding center, in the form $v_\parallel \equiv \underset{\sim}{v} \cdot \hat{\underset{\sim}{b}}$, is not yet here implied; it will follow from one of the Lagrangian equations.

Equation (2.1) is a plausible ansatz. The equations of motion following from this Lagrangian can be proved to be correct by showing them to agree, to leading orders in ε, with the theory of Morozov and Solov'ev [1]. See Appendices A and C. This comparison is easy and is, hence, left to the reader.

The Lagrangian equations are [5]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_i}\right) = \frac{\partial L}{\partial z_i}, \tag{2.7}$$

where the z_i are arbitrary coordinates, and $L = L(t, z_i, \dot{z}_i)$. Here $\{z_i\} \equiv \{\underset{\sim}{x}, u_\parallel\}$ is chosen. As a parameter $J_\perp$ is not included among the z_i. The conjugated momenta are defined as

$$p_i \equiv \frac{\partial L}{\partial \dot{z}_i}. \tag{2.8}$$

One notes that the momentum $p_\parallel$ associated with $u_\parallel$ does not exist since $\partial L / \partial \dot{u}_\parallel$ vanishes identically. However,

$$\underset{\sim}{p} \equiv \frac{\partial L}{\partial \underset{\sim}{v}} \equiv \frac{e}{c} \underset{\sim}{A}^* \equiv m_0 u_\parallel \hat{\underset{\sim}{b}} + \frac{e}{c} \underset{\sim}{A} \tag{2.9}$$

exists and, according to eq.(2.7), is conserved if $\partial L/\partial \underset{\sim}{x} \equiv \nabla L = 0$. In inhomogeneous fields one or two spatial derivatives of L vanish at most so that only special components of $\underset{\sim}{p}$, to be expressed in appropriately transformed coordinates, may be conserved. We shall not need to transform to a Hamiltonian representation [4] in what follows. However, the definition of "modified fields" $\underset{\sim}{B}^*$ and $\underset{\sim}{E}^*$ ([2], [3]) will prove useful, viz.

$$\underset{\sim}{B}^* \equiv \text{curl}\ \underset{\sim}{A}^* \equiv \underset{\sim}{B} + \frac{m_0 c}{e} u_\parallel\ \text{curl}\ \hat{\underset{\sim}{b}}\ , \tag{2.10}$$

$$\underset{\sim}{E}^* \equiv -\nabla\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}^*}{\partial t} \equiv \underset{\sim}{E} - \frac{m_0}{e} u_\parallel \frac{\partial \hat{\underset{\sim}{b}}}{\partial t}\ . \tag{2.11}$$

It should be observed that time- and space-derivatives are always formed with $u_\parallel$ (and $J_\perp$) kept constant. The "modified homogeneous Maxwell equations", viz.

$$\text{div}\ \underset{\sim}{B}^* = 0\ , \tag{2.12}$$

$$\frac{\partial \underset{\sim}{B}^*}{\partial t} = -c\ \text{curl}\ \underset{\sim}{E}^* \tag{2.13}$$

follow and will prove useful. We shall also need the parallel component of $\underset{\sim}{B}^*$, viz.

$$B_\parallel^* \equiv \hat{\underset{\sim}{b}} \cdot \underset{\sim}{B}^* \equiv B + \frac{m_0 c}{e} u_\parallel \left(\hat{\underset{\sim}{b}} \cdot \text{curl}\ \hat{\underset{\sim}{b}}\right). \tag{2.14}$$

Let us consider the components of eq.(2.7). The equation

$$\frac{\partial L}{\partial u_\parallel} = 0 \tag{2.15}$$

yields the relation

$$\underset{\sim}{v} \cdot \hat{\underset{\sim}{b}} = v_\parallel\ . \tag{2.16}$$

The other guiding center equations of motion are obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underset{\sim}{v}}\right) = \frac{\partial L}{\partial \underset{\sim}{x}} \equiv \nabla L , \tag{2.17}$$

which reads

$$\frac{e}{c}\frac{d\underset{\sim}{A}^*}{dt} = \frac{e}{c}\nabla\left(\underset{\sim}{A}^*\cdot\underset{\sim}{v}\right) - e\nabla\Phi - \nabla W_K . \tag{2.18}$$

By using

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla + \dot{u}_{\parallel}\frac{\partial}{\partial u_{\parallel}} \tag{2.19}$$

and several of the above relations this can be transformed to yield

$$m_0\,\dot{u}_{\parallel}\,\hat{\underset{\sim}{b}} = \left(e\underset{\sim}{E}^* - \nabla W_K\right) + \frac{e}{c}\,\underset{\sim}{v}\times\underset{\sim}{B} . \tag{2.20}$$

This equation is the starting point of the following. It yields an equation for the guiding center drift velocity $\underset{\sim}{v}_{\perp}$ and another one for the "parallel acceleration" $\dot{u}_{\parallel}$.

By forming the cross product of eq.(2.20) with $\hat{\underset{\sim}{b}}$ one obtains for $\underset{\sim}{v}$:

$$B_{\parallel}^*\,\underset{\sim}{v} = v_{\parallel}\,\underset{\sim}{B}^* + \frac{c}{e}\left(e\underset{\sim}{E}^* - \nabla W_K\right)\times\hat{\underset{\sim}{b}} , \tag{2.21}$$

and for $\underset{\sim}{B}^*$:

$$v_\parallel \underset{\sim}{B}^* = B_\parallel^* \underset{\sim}{v} - \frac{c}{e}\left(e\underset{\sim}{E}^* - \nabla W_K\right) \times \hat{\underset{\sim}{b}} \,. \tag{2.22}$$

The latter equation will be needed below. On inserting the explicit expressions for $\underset{\sim}{E}^*$ and $\underset{\sim}{B}^*$, and using

$$\nabla W_K = \frac{m_0 J_\perp}{2\gamma} \nabla B \equiv \mu \nabla B \tag{2.23}$$

the guiding center velocity is obtained, viz.

$$\underset{\sim}{v} = v_\parallel \hat{\underset{\sim}{b}} + \frac{c}{eB_\parallel^*}\left(e\underset{\sim}{E}^* - \nabla W_K - m_0 u_\parallel v_\parallel \frac{\partial \hat{\underset{\sim}{b}}}{\partial s}\right) \times \hat{\underset{\sim}{b}} \tag{2.24}$$

or, alternatively,

$$\underset{\sim}{v} = v_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{v}_E^* + \underset{\sim}{v}_{\nabla B}^* + \underset{\sim}{v}_k^* + \underset{\sim}{v}_{CD}^* \,, \tag{2.24a}$$

the components of the drift velocity being defined as

$$\underset{\sim}{v}_E^* \equiv \frac{c}{B_\parallel^*} \underset{\sim}{E} \times \hat{\underset{\sim}{b}} \,, \tag{2.25}$$

$$\underset{\sim}{v}_{\nabla B}^* \equiv \frac{J_\perp}{2\gamma \Omega_0^*} \hat{\underset{\sim}{b}} \times \nabla B \equiv \frac{\mu}{m_0 \Omega_0^*} \hat{\underset{\sim}{b}} \times \nabla B \,, \tag{2.26}$$

$$\underset{\sim}{v}_k^* \equiv \frac{u_\parallel v_\parallel}{\Omega_0^*} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} \,, \tag{2.27}$$

$$\underset{\sim}{v}_{CD}^* \equiv \frac{u_\parallel}{\Omega_0^*} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} \,, \tag{2.28}$$

with the definitions

$$\Omega_0^* \equiv \frac{e B_\parallel^*}{m_0 c} , \qquad (2.29)$$

$$\frac{\partial}{\partial s} \equiv \hat{\underset{\sim}{b}} \cdot \nabla . \qquad (2.30)$$

By forming the scalar product of eq.(2.20) with $\underset{\sim}{B}^*$ one obtains a first expression for $\dot{u}_\parallel$, viz.

$$\dot{u}_\parallel = \frac{1}{m_0 B_\parallel^*} \underset{\sim}{B}^* \cdot \left(e \underset{\sim}{E}^* - \nabla W_K \right) . \qquad (2.31)$$

By using eq.(2.22) for $\underset{\sim}{B}^*$ this is transformed in a remarkable way, viz.

$$\dot{u}_\parallel = \frac{1}{m_0 V_\parallel} \underset{\sim}{V} \cdot \left(e \underset{\sim}{E}^* - \nabla W_K \right) . \qquad (2.32)$$

On decomposing $\underset{\sim}{V}$, viz. $\underset{\sim}{V} = V_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{V}_\perp$, and using eqs.(2.21), (2.11), and (2.23) one finally obtains

$$\dot{u}_\parallel = \frac{e}{m_0} E_\parallel - \frac{J_\perp}{2\gamma} \frac{\partial B}{\partial s} + u_\parallel \underset{\sim}{V} \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} , \qquad (2.33)$$

with

$$\frac{J_\perp}{2\gamma} \equiv \frac{\mu}{m_0} . \qquad (2.34)$$

The relation from eq.(2.24), viz.

$$\frac{1}{m_0 V_\parallel} \underset{\sim}{V}_\perp \cdot \left(e\underset{\sim}{E}^* - \nabla W_k \right) = u_\parallel \underset{\sim}{V} \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} , \qquad (2.35)$$

is important in deriving eq.(2.33).

Next the energy equation is to be derived. On using eqs.(2.19) and (2.32) one obtains

$$\frac{dW_K}{dt} = e \underset{\sim}{E}^* \cdot \underset{\sim}{V} + \frac{\partial W_K}{\partial t} . \qquad (2.36)$$

A more explicit expression is found by inserting $\underset{\sim}{E}^*$ from eq.(2.11) and observing that

$$\frac{\partial W_K}{\partial t} = \frac{m_0 J_\perp}{2\gamma} \frac{\partial B}{\partial t} \equiv \mu \frac{\partial B}{\partial t} , \qquad (2.37)$$

whence the energy equation reads

$$\frac{dW_K}{dt} = e \underset{\sim}{E} \cdot \underset{\sim}{V} - \underset{\sim}{\mu} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} , \qquad (2.38)$$

with the definition

$$\underset{\sim}{\mu} \equiv - \frac{m_0 J_\perp}{2\gamma} \hat{\underset{\sim}{b}} + \frac{m_0 u_\parallel}{B} \underset{\sim}{V}_\perp . \qquad (2.39)$$

In order to confirm conservation of energy in time-independent fields we put $\partial/\partial t = 0$ in eq.(2.38), to obtain

$$\frac{d}{dt}\left(W_K + e\phi\right) = 0, \tag{2.40}$$

or $W_K + e\phi$ = const along orbits.

Liouville's theorem remains to be proved. Generally, the correct phase space volume element $d\tau$ must be constructed by transforming to a Hamiltonian formalism [4]. Here, however, we use an intelligent guess, viz.

$$d\tau \equiv \pi B_{\parallel}^{*}\, d^3x\; du_{\parallel}\; dJ_{\perp}, \tag{2.41}$$

and prove $d\dot{\tau} \equiv d(d\tau)/dt = 0$. The perpendicular adiabatic invariant $J_{\perp}$ has now been included among the phase space coordinates $\{\alpha_i\} \equiv \{\underset{\sim}{x}, u_{\parallel}, J_{\perp}\}$, which makes no difference in the calculation, since $\dot{J}_{\perp} = 0$. It is known [3] that $d\dot{\tau} = 0$ holds if

$$\delta \equiv \frac{\partial B_{\parallel}^{*}}{\partial t} + \mathrm{div}\left(B_{\parallel}^{*}\underset{\sim}{v}\right) + \frac{\partial}{\partial u_{\parallel}}\left(B_{\parallel}^{*}\dot{u}_{\parallel}\right) = 0 \tag{2.42}$$

is satisfied. Here $\underset{\sim}{v}$ must be taken from eq.(2.21) and $\dot{u}_{\parallel}$ from eq.(2.31). Let us prove eq.(2.42) to be true. One has

$$\frac{\partial B_{\parallel}^{*}}{\partial t} = \underset{\sim}{B}^{*}\cdot\frac{\partial\hat{\underset{\sim}{b}}}{\partial t} + \hat{\underset{\sim}{b}}\cdot\frac{\partial\underset{\sim}{B}^{*}}{\partial t}, \tag{2.43}$$

$$\mathrm{div}\left(B_{\parallel}^{*}\underset{\sim}{v}\right) = \underset{\sim}{B}^{*}\cdot\nabla v_{\parallel} + c\,\hat{\underset{\sim}{b}}\cdot\mathrm{curl}\,\underset{\sim}{E}^{*}$$
$$- \frac{c}{e}\left(e\underset{\sim}{E}^{*} - \nabla W_K\right)\cdot\mathrm{curl}\,\hat{\underset{\sim}{b}}, \tag{2.44}$$

$$\frac{\partial}{\partial u_{\parallel}}\left(B_{\parallel}^{*}\dot{u}_{\parallel}\right) = \frac{1}{m_0}\frac{\partial}{\partial u_{\parallel}}\left[\left(e\underset{\sim}{E}^{*} - \nabla W_K\right)\cdot\underset{\sim}{B}^{*}\right]. \tag{2.45}$$

Hence the l.h.s. of eq.(2.42) is transformed to read

$$\delta \equiv \frac{e}{m_0}\left\{\frac{\partial \underset{\sim}{E}^*}{\partial u_\parallel}+\frac{m_0}{e}\frac{\partial \hat{\underset{\sim}{b}}}{\partial t}\right\}\cdot \underset{\sim}{B}^*$$

$$+\frac{1}{m_0}\left\{\frac{\partial \underset{\sim}{B}^*}{\partial u_\parallel}-\frac{m_0 c}{e}\,\mathrm{curl}\,\hat{\underset{\sim}{b}}\right\}\cdot\left(e\underset{\sim}{E}^*-\nabla W_K\right). \qquad (2.46)$$

In doing this transformation eqs.(2.12) and (2.13) have also been used. Inspection of eqs.(2.10) and (2.11) for the "modified fields" shows that both curly brackets in eq.(2.46) vanish. This proves the validity of eq.(2.42) and, hence, that of Liouville's theorem concerning the $d\tau$ of eq.(2.41).

In this section we have derived a relativistic guiding center mechanics from a Lagrangian. An energy theorem and a Liouville's theorem are satisfied. The explicit results are given by eqs.(2.16), (2.24) through (2.28), (2.33), (2.38) with (2.39) and (2.40), (2.41) with $d\dot{\tau}=0$ or (2.42). As mentioned, the correctness of this theory to leading orders in ε may be proved by comparing with Morozov and Solov'ev's theory [1] as given in Appendix A (below).

3. Collisionless Relativistic Kinetic Equation for Guiding Centers

We give a collisionless kinetic theory for the relativistic guiding center mechanics established in Sec.2. The phase space is 5 dimensional, with coordinates $\{\alpha_i\} \equiv \{\underset{\sim}{x},\, u_\parallel,\, J_\perp\}$, $i = 1$ to 5. The volume element in phase space, $d\tau$, is given by eq.(2.41) and, hence, is Liouvillian, i.e. $d\dot{\tau} = 0$. A guiding center distribution function f is defined by

$$dN \equiv f\, d\tau\,, \tag{3.1}$$

with $f = f(t, \underset{\sim}{x}, u_\parallel, J_\perp)$, N being the number of guiding centers (in phase space). The collisionless kinetic equation expresses the conservation of dN in a volume element $d\tau$ that moves with the guiding centers, i.e.

$$\frac{d}{dt}(dN) \equiv \frac{d}{dt}(f\, d\tau) = 0\,. \tag{3.2}$$

This equation can be reformulated [3] to read

$$\frac{\partial}{\partial t}(B_\parallel^* f) + \mathrm{div}(B_\parallel^* f \underset{\sim}{v}) + \frac{\partial}{\partial u_\parallel}(B_\parallel^* f\, \dot{u}_\parallel) = 0. \tag{3.3}$$

This form of the kinetic equation holds independent of whether $d\tau$ is Liouvillian or not. Since our $d\tau$ <u>is</u> Liouvillian eq.(3.3) can be simplified to read

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \dot{u}_\parallel \frac{\partial f}{\partial u_\parallel} = 0. \tag{3.4}$$

Here $\underset{\sim}{v}$ must be taken from eqs.(2.24) through (2.28) and $\dot{u}_\parallel$ from eq.(2.33). The transition from eq.(3.3) to eq.(3.4) follows from eq.(2.42). In time-independent fields equilibrium distribution functions have the form

$$f_0 = f_0 \left(W_K + e\Phi,\ J_\perp,\ \text{other integrals of the motion} \right). \qquad (3.5)$$

When moment equations are to be derived from the kinetic equation the use of eq.(3.3) is to be preferred. We shall derive the equation of continuity and an energy equation. On defining the guiding center density

$$n \equiv \int d\tau_v \, f \qquad (3.6)$$

and the guiding center flux density

$$\underset{\sim}{\Gamma} \equiv \int d\tau_v \, f \, \underset{\sim}{v} \, , \qquad (3.7)$$

with the definition of the volume element in velocity space as

$$d\tau_v \equiv \pi B_\parallel^* \, du_\parallel \, dJ_\perp \, , \qquad (3.8)$$

integration of eq.(3.3) over $du_\parallel \, dJ_\perp$ yields

$$\frac{\partial n}{\partial t} + \operatorname{div} \underset{\sim}{\Gamma} = 0 . \qquad (3.9)$$

Here, one has assumed that $(B_\parallel^* \dot{u}_\parallel f) \to 0$ for $|u_\parallel| \to \infty$. The ranges of integration are, of course, $-\infty < u_\parallel < +\infty$ and $0 \leq J_\perp < \infty$.

In order to formulate the energy equation let us define the kinetic energy density

$$D \equiv \int d\tau_v \, f \, W_K \, , \qquad (3.10)$$

the kinetic energy flux density, viz.

$$\underset{\sim}{F}_1 \equiv \int d\tau_v \, f \, \underset{\sim}{v} \, W_K \, , \qquad (3.11)$$

and the magnetic moment density

$$\underset{\sim}{M} \equiv \int d\tau_v \, f \, \underset{\sim}{\mu} \, , \tag{3.12}$$

with $\underset{\sim}{\mu}$ to be taken from eq.(2.39).

Multiplying eq.(3.3) by $W_K \, du_\parallel \, dJ_\perp$ and integrating over ($u_\parallel$, $J_\perp$) space yields

$$\frac{\partial D}{\partial t} + \operatorname{div} \underset{\sim}{F}_1 = \int d\tau_v \, f \, \frac{dW_K}{dt} \tag{3.13}$$

or [eq.(2.38)]

$$\frac{\partial D}{\partial t} + \operatorname{div} \underset{\sim}{F}_1 = e \underset{\sim}{E} \cdot \underset{\sim}{\Gamma} - \underset{\sim}{M} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} \, . \tag{3.14}$$

It is desirable to transform eq.(3.14) so that the new right-hand side has the form $\underset{\sim}{E} \cdot \underset{\sim}{j}_{eff}$, where $\underset{\sim}{j}_{eff}$ is the "effective guiding center current density" [3]. In this way, the guiding center theory can be associated with Maxwell's equations. By using

$$\frac{\partial \underset{\sim}{B}}{\partial t} = - c \operatorname{curl} \underset{\sim}{E} \tag{3.15}$$

and doing a partial differentiation one obtains

$$\frac{\partial D}{\partial t} + \operatorname{div} \underset{\sim}{F} = \underset{\sim}{E} \cdot \left(e \underset{\sim}{\Gamma} + c \operatorname{curl} \underset{\sim}{M} \right) , \tag{3.16}$$

with the definition of the effective current density (of one single guiding center component) as

$$\underset{\sim}{j}_{eff} \equiv e \underset{\sim}{\Gamma} + c \, \mathrm{curl} \, \underset{\sim}{M} \tag{3.17}$$

and of the effective energy flux density as

$$\underset{\sim}{F} \equiv \underset{\sim}{F}_1 + c \, \underset{\sim}{M} \times \underset{\sim}{E} \, . \tag{3.18}$$

The total electric current density of the guiding center plasma is obtained by

$$\underset{\sim}{j}_{tot} \equiv \sum_{\alpha} \left(e_\alpha \underset{\sim}{\Gamma}_\alpha + c \, \mathrm{curl} \, \underset{\sim}{M}_\alpha \right), \tag{3.19}$$

i.e. by summing over the plasma components, e.g. $\alpha = i, e$. Then

$$\sum_{\alpha} \left(\frac{\partial D_\alpha}{\partial t} + \mathrm{div} \, \underset{\sim}{F}_\alpha \right) = \underset{\sim}{E} \cdot \underset{\sim}{j}_{tot} \, . \tag{3.20}$$

The energy equation for the electromagnetic fields is given by

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \left(E^2 + B^2 \right) + \frac{c}{4\pi} \, \mathrm{div} \left(\underset{\sim}{E} \times \underset{\sim}{B} \right) = - \underset{\sim}{E} \cdot \underset{\sim}{j}_{tot} \, . \tag{3.21}$$

Summing eqs.(3.20) and (3.21) yields conservation of total energy. The results of this section are very similar to those obtained in the non-relativistic case [3].

4. Conclusion

A consistent relativistic guiding center mechanics has been derived from a guiding center Lagrangian. It conserves energy in time-independent fields and obeys a Liouville's theorem for an appropriate volume element in guiding center phase space. The new theory agrees to leading orders in $\varepsilon \equiv r_g/L$ with the earlier theory by Morozov and Solov'ev [1] and reduces to Littlejohn's theory [2], [3], [4] in the non-relativistic limit. Collisionless kinetic theory and moment equations have also been considered. Drift scaling was assumed throughout. The new theory and the other theories mentioned all lack Lorentz invariance, or Galilei invariance, respectively; the structural reason for this has been explained in ref.[6].

The new relativistic guiding center theory can find application to run-away electrons in laboratory plasmas and to relativistic particles in space. In fact, run-away drifts in time-independent electromagnetic fields were considered by Zehrfeld et al. [7] on the basis of the earlier relativistic guiding center theory [1]. If needed, the guiding center kinetic equation of Sec.3 could be supplemented by a relativistic collision term.

The Lagrangian formalism introduced by Littlejohn [4] in the non-relativistic problem and generalized to the relativistic one here is not only concise and elegant; it also makes clear that the guiding center equations of motion are strongly interdependent. In particular, one would destroy some or all symmetries of the theory (or even create a singularity at $u_\parallel \rightarrow 0$) if one tried to approximate the drift velocity $\underset{\sim}{V}_\perp$ by dropping one or several of the magnetic drift velocities, viz. $\underset{\sim}{V}^*_{\nabla B}$, $\underset{\sim}{V}^*_{\kappa}$, and/or $\underset{\sim}{V}^*_{CD}$. This point was earlier discussed in some detail in the case of the non-relativistic guiding center theory [3].

This paper has limited itself to leading-order approximations in ε . Higher-order terms, e.g. $\underset{\sim}{V}^*_{CD}$ are included as far as they are necessary for the required symmetries of the theory, i.e. the conservation theorems. Littlejohn's non-relativistic work [4] is, within its subject, more comprehensive in that the guiding center Lagrangian is systematically derived from the particle Lagrangian up to higher orders in ε . This has not been done in our relativistic case here since it was not needed for proving the correctness of our theory. If desired, this could be done along the lines followed by Littlejohn [4] in the non-relativistic case.

Acknowledgement

This work was motivated by Littlejohn's paper on Variational Principles of Guiding Center Motion [4]. I am very grateful to R.G. Littlejohn for making this paper available to me prior to publication. I also thank D. Correa Restrepo for pointing out ref.[1].

Appendix A. Summary of the Relativistic Guiding Center Theory by Mozorov and Solov'ev

The theory by Morozov and Solov'ev [1] is defined by the following results:

$$\dot{J}_\perp = 0 , \tag{A.1}$$

$$\dot{W}_k = e\,\underset{\sim}{E}\cdot\underset{\sim}{V} + \frac{m_0 J_\perp}{2\gamma}\frac{\partial B}{\partial t} , \tag{A.2}$$

$$\underset{\sim}{V} = V_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{V}_E + \underset{\sim}{V}_{\nabla B} + \underset{\sim}{V}_K , \tag{A.3}$$

with the definitions

$$\underset{\sim}{V}_E \equiv \frac{c}{B}\,\underset{\sim}{E}\times\hat{\underset{\sim}{b}} , \tag{A.4}$$

$$\underset{\sim}{V}_{\nabla B} \equiv \frac{J_\perp}{2\gamma\Omega_0}\,\hat{\underset{\sim}{b}}\times\nabla B , \tag{A.5}$$

$$\underset{\sim}{V}_K \equiv \frac{V_\parallel u_\parallel}{\Omega_0}\,\hat{\underset{\sim}{b}}\times\frac{\partial\hat{\underset{\sim}{b}}}{\partial s} , \tag{A.6}$$

$$\Omega_0 \equiv \frac{eB}{m_0 c} \quad . \tag{A.7}$$

Otherwise the notation is that of Sec.2. In particular eqs.(2.3) through (2.6) and (2.16) for W_K, γ, $u_\parallel$, $V_\parallel$, $\underset{\sim}{V} \cdot \hat{\underset{\sim}{b}}$ are in effect. Morozov and Solov'ev [1] do not give an explicit expression for $\dot{u}_\parallel$, but this can be supplemented, viz.

$$\dot{u}_\parallel = \frac{e}{m_0} E_\parallel - \frac{J_\perp}{2\gamma} \frac{\partial B}{\partial s} + u_\parallel \underset{\sim}{V} \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} \quad . \tag{A.8}$$

For time-independent fields conservation of energy follows from eq.(A.3), viz.

$$W_K + e\Phi = \text{const} \tag{A.9}$$

along orbits. It is seen that eqs.(A.1) and (A.8) agree exactly with the corresponding relations of Sec.2, while eq.(A.2) and $\underset{\sim}{V}$ of eqs.(A.3) through (A.6) only deviate by terms of $O(\varepsilon^2)$, if drift ordering (see Appendix C) is assumed. In particular, $\underset{\sim}{V}^*_{CD}$ of eq.(2.28) is of order ε^2. Note that the agreement of eq.(A.8) with eq.(2.33) is only formal, since the two quantities $\underset{\sim}{V}$ are not the same in the two equations. Physically, the discrepancy is again of $O(\varepsilon^2)$. As mentioned in Sec.1, this theory is not Lorentz invariant and it does not generally satisfy a Liouville's theorem. The lack of Lorentz invariance is explained in ref.[6].

Appendix B. Littlejohn's Non-relativistic Guiding Center Theory Summarized

This summary is restricted to the special form Littlejohn's theory [2] assumes when drift scaling (see Appendix C) is assumed [3], [4]. Littlejohn [4] uses an extended Lagrangian that contains the gyro-motion, viz.

$$\tilde{L} \equiv \frac{e}{c} \underset{\sim}{A}^{*} \cdot \underset{\sim}{v} - e\phi - \mu B - \frac{m}{2} v_{\parallel}^{2} + \frac{mc}{e} \mu \dot{\theta}, \quad \text{(B.1)}$$

where θ is the gyration angle and μ is one of the independent variables rather than a parameter. We prefer to start with an abbreviated Lagrangian, viz.

$$L \equiv \frac{e}{c} \underset{\sim}{A}^{*} \cdot \underset{\sim}{v} - e\phi - \mu B - \frac{m}{2} v_{\parallel}^{2}, \quad \text{(B.2)}$$

which does not contain the gyro-motion and where μ is now a parameter rather than one of the independent variables, which are t, $\underset{\sim}{x}$, $\underset{\sim}{v} \equiv \dot{\underset{\sim}{x}}$, and $v_{\parallel}$. Here it is μ that is the (perpendicular) adiabatic invariant, i.e. $\dot{\mu} = 0$. The modified vector potential is now

$$\underset{\sim}{A}^{*} \equiv \underset{\sim}{A}(t, \underset{\sim}{x}) + \frac{mc}{e} v_{\parallel} \hat{\underset{\sim}{b}}(t, \underset{\sim}{x}), \quad \text{(B.3)}$$

while the kinetic energy is, of course,

$$W_K \equiv \frac{m}{2} v_{\parallel}^{2} + \mu B. \quad \text{(B.4)}$$

The Lagrangian equations yield

$$\underset{\sim}{v} \cdot \hat{\underset{\sim}{b}} = v_{\parallel} \quad \text{(B.5)}$$

and the other equations of motion, viz.

$$\dot{v}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m}\frac{\partial B}{\partial s} + v_{\parallel}\, \mathbf{v} \cdot \frac{\partial \hat{\mathbf{b}}}{\partial s} , \tag{B.6}$$

$$\frac{dW_K}{dt} = e\, \mathbf{E} \cdot \mathbf{v} - \boldsymbol{\mu} \cdot \frac{\partial \mathbf{B}}{\partial t} , \tag{B.7}$$

with

$$\boldsymbol{\mu} \equiv -\mu \hat{\mathbf{b}} + \frac{m}{B} v_{\parallel} \mathbf{v}_{\perp} , \tag{B.8}$$

and

$$\mathbf{v} = v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}^{*}_{E} + \mathbf{v}^{*}_{\nabla B} + \mathbf{v}^{*}_{\kappa} + \mathbf{v}^{*}_{CD} , \tag{B.9}$$

with

$$\mathbf{v}^{*}_{E} \equiv \frac{c}{B^{*}_{\parallel}} \mathbf{E} \times \hat{\mathbf{b}} , \tag{B.10}$$

$$\mathbf{v}^{*}_{\nabla B} \equiv \frac{\mu}{m\Omega^{*}} \hat{\mathbf{b}} \times \nabla B , \tag{B.11}$$

$$\underset{\sim}{V}^*_K \equiv \frac{V_\parallel^2}{\Omega^*} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} , \tag{B.12}$$

$$\underset{\sim}{V}^*_{CD} \equiv \frac{V_\parallel}{\Omega^*} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} , \tag{B.13}$$

with

$$B^*_\parallel \equiv B + \frac{mc}{e} V_\parallel \; \hat{\underset{\sim}{b}} \cdot \mathrm{curl} \, \hat{\underset{\sim}{b}} , \tag{B.14}$$

$$\Omega^* \equiv e B^*_\parallel / mc . \tag{B.15}$$

The energy theorem reads

$$\frac{d}{dt} \left(W_K + e\Phi \right) = 0 . \tag{B.16}$$

The Liouville's theorem is satisfied for

$$d\tau \equiv \frac{2\pi}{m} B^*_\parallel \, d^3x \; dv_\parallel \; d\mu , \tag{B.17}$$

and a rational kinetic theory can therefore be derived [3]. Littlejohn's theory [2], [3], [4] is not Galilei invariant, as is explained in [6].

Appendix C. Relativistic Drift Scaling

The expansion parameter $\varepsilon \ll 1$ is defined as the ratio between the gyro-radius r_g and a typical macroscopic length L, i.e. $\varepsilon \equiv r_g / L$. Here

$$r_g = \frac{w_\perp}{\Omega} = \frac{\gamma w_\perp}{\Omega_0} \sim \frac{\gamma v_{th}}{\Omega_0} , \tag{C.1}$$

where v_{th} is a typical thermal velocity. For a time t defined by L/v_{th}, which is of the order of a bounce time in a toroidal magnetic field, it follows that

$$\Omega t = O(\varepsilon^{-1}) . \tag{C.2}$$

Drift ordering assumes that

$$\frac{c \underset{\sim}{E}}{v_{th} B} = O(\varepsilon) . \tag{C.3}$$

For $E_\parallel$ this implies

$$\frac{e E_\parallel t}{m_0 \gamma v_{th}} = O(1) . \tag{C.4}$$

It follows that

$$\frac{v^*_E}{v_{th}} \sim \frac{v^*_{\nabla B}}{v_{th}} \sim \frac{v^*_\kappa}{v_{th}} = O(\varepsilon) , \tag{C.5}$$

while

$$\frac{v_{CD}^{*}}{v_{th}} = O(\varepsilon^2) \tag{C.6}$$

owing to

$$\frac{\partial \hat{\underset{\sim}{b}}}{\partial t} = \frac{1}{B}\frac{\partial \underset{\sim}{B}}{\partial t} - \hat{\underset{\sim}{b}}\,\frac{\partial B}{\partial t} \tag{C.7}$$

and

$$\frac{1}{\Omega B}\frac{\partial \underset{\sim}{B}}{\partial t} = -\frac{c}{\Omega B}\,\mathrm{curl}\,\underset{\sim}{E} = O(\varepsilon^2). \tag{C.8}$$

From eq.(2.4) it follows that

$$\frac{J_\perp B}{\gamma^2 v_{th}^2} = \frac{2\mu B}{m_0 \gamma\, v_{th}^2} = O(1). \tag{C.9}$$

It is important to note that for several dimensionless quantities no ε-scaling is defined, e.g. for E/B, v_{th}/c, $e\Phi/W_K$, and others. The magnitude of these quantities is inasmuch arbitrary.

It is sometimes useful to employ a <u>dimensional representation</u> of the above drift scaling. One may then put

$$L \sim v_{th} \sim t \sim \frac{B}{c} \sim \frac{c\mu}{m_0\gamma} \sim \frac{c\,J_\perp}{\gamma^2} = O(1), \tag{C.10}$$

$$\frac{m_0\gamma}{e} \sim \frac{1}{\Omega} \sim \frac{\gamma}{\Omega_0} \sim r_g \sim \underset{\sim}{E} = O(\varepsilon)\,, \tag{C.11}$$

whence

$$\underset{\sim}{V}^*_E \sim \underset{\sim}{V}^*_{\nabla B} \sim \underset{\sim}{V}^*_{\kappa} = O(\varepsilon) \tag{C.12}$$

and

$$\underset{\sim}{V}^*_{GD} = O(\varepsilon^2)\,. \tag{C.13}$$

Again, several quantities remain <u>unscaled</u>, e.g. c , B , μ , $J_\perp$, γ , m_0 , e . While $B^*_\parallel / c$ and B/c are $O(1)$, their difference is $O(\varepsilon)$, viz.

$$\frac{B^*_\parallel}{c} = \frac{B}{c} + O(\varepsilon)\,. \tag{C.14}$$

Using these relations it can easily be shown that the theory of Sec.2 and the theory by Morozov and Solov'ev [1], as cited in Appendix A, agree to leading orders in ε . When comparing vector equations, as the ones for $\underset{\sim}{V}$, the parallel and perpendicular components must be compared separately.

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