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Spectrum of a Resistive Plasma Slab

D. Lortz and G.O. Spies

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Abstract:

The resistive perturbation of the Alfvén continuum of incompressible ideal magnetohydrodynamics is derived for an arbitrary static plane slab equilibrium. If the unperturbed continuum does not contain the origin, the perturbed eigenfrequencies fill a system of curves in the stable part of the complex plane which are independent of the resistivity.

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

The frequency spectrum of the linearized motion of a plasma is perhaps one of the most useful tools in a variety of applications such as stability, wave propagation and heating. In the present note we derive the influence of small resistivity on the Alfvén continua of ideal magnetohydrodynamics. Most previous investigations of this problem (for instance [1]- [4]) focussed on unstable resistive modes whose eigenfrequencies emerge from the origin due to the presence of "singular surfaces". In contrast, we investigate all modes which emerge from a continuum which does not contain the origin (there are no singular surfaces). Even though we consider only the simplest possible case, viz. the incompressible motion about a static plane slab equilibrium, we believe that our results are representative.

The equations of resistive incompressible magnetohydrodynamics, when linearized about a static equilibrium subject to $\nabla P + \vec{B} \times \operatorname{curl} \vec{B} = 0$ and $\operatorname{div} \vec{B} = 0$, are

$$\left.\begin{aligned} &\rho \frac{\partial \vec{u}}{\partial t} + \nabla p + \vec{B} \times \operatorname{curl} \vec{b} + \vec{b} \times \operatorname{curl} \vec{B} = 0 , \\ &\frac{\partial \vec{b}}{\partial t} + \eta \operatorname{curl} \operatorname{curl} \vec{b} - \operatorname{curl} (\vec{u} \times \vec{B}) = 0 , \\ &\operatorname{div} \vec{u} = 0 , \quad \operatorname{div} \vec{b} = 0 . \end{aligned}\right\} \qquad (1)$$

Here p and $\vec{b}$ are the perturbations of the equilibrium pressure P and magnetic field $\vec{B}$, $\vec{u}$ is the flow velocity, and ρ and η (both assumed to be constants) are the mass density and resistivity. We impose the boundary conditions pertaining to a perfectly conducting rigid wall, $\vec{u}_n = 0$, $\vec{b}_n = 0$, and $\operatorname{curl}_t \vec{b} = 0$ (the subscripts n and t denote normal and tangential components). Since resistive diffusion is ignored, the system (1) is meaningful only for small η. Thus, terms $O(\eta^{\alpha})$ with $\alpha \geq 1$ will be neglected.

Considering a slab equilibrium, characterized by $\partial/\partial y = \partial/\partial z = 0$, $B_x = 0$, and $P + \frac{1}{2} B^2 = const$, we Fourier decompose the system (1) by putting the perturbations proportional to exp $(\sigma t + ik_y y + ik_z z)$. With the abbreviations $u = -i\,u_x$, $b = b_x$, $F = \vec{k}\cdot\vec{B}$, $\vec{k} = (0, k_y, k_z)$, and $\Delta = d^2/dx^2 - k^2$, a subset of equations can be written as [1]

$$\left.\begin{aligned} &\sigma\rho\Delta u + F\Delta b + F'' b = 0\,, \\ &\eta\Delta b - \sigma b - F u = 0\,, \end{aligned}\right\} \qquad (2)$$

where primes denote derivatives with respect to x. The boundary conditions are $u = 0$ and $b = 0$ at $x = 0$ and $x = L$. In what follows we assume that there are no singular surfaces, i.e., that $F(x)$ vanishes nowhere. In other words, we assume that $B(x)$ vanishes nowhere, and consider only those wave vectors $\vec{k}$ which are nowhere perpendicular to $\vec{B}$. If there is no magnetic shear (i.e., if $\vec{B}$ is unidirectional), F is either identically zero or it has no zeroes at all. Since the system (2) is trivial if $F \equiv 0$, we thus have a genuine restriction upon $\vec{k}$ only if $\vec{B}$ has shear.

Introducing dimensionless variables by $x = L\bar{x}$, $k^2 = \bar{k}^2/L^2$, $F = F_o\bar{F}$, $\sigma = \bar{\sigma} F_o \rho^{-\frac{1}{2}}$, and $\eta = \bar{\eta} L^2 F_o \rho^{-\frac{1}{2}}$ (F_o is a characteristic value of F), then eliminating u from the system (2) and omitting the bars, we obtain the fourth-order equation

$$\left(\eta\Delta\frac{1}{F}\Delta F + \frac{d}{dx}A\frac{d}{dx} - k^2 A\right)\frac{1}{F}b = 0\,, \qquad (3)$$

where $A = -\sigma - F^2/\sigma$. Since the boundary conditions are $b = 0$ and $b'' = 0$ at $x = 0$ and $x = 1$, the dispersion relation is

$$\begin{vmatrix} b_1''(0) & b_2''(0) & b_3''(0) & b_4''(0) \\ b_1''(1) & b_2''(1) & b_3''(1) & b_4''(1) \\ b_1(0) & b_2(0) & b_3(0) & b_4(0) \\ b_1(1) & b_2(1) & b_3(1) & b_4(1) \end{vmatrix} = 0, \quad (4)$$

where $b_1, \ldots, b_4$ are any four independent solutions.

If $\eta = 0$, the zeroes of A are singular points of Eq.(3). Correspondingly, the ranges of the functions $\sigma(x) = \pm i F(x)$ form continuous spectra (the Alfvén continua). Discrete eigenvalues do not exist. In contrast, the resistive spectrum is purely discrete because Eq. (3) has no singular points if $\eta \neq 0$.

If $\eta \ll 1$ and $k^2 = O(1)$, b_3 and b_4 can be chosen to satisfy the ideal equation (i.e., Eq.(3) with $\eta = 0$), thus varying on the equilibrium scale; the remaining two solutions b_1 and b_2 can be chosen as $b = \int y\,dx$, where y satisfies

$$\eta y'' + A y = 0 \quad (5)$$

thus varying fast. Since $b_3(0)b_4(1) - b_3(1)b_4(0) \neq 0$ (the ideal spectrum contains no discrete eigenvalues), and since b_1'' and b_2'' are large compared to the other terms in the dispersion relation (4), the latter reduces asymptotically to

$$y_1'(0)\, y_2'(1) - y_1'(1)\, y_2'(0) = 0 \quad (6)$$

Thus, the eigenvalues are determined from Eq.(5) with the boundary conditions $y' = 0$.

Multiplying Eq. (5) with the complex conjugate of y and averaging over x we obtain the quadratic equation $\sigma^2 \langle |y|^2 \rangle + \sigma \eta \langle |y'|^2 \rangle + \langle F^2 |y|^2 \rangle = 0$. This implies $Re\, \sigma < 0$ and (for complex eigenvalues) $F^2_{min} \leq |\sigma|^2 \leq F^2_{max}$. Hence all eigenmodes are stable and damped, and the spectrum is restricted to the negative real axis and to the annulus which is traced out by the Alfvén continuum when rotated about the origin.

Assuming from now on that F(x) is monotonic, we introduce a new independent variable z by [5]

$$z = \eta^{-\frac{1}{3}} \left(\frac{3}{2} i \alpha \right)^{\frac{2}{3}} , \quad \alpha(x,\sigma) = \int_{x_0(\sigma)}^{x} dx \sqrt{A(x,\sigma)} , \qquad (7)$$

where x_0 solves $\sigma = i F(x)$, so that $z(x)$ is analytic near the interval $0 \leq x \leq 1$. The new dependent variable $w = y \sqrt{z'(x)}$ then satisfies asymptotically the Airy equation

$$\frac{d^2 w}{dz^2} - z w = 0 \qquad (8)$$

Since z is large for $x = 0$ and $x = 1$ as long as $Re\, \sigma$ is finite, we may use the asymptotic representations of the Airy functions [6] $Ai(z) \sim z^{-\frac{1}{4}} exp\left(\pm 2 z^{\frac{3}{2}} / 3 \right)$ for eigenvalues which are not too close to the ideal continuum.

Due to Stokes' phenomenon [7] the Airy functions, for large

z , change abruptly across the rays $\arg z = (2n+1)\pi/3$. Therefore, the dispersion relation (6) has the following three branches:

$$\left.\begin{aligned} &1)\quad \alpha_0 = -\eta^{\frac{1}{2}}(n+\tfrac{1}{4})\pi, \quad \mathrm{Im}\,\alpha_1 > 0 \\ &2)\quad \alpha_1 = \eta^{\frac{1}{2}}(n+\tfrac{1}{4})\pi, \quad \mathrm{Im}\,\alpha_0 > 0 \\ &3)\quad \alpha_1 - \alpha_0 = \eta^{\frac{1}{2}} n\pi, \quad \mathrm{Im}\,\alpha_i < 0 \end{aligned}\right\} \qquad (9)$$

where $\alpha_i(\sigma) = \alpha(i,\sigma)$, and n is a positive integer. The eigenvalues of the first two branches (n smaller than some number $O(\eta^{-\frac{1}{2}}$)) are on two curves which start at the two edges of the ideal continuum (forming angles of $\pi/6$ with the imaginary axis) and meet at the "triple point" where both α_0 and α_1 are real. The eigenvalues of the third branch (larger n) are on a curve which starts at the triple point to meet its complex conjugate somewhere at the negative real axis, and then continues along the real axis to both sides. For $n \to \infty$ the eigenvalues accumulate at both the origin and infinity according to $\sigma \to -\langle F\rangle^2/\eta n^2\pi^2$ and $\sigma \to -\eta n^2\pi^2$. The complex eigenvalues in the upper half plane are shown in the Figure for $\eta = 10^{-5}$ and $F = 1+3x$. This is a case in which a numerical integration of Eq. (3) is difficult because η is too small. However, we have found excellent agreement between numerical evaluations of Eq. (3) and our asymptotic formulas (9) for $\eta = 1.25\times10^{-4}$, even for the eigenvalues with small n ($n \geq 2$).

In summary, we have reduced the spectral problem to quadratures, and we have gained a qualitative picture of the entire spectrum. In particular, we have shown that shearless slab equilibria (without magnetic nulls) are resistively stable, and that the damping of the Alfvén modes is $O(1)$ as $\eta \to 0$. This supports the conjecture

[8] that ideally stable equilibria without shear (and without magnetic nulls) remain resistively stable, and it may have impact upon the theory of Alfvén wave heating.

Details of the present theory and, in addition, the boundary layer modes ($Re\,\sigma \to 0$) as well as the ballooning modes ($k^2 \to \infty$) will be described in a forthcoming paper.

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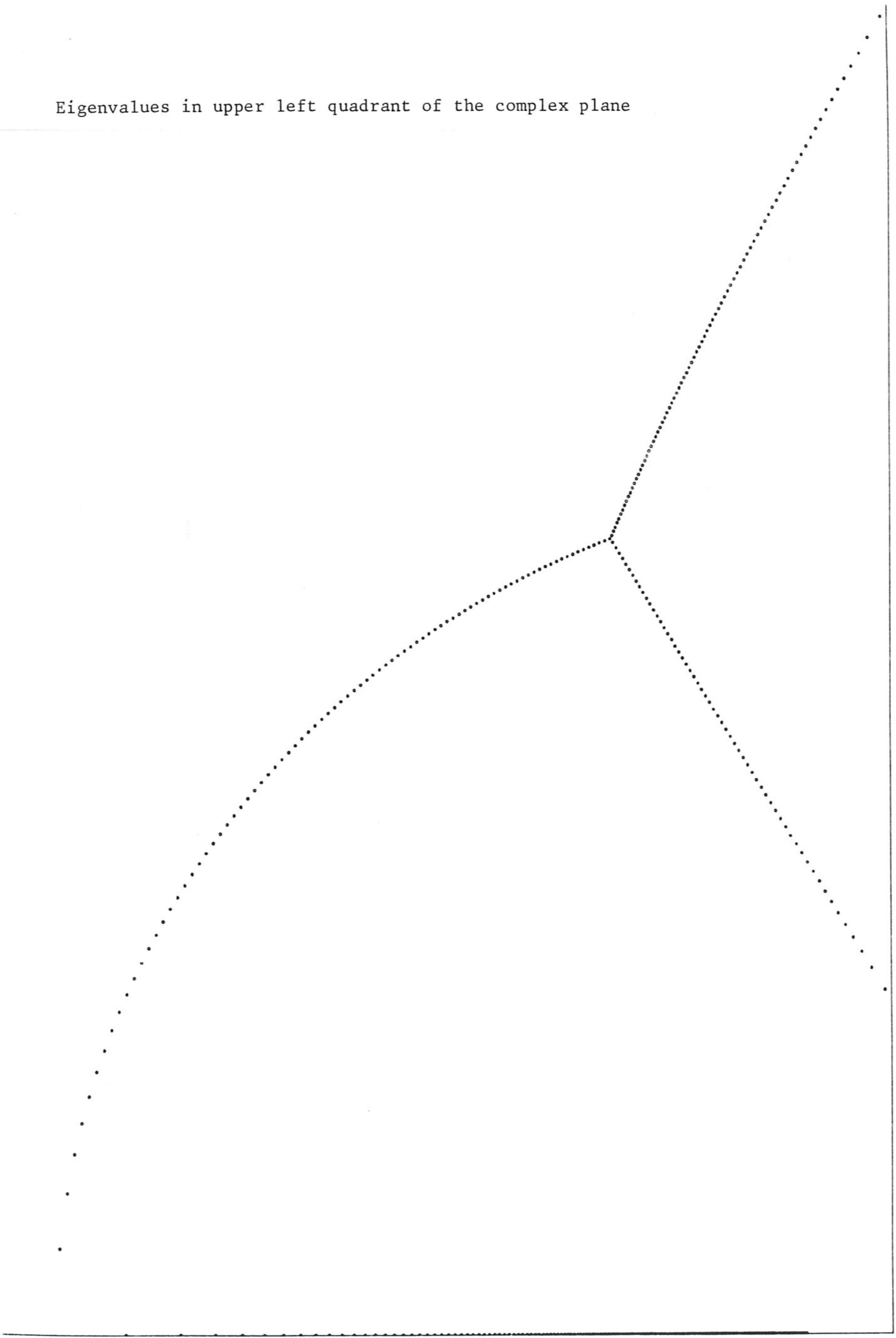
Eigenvalues in upper left quadrant of the complex plane



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