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# Remarks on Stochastic Acceleration

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## Abstract

Stochastic acceleration and turbulent diffusion are strong turbulence problems since no expansion parameter exists. Hence the problem of finding rigorous results is of major interest both for checking approximations and for reference models.

Since we have found [1)] a way of constructing such models in the turbulent diffusion case the question of the extension to stochastic acceleration now arises.

The paper offers some possibilities illustrated by the case of "stochastic free fall" which may be particularly interesting in the context of linear response theory.

## 1. Where the problem arises

In a hot plasma, for example, the heat flux is carried by the charged particles. Owing to the presence of instabilities these particles are driven by electric fields, which can be considered as stochastic, i.e. resulting from superpositioning of various unstable modes with random phases. The transition probability of a test particle under such conditions is thus of interest.

Two limiting cases can be distinguished:

- The magnetically dominated region.

  For gyrating particles we essentially expect a drift according to

$$\frac{d\underline{x}}{dt} = \frac{e}{m}\,\underline{E}(\underline{x},t) \times \underline{B}_o \tag{1}$$

  where the electric field ought to be considered as a stochastic one. This kind of problem is of the type termed "turbulent diffusion". In an earlier paper the author started a rigorous treatment.[1)]

- The electrically dominated region

  The dynamics of a test particle is then governed by a Langevin equation of the form

$$\frac{d^2\underline{x}}{dt^2} = \frac{e}{m}\,\underline{E}(\underline{x},t) \tag{2}$$

  which, unfortunately, cannot be split into two equations of the former type in a general way. The previous methods thus cannot be applied directly to the present situation. It should be added that stochastic acceleration is of widespread interest extending to the diffusion of finite-mass particles, e.g. an aircraft landing in turbulent air.

In the following, an attempt is made to extend the method of turbulent diffusion to the present situation as far is reasonable.

## 2. The "winter" case

If we assume that the right-hand side of eq. (2) does not depend explicitly on $\underline{x}$, it seems natural to describe the stochasticity in terms of a suitable Gaussian distribution (Einstein and Hopf). Subtracting the mean field (linear response theory), we end up with an equation of the type

$$\frac{d^2\underline{x}}{dt^2} = -\beta \frac{d\underline{x}}{dt} + \underline{A}(t) \tag{3}$$

which has been extensively treated in the context of the theory of Brownian motion (Wang & Uhlenbeck, Doob)[2]. Such a treatment even allows for any causal dependence of the right-hand side upon $\underline{x}$, as is explicitly demonstrated with the Brownian oscillator as well as with any other theory based on Fokker-Planck equations.

The other extreme would be to have the right-hand side not depend explicitly on t, but merely on $\underline{x}$ in a purely stochastic manner. The author has already treated this "time-independent" situation to some extent in the case of turbulent diffusion, where it was referred to as "frozen-in" turbulence. The present situation is thus called the "winter" case.

In the following, attention is essentially confined to the one-dimensional case. The corresponding differential equation,

$$\frac{d^2 \underline{x}}{dt^2} = \frac{e}{m} \underline{E}(\underline{x}) \tag{4}$$

$$= - \frac{\partial V}{\partial \underline{x}} \tag{5}$$

can easily be integrated if we proceed naively without reference to a possibly underlying Wiener process (thus neglecting the discussion around Ito-Stratonovich). We end up with two constants of integration, the energy and the "starting time":

$$E = \tfrac{1}{2} v^2 + V(x) = \tfrac{1}{2} v_0^2 + V(x_0) \tag{6}$$

$$- t_0 = \rho(x|v_0) - t \tag{7}$$

with

$$\rho(x|v_0) = \int_{x_0}^{x} \frac{dx'}{\sqrt{v_0^2 + 2W(x')}} \tag{8}$$

where

$$W(x) = 2(V(x_0) - V(x)). \tag{9}$$

Hence we find ourselves in a situation which closely corresponds to that of turbulent diffusion. However, the "frozen-in" case is replaced by a whole family of such cases depending on a parameter $v_o$. This is the situation we are dealing with in "winter" times. We have only to identify our previous S with $-t_o(x/v_o)$.

## 3. Family problems

S, or in our case $\varrho$ , depends on two parameters, $v_o$ and x. Unfortunately, the dependence on the initial velocity is not completely stochastic but causal in a rather trivial way. Accordingly, we are dealing with a family of stochastic x-functions. Instead of $\varrho$ we use

$$\hat{\varrho}(x|\alpha) = \int^{x} \sqrt{\alpha + W(x')}\, dx' \tag{1o}$$

from which we obtain $\varrho$ by differentiation with respect to $\alpha$. In principle, this family can be characterized by the differential equation

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial\hat{\varrho}}{\partial x}\right)^2 = 1 \tag{11}$$

which ought to be considered as a "boundary condition" for the statistics of $\hat{\varrho}$ . However, it seems difficult to pursue this line. Hence we may proceed as follows: Let us assume that we know the statistics for a given fixed value of $\alpha$ . We may then ask whether we can obtain at least some information for another value. In fact, this is possible at least for the lower moments and hence we may apply the ASUCON [1]) method in order to obtain the scaling laws for the corresponding diffusion. This can be illustrated by the formula

$$\langle \hat{\varrho}(x|\beta) \rangle = \int^{x} dx' \left\langle \sqrt{\beta - \alpha + \hat{\varrho}_x^2(x',\alpha)} \right\rangle \tag{12}$$

## 4. Scattering

If the potential energy is essentially bounded (vanishing probability for very large values of V), rather fast particles will not get trapped. If we thus assume $\hat{\varrho}(x|\alpha \to \infty)$ to be known, this immediately leads to a simple relation for our potential after expansion of the square root:

$$\hat{\varrho}(x|\alpha) = \sqrt{\alpha}\, x + \frac{1}{2\sqrt{\alpha}} \int^{x} W(x')\, dx' \qquad (13)$$

Hence from the statistics of $\hat{\varrho}$ we also obtain that of W, which in turn allows us to calculate, for example, the average value of $\hat{\varrho}(x|\alpha)$ for any other finite initial velocity by means of

$$\langle \hat{\varrho}(x|\alpha) \rangle = \int^{x} dx' \int \mathrm{prob}(x' : W) \sqrt{\alpha + W}\, dW \qquad (14)$$

This formula only makes sense as long as trapping does not come into play, e.g. if the corresponding potential is a monotonically stochastic function or if the initial energies $\alpha$ are large enough. - The problems connected with trapping will first be considered in the context of turbulent diffusion in a forthcoming paper before it is applied to the present situation.

## 5. The stochastic "free fall"

The concept of linear response is connected with the application of an external constant field of force. A test particle under consideration will experience, in addition to the agitation of its surroundings (the field particles), a somewhat disturbed external field due to the perturbation of the field particles (polarization effects, etc.). Hence we are led to consider stochastic potentials which, at least in the mean, behave like the external one:

$$\langle V(x)\rangle \sim -x \tag{15}$$

Accordinglv, we may designate this problem as "free fall" with noise.

As an example which allows scattering states we choose:

$$-2V = \gamma x + u^2(x) \qquad \langle u(x_1)\, u(x_2)\rangle = e^{-|x_1-x_2|} \tag{16}$$

where u is the Ornstein-Uhlenbeck process. Using the ASUCON method, we obtain for the dwell time

$$\bar{t} = \langle \rho(x)\rangle = \int_0^x \frac{dx'}{\sqrt{2\pi(1-e^{-x'})}} \int \frac{e^{-\frac{u^2}{2(1-e^{-x})}}}{\sqrt{\alpha+\gamma x+u^2}}\, du \tag{17}$$

which yields

$$= \int_0^x \frac{dx'}{\sqrt{2\pi(1-e^{-x'})}}\, e^{\frac{\alpha+\gamma x'}{4(1-e^{-x'})}}\, K_0\!\left(\frac{\alpha+\gamma x'}{4(1-e^{-x'})}\right) \tag{18}$$

For large values of x this tends to

$$\bar{t} \sim x \qquad (\gamma = 0)$$
$$\sim \sqrt{\gamma x} \qquad (\gamma \neq 0) \tag{19}$$

which is in keeping with the causal situation.

If we had chosen instead of u the Wiener process:

$$2V = -w^2(x) \qquad \langle w^2(x) \rangle = x \tag{20}$$

we would have obtained

$$\bar{t} = \int_0^x \frac{dx'}{\sqrt{2\pi x'}} e^{\frac{\alpha}{4x'}} K_0\left(\frac{\alpha}{4x'}\right) \tag{21}$$

For small x, using asymptotic expansions of $K_o$, we get

$$\bar{t} \sim \frac{1}{v_0}\left(x - \frac{x^2}{4v_0^2}\right) \tag{22}$$

for the dwell time behaviour, whereas for large values of x, since $K_o$ behaves essentially logarithmically, we obtain

$$\bar{t} \sim \sqrt{2x}\ \frac{\ln x}{\sqrt{\pi}} \tag{23}$$

This formula shows a surprising deviation from the expected behaviour ($t \sim \sqrt{2x}$) in the causal situation. In a certain sense, this reminds one of the law of iterated logarithms. [3) ] So far, however, the connection is an open question.

For a better understanding of this result, we compare it with the motion of the mean value: From the equation of motion

$$\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} \tag{24}$$

we obtain by averaging

$$\frac{d^2}{dt^2}\langle x\rangle = 1 \tag{25}$$

and hence the conventional result

$$\langle x\rangle \sim \frac{1}{2}t^2 \tag{26}$$

The centre of mass of a cloud of test particles would thus behave normally, which is evidently wrong where the movement of a quantile is concerned, this being the case when $\bar{t}$ is calculated.

## 6. Fokker-Planck techniques

Very often the underlying stochasticity is based on Wiener processes. In the last example this was precisely the case, even for the Ornstein-Uhlenbeck process. One may ask whether it is also possible to obtain expressions for the whole transition probabilities themselves. With conventional techniques, the answer is [3) 1)]

$$\frac{\partial p}{\partial t} + \sqrt{\alpha + w^2}\,\frac{\partial p}{\partial x} = \frac{1}{2}\,\frac{\partial^2}{\partial w^2}\sqrt{\alpha + w^2}\,p \tag{27}$$

with the initial condition

$$p\,(t=0;\,x,w) = \delta(x)\,\delta(w) \tag{28}$$

for this family of frozen-in situations ("winter case").

If no analytical solution of such Fokker-Planck equations can be found, one may doubt the advantage they offer compared with the direct Monte-Carlo treatment of (5). However, the latter introduces two sources of error:

i) The representation of a statistical ensemble by a finite sample.

ii) The numerical integration.

At least the first point is avoided, if the Fokker-Planck equation is used as a starting point - unless it is again treated by Monte-Carlo methods.

No discussion of equation (27) will be given here. It is merely stated that extension is possible even to cases which are time dependent.

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