MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Consistent Guiding Center Drift Theories

H.K. Wimmel

IPP 6/212 April 1982

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

IPP 6/212 H.K. Wimmel Consistent Guiding Center Drift Theories

April 1982

(in English)

Abstract

Various guiding-center drift theories are presented that are optimized in respect of consistency. They satisfy exact energy conservation theorems (in time-independent fields), Liouville's theorems, and appropriate power balance equations. A theoretical framework is given that allows direct and exact derivation of associated drift-kinetic equations from the respective guiding-center drift-orbit theories. These drift-kinetic equations are listed. Northrop's non-optimized theory is discussed for reference, and internal consistency relations of G.C. drift theories are presented.

Contents

1. Introduction

When guiding-center drift theories [i.e. theories with $\underset{\sim}{E} = 0(\varepsilon)$] or guiding center theories [i.e. theories with $\underset{\sim}{E}_{\perp} = 0(1)$, $\underset{\sim}{E}_{\parallel} = 0(\varepsilon)$] are derived from non-relativistic particle mechanics by an expansion in the small parameter ε and gyro-averaging (see Appendix A and Refs. [1, 3, 4]) the following situation obtains. The particle motion in an electromagnetic field is described by six independent first-order equations of motion, but several dependent equations are equally important (see below) because they express certain internal relations and symmetries. When ε-expansion and truncation are performed, the dependent relations are usually lost as exact equations if appropriate precautions are not taken (see Sec. 3). This leads to a mutilated type of G.C. orbit theories that are, moreover, unsuitable as a basis of rational G.C. kinetic theories (see Secs. 2, 3, and 7).

In this paper optimized guiding-center drift theories are presented, i.e. ones that have most of the symmetry relations exactly preserved. The advantages of such theories will be shown; in particular, they allow the exact and direct derivation of associated G.C. drift-kinetic equations from G.C. drift mechanics (see Sec. 2). The first advance in this direction was made by Boozer [2] ; but a real breakthrough was accomplished by Littlejohn [3, 4] . These two authors focus their attention on the G.C. energy conservation theorem (in time-

-independent fields) and on Liouville's theorem. These two theorems are in fact indispensable for constructing rational G.C. drift-kinetic theories in which equilibrium distribution functions assume a simple form (see Sec. 2). Littlejohn [4] mentioned another point in favor of Liouville's theorem, viz. that its validity excludes the occurrence of limit cycles and strange attractors, in agreement with Hamiltonian particle dynamics.

If conservation of energy is also to hold for the drift-kinetic G.C. plasma, i.e. for the corresponding continuous system in phase space, then an exact power balance equation (of an appropriate form) for single G.C. particles must be satisfied. It has the following form (see Sec. 2 and Appendix B):

$$\dot{W}_K \equiv \frac{dW_K}{dt} = e\,\underset{\sim}{E}\cdot\underset{\sim}{v} - \underset{\sim}{\mu}\cdot\frac{\partial \underset{\sim}{B}}{\partial t}\,, \tag{1.1}$$

where W_K is the G.C. kinetic energy, e.g. to lowest order in ε,

$$W_K \equiv \frac{m}{2}v_{\parallel}^2 + \mu B\,, \tag{1.2}$$

$\underset{\sim}{\mu}$ is the vectorial magnetic moment, μ is the scalar magnetic moment, $\underset{\sim}{v}$ is the G.C. velocity (without the gyration), and $v_{\parallel}$ is the "parallel" component of $\underset{\sim}{v}$. (For more details see Secs. 2, 3 and Appendix B). Equation (1.1) implies energy conservation of single G.C. particles in time-independent fields and, at the same time, conservation of total energy of a system consisting of a G.C. plasma and its electromagnetic fields (see Sec. 2).

In special systems, with appropriate spatial symmetries, canonical momenta are also conserved, and exact preservation of these conservation theorems in G.C. drift theories may be desired. This question will not be considered here. Another important symmetry, Galilei invariance, can be artificially postulated for non-relativistic mechanics of charged particles. It is an interesting question whether this symmetry can be preserved when constructing a G.C. drift theory; but, again, considera-tion of this would exceed the scope of this investigation. The optimized G.C. theories and G.C. drift theories presented in the literature [2, 3, 4] and in this paper are not Galilei invariant. It may well be that formal restrictions exist which forbid the existence of optimized Galilei-invariant G.C. theories and G.C. drift theories.

The following three categories are important for classifying "derived theories" that are approximations of more accurate "generating theories".

1. "Accuracy" describes the degree of agreement between the derived theory and the generating theory, e.g. expressed by the truncation order in ε.

2. "Intrinsic symmetry" (or "intrinsic consistency") indicates the availability of exact integrals, conservation theorems, and other symmetries.

3. "Extrinsic consistency" describes what symmetry properties of the generating theory are preserved by a derived theory. This category must be distinguished from category 1.

In order to visualize these categories, one may pick relativistic mechanics and non-relativistic mechanics as generating and derived theories, respectively, with $\varepsilon \equiv V/c$. In this paper the categories refer to non-relativistic mechanics (generating) and G.C. drift-orbit theories (derived). We shall focus our attention on categories 2 and 3, which are usually not properly taken into account. The term "exact", e.g. in "exact energy conservation" and the like, will be used in the sense of category 2.

Section 2 gives the theoretical framework that allows direct and exact derivation of associated drift-kinetic equations from the respective G.C. drift-orbit theories. Sections 3 through 6 document individual G.C. drift theories. Section 7 presents the internal consistency relations of G.C. drift theories. In particular, G.C. theories that take only $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts into account are critically examined. Section 8 presents the conclusions. Appendix A explains the drift ordering, Appendix B derives the G.C. power balance equation, and Appendix C derives the leading-order expression for the "effective current density" of a G.C. component.

Throughout the paper Gaussian cgs units will be used so that Maxwell's equations have the form

$$\frac{\partial \underset{\sim}{B}}{\partial t} = - c \, \nabla \times \underset{\sim}{E} \quad , \tag{1.3}$$

$$\frac{\partial \underset{\sim}{E}}{\partial t} = c \, \nabla \times \underset{\sim}{B} - 4\pi \underset{\sim}{j} \, , \tag{1.4}$$

$$\nabla \cdot \underset{\sim}{B} = 0 , \tag{1.5}$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi \rho , \tag{1.6}$$

and the Lorentz force is given by

$$m \dot{\underset{\sim}{V}} = e \left(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{V} \times \underset{\sim}{B} \right) , \tag{1.7}$$

with the gyro-frequency defined by $\Omega = eB/mc$.

2. Drift-kinetic theories from guiding-center drift mechanics

It is preferable to derive G.C. drift-kinetic equations direct from the G.C. equations of motion (rather than by expansion of particle kinetic equations) because then the G.C. orbits are exact characteristics of the resulting drift-kinetic equations. This is a special application of the general principle that theories with exact symmetries and exact internal consistency relations are more valuable than others where such relations are absent. We give here a theoretical framework that permits constructing drift-kinetic equations from G. C. orbits (in G.C. phase space), including such cases where a Liouville's theorem is not available.

Consider a 5-dimensional G.C. phase space with coordinates α_i (i = 1 to 5). Later on the α_i will be specialized to become $\{\alpha_i\} = \{\underset{\sim}{x}, v_\parallel, \mu\}$, where $\underset{\sim}{x}$ is the G.C. position, $v_\parallel$ is the G.C. velocity component parallel to $\underset{\sim}{B}$ at the G.C. position, and μ is the (lowest-order) magnetic moment (see Sec. 3). The volume element in G.C. phase space shall be defined as

$$d\tau \equiv J(\alpha_i, t) \prod_i d\alpha_i \equiv J\, d\tau_0 , \qquad (2.1)$$

i.e. with

$$d\tau_0 \equiv \prod_i d\alpha_i . \qquad (2.2)$$

The collisionless G.C. drift-kinetic equation (DKE) follows from the requirement that the number

$$dN \equiv f\, d\tau \tag{2.3}$$

of guiding centers in the co-moving volume element $d\tau$ be constant in time, viz.

$$\frac{d}{dt}\left(f\, d\tau\right) = 0. \tag{2.4}$$

Here

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \sum_i \dot{\alpha}_i \frac{\partial}{\partial \alpha_i} \tag{2.5}$$

is the total time derivative in phase space, and $\dot{\alpha}_i \equiv d\alpha_i/dt$ is the total time derivative of α_i. One has, of course, $\partial\alpha_i/\partial t = 0$, $\partial\alpha_i/\partial\alpha_j = \delta_{ij}$ (δ_{ij} = Kronecker symbol). The quantities $\dot{\alpha}_i$ are functions of time and "phase", i.e.

$$\dot{\alpha}_i = \dot{\alpha}_i\left(t, \alpha_1 \ldots \alpha_5\right); \tag{2.5a}$$

see Secs. 3 through 6. By using the relation [5]

$$\frac{d}{dt}(d\tau) = \left\{\sum_i \frac{\partial \dot{\alpha}_i}{\partial \alpha_i} + \frac{1}{J}\frac{dJ}{dt}\right\} d\tau \tag{2.6}$$

eq. (2.4) can be transformed to yield

$$\frac{df}{dt} + Sf = 0, \tag{2.7}$$

with S defined by

$$S \equiv \frac{1}{d\tau}\frac{d}{dt}(d\tau) = \sum_i \frac{\partial \dot{\alpha}_i}{\partial \alpha_i} + \frac{1}{J}\frac{dJ}{dt}. \tag{2.8}$$

Equation (2.7) is the collisionless drift-kinetic equation. The quantity S vanishes identically when Liouville's theorem applies with respect to the chosen $d\tau$ (or J). The left-hand side of the DKE is then a total time derivative in phase space, whence f = const along phase space trajectories. If $S \not\equiv 0$, then f varies along phase space trajectories. It is important to note that eqs. (2.7), (2.8) can be transformed to yield

$$\frac{\partial}{\partial t}(Jf) + \sum_i \frac{\partial}{\partial \alpha_i}\left(\dot{\alpha}_i Jf\right) = 0, \tag{2.9}$$

as an alternative form of the DKE, and S can be written in the form

$$S = \frac{1}{J}\left\{\frac{\partial J}{\partial t} + \sum_i \frac{\partial}{\partial \alpha_i}\left(\dot{\alpha}_i J\right)\right\}, \tag{2.10}$$

as an alternative to eq. (2.8). The form of eq. (2.9) is the same whether Liouville's theorem is satisfied ($S \equiv 0$) or not ($S \not\equiv 0$). In the form of eq. (2.9) the DKE is better adopted for forming moments to obtain the equation of continuity, etc., while eq. (2.7) is more suitable for obtaining solutions for the distribution function f.

In order to derive the equation of continuity, we specialize $\{\alpha_i\} \rightarrow \{\underset{\sim}{x}, v_\parallel, \mu\}$, $\dot{\underset{\sim}{x}} \equiv \underset{\sim}{v}$. The phase space volume element is factored thus:

$$d\tau = d\tau_x \cdot d\tau_v , \tag{2.11}$$

with $d\tau_x \equiv d^3x$ and $d\tau_v \equiv J\, dv_\parallel\, d\mu$. In addition, one defines the G.C. density

$$n \equiv \int f\, d\tau_v \equiv \int Jf\, dv_\parallel\, d\mu \tag{2.12}$$

and the G.C. flux density (remember $\dot{\underset{\sim}{x}} \equiv \underset{\sim}{v}$)

$$\underset{\sim}{\Gamma} \equiv \int \dot{\underset{\sim}{x}}\, f\, d\tau_v \equiv \int \dot{\underset{\sim}{x}}\, Jf\, dv_\parallel\, d\mu . \tag{2.12a}$$

Here the ranges of integration are $-\infty < v_\parallel < \infty$ and $0 \leq \mu < \infty$. By multiplying eq. (2.9) by $dv_\parallel \, d\mu$ and integrating over $(v_\parallel, \mu)$ space one obtains

$$\frac{\partial n}{\partial t} + \nabla \cdot \underset{\sim}{\Gamma} + \int_0^\infty d\mu \int_{-\infty}^{+\infty} dv_\parallel \frac{\partial}{\partial v_\parallel} \left(\dot{v}_\parallel J f \right) = 0, \qquad (2.13)$$

where $\dot{\mu} = 0$ has been used. The $v_\parallel$ integral vanishes whenever $(\dot{v}_\parallel J f)$ goes to zero for $v_\parallel \rightarrow \pm \infty$, which leaves one with the equation of continuity.

The G.C. drift fluid described by eqs. (2.7) and (2.9) also satisfies an exact energy theorem if the guiding centers obey an exact power balance equation of an appropriate form, viz. (see Appendix B)

$$\dot{W}_K \equiv \frac{dW_K}{dt} = e \underset{\sim}{E} \cdot \underset{\sim}{v} - \underset{\sim}{\mu} \cdot \frac{\partial \underset{\sim}{B}}{\partial t}, \qquad (2.14)$$

where W_k is the kinetic energy of a G.C., defined as

$$W_K \equiv \frac{m}{2} v_\parallel^2 + \mu B, \qquad (2.15)$$

and $\underset{\sim}{\mu} = -\mu \hat{\underset{\sim}{b}}$ $[\hat{\underset{\sim}{b}} \equiv \underset{\sim}{B}/B]$ is the vectorial magnetic moment of a G.C. Equation (2.14) implies that the total energy of a G.C. is conserved in stationary fields, viz.

$$W \equiv e\Phi + \frac{m}{2} v_\parallel^2 + \mu B = \text{const}, \qquad (2.16)$$

where $\nabla \Phi = -\underset{\sim}{E}$. One should note that the exact G.C. velocity $\underset{\sim}{v}$ (not merely $v_\parallel \hat{\underset{\sim}{b}}$) must be used in eq. (2.14) in order to obtain eq. (2.16). Exact energy conservation and validity of a Liouville's theorem (see below) are both necessary conditions for expressing equilibrium distribution functions f_0 solely by constants of the motion. In fact, any normalizable $f_0(W, \mu)$ is then an equilibrium distribution function, while $f_0(\mu)$ is not normalizable and therefore not a distribution function at all. (It yields infinite moments, e.g. infinite densities owing to the $v_\parallel$ integration).

If eq. (2.9) is multiplied by $W_K \, dv_\parallel \, d\mu$ and integrated over $\{v_\parallel, \mu\}$ space, the following exact energy theorem for the G.C. fluid follows:

$$\frac{\partial D}{\partial t} + \nabla \cdot \underset{\sim}{F} = \underset{\sim}{E} \cdot \left(e \underset{\sim}{\Gamma} + c \, \nabla \times \underset{\sim}{M} \right). \tag{2.17}$$

Here

$$D \equiv \int W_K \, f \, d\tau_v \equiv \int W_K \, J f \, dv_\parallel \, d\mu \tag{2.18}$$

is the kinetic energy density, and

$$\underset{\sim}{F} \equiv \underset{\sim}{F}_1 + c \, \underset{\sim}{M} \times \underset{\sim}{E} \tag{2.19}$$

is the effective energy flux, with the definitions

$$\underset{\sim}{F}_1 \equiv \int \underset{\sim}{v} \, W_K \, f \, d\tau_v \equiv \int \underset{\sim}{v} \, W_K \, J f \, dv_\parallel \, d\mu \,, \tag{2.20}$$

$$\underset{\sim}{M} \equiv \int \underset{\sim}{\mu} \, f \, d\tau_v \equiv \int \underset{\sim}{\mu} \, J f \, dv_\parallel \, d\mu \,, \tag{2.21}$$

the latter being the magnetic moment density. The bracket on the r.h.s. of eq. (2.17) must be identified with the effective electric current density $\underset{\sim}{j}_{eff}$ of a single G.C. fluid, viz.

$$\underset{\sim}{j}_{eff} \equiv e\underset{\sim}{\Gamma} + c\nabla\times\underset{\sim}{M} . \qquad (2.17a)$$

This is necessary for establishing a conservation theorem of total energy, including field energy (see below). It is shown in Appendix C that eq. (2.17a) agrees to leading order in ε with the true current density of a charged-particle plasma component. The G.C. energy flux density must be identified with $\underset{\sim}{F}$ rather than $\underset{\sim}{F}_1$ [eq. (2.19)] in order to make eq. (2.17) compatible with Maxwell's equations (see below). The term $c\,\underset{\sim}{M}\times\underset{\sim}{E}$ in eq. (2.19) describes a "difference effect" arising from changing the mode of description. Mode 1 uses a Taylor expansion of $\underset{\sim}{E}$ about the G.C. position (see Appendix B), while Mode 2 employs the effective current density as expressed by a Taylor expansion of $\underset{\sim}{M}$. Without this change of description (effected by a partial differentiation) the energy theorem would read

$$\frac{\partial D}{\partial t} + \nabla\cdot\underset{\sim}{F}_1 = e\underset{\sim}{E}\cdot\underset{\sim}{\Gamma} + c\underset{\sim}{M}\cdot(\nabla\times\underset{\sim}{E}). \qquad (2.17b)$$

Obviously, this equation is formally less well adapted to the energy theorem valid for the electromagnetic field, viz. eq. (2.24) below.

We define the total effective electric current density of the G.C. plasma, with the components $\alpha = i, e$, by

$$\underset{\sim}{j}_{tot} \equiv \sum_\alpha \underset{\sim}{j}_{eff} \equiv \sum_\alpha \left(e_\alpha\underset{\sim}{\Gamma}_\alpha + c\nabla\times\underset{\sim}{M}_\alpha\right). \qquad (2.22)$$

The energy theorem of the G.C. plasma then reads

$$\sum_\alpha\left(\frac{\partial D_\alpha}{\partial t} + \nabla\cdot\underset{\sim}{F}_\alpha\right) = \underset{\sim}{E}\cdot\underset{\sim}{j}_{tot} . \qquad (2.23)$$

On replacing $\underset{\sim}{j}$ by $\underset{\sim}{j}_{tot}$ in Maxwell's equations, too, the energy theorem for the electromagnetic field reads

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(E^2 + B^2\right) + \frac{c}{4\pi}\nabla\cdot\left(\underset{\sim}{E}\times\underset{\sim}{B}\right) = -\underset{\sim}{E}\cdot\underset{\sim}{j}_{tot}\,. \tag{2.24}$$

Contrary to convention in the theory of diamagnetic media, it is not appropriate here to move the magnetic moment density $\underset{\sim}{M}$ to the left-hand side of the Maxwell's equations and of eq. (2.24) by introducing a new field $\underset{\sim}{H} \equiv \underset{\sim}{B} - 4\pi\underset{\sim}{M}$. Adding eqs. (2.23) and (2.24) yields an energy conservation theorem for the system consisting of the G.C. plasma and the electromagnetic fields.

It is not useful to derive also a conservation theorem of momentum (by forming a first moment of the drift-kinetic equation). This is so because a simple expression for $\dot{\underset{\sim}{v}}$ would be needed [by analogy with $\dot{W}_k$ of eq. (2.14)] in order to obtain a useful momentum theorem. Such a simple expression for $\dot{\underset{\sim}{v}}$ is not available in G.C. drift theories because the simple equation of motion of a charged particle has been thoroughly complicated by the G.C. expansion in ε and by using explicit expressions for $\underset{\sim}{v}_{\perp}$.

The above treatment demonstrates the following points: The derivation of drift-kinetic equations direct from G.C. orbits together with an appropriate form of the power balance equation for a single G.C. particle are the basic starting points of an exactly consistent G.C. drift-kinetic theory. Independent of the validity of a Liouville's theorem, such a theory has exact conservation theorems for the G.C. particle numbers and for total energy; in addition, there is exact energy conservation for single G.C. particles in time-independent fields. Exact conservation of the number of G.C. particles is necessary in order to avoid contradiction with Maxwell's equations (which imply exact conservation of charge) without being forced to expand Maxwell's equations and electromagnetic fields in the G.C. parameter ε. Exact energy conservation seems important in order to prevent a theory from yielding spurious low-frequency instabilities that might arise from violation of exact energy conservation.

Contrary to the above conservation theorems, the availability of a Liouville's theorem is more important for practical reasons. If Liouville's theorem holds (for a certain choice of $d\tau$), any normalizable distribution function $f_0(c_\nu)$ that only depends on the values of constants of the motion, c_ν, of G.C. particles is an equilibrium distribution function, i.e. it satisfies

$$\frac{\partial f_0}{\partial t} = 0 \qquad (2.25)$$

and eq. (2.7), with S = 0, i.e.

$$\frac{d f_0}{d t} = 0 \, . \qquad (2.26)$$

It suffices, of course, for a Liouville's theorem to be available in time-independent fields in order to construct equilibrium distribution functions in this simple fashion (see Sec. 6).

In the alternative case, when a Liouville's theorem is not available, the equilibrium distribution functions f_o are no longer constant along G.C. trajectories in phase space. Hence they can no longer be expressed as $f_o(c_\nu)$, with c_ν being constants of the motion. In particular, a Maxwell-Boltzmann "distribution function" is now not a legitimate distribution function at all because it does not solve the time-independent G.C. drift-kinetic equation. This may also be important for numerical computations of drift orbits when a statistical evaluation is intended by postulating a distribution function.

In order to determine equilibrium distribution functions in the case of $S \not\equiv 0$ (no Liouville's theorem available), eq. (2.7) must be solved in the form

$$\sum_i \dot{\alpha}_i \frac{\partial f_o}{\partial \alpha_i} + \frac{f_o}{J} \sum_i \frac{\partial}{\partial \alpha_i} \left(\dot{\alpha}_i J \right) = 0, \qquad (2.27)$$

with $\partial f_o / \partial t = 0$ and, because the (self-consistent) fields are then time-independent, $\partial J / \partial t = 0$. The solution can be facilitated by choosing $\{\alpha_i\} = \{\underset{\sim}{x}, W, \mu\}$, with $\dot{W} = \dot{\mu} = 0$.

3. Northrop's non-optimized G.C. drift theory and partial optimization thereof

The optimized theories of the forthcoming sections agree to the leading orders of their terms with Northrop's non-optimized theory [1] when the latter is adapted to drift scaling, i.e. $\underset{\sim}{E} = O(\varepsilon)$. In order to enable the reader to verify this agreement, Northrop's adapted theory is listed here and its properties are discussed. In addition, an improved theory is presented that is a partially optimized modification of Northrop's theory. Appendix A should be consulted for details of drift ordering and questions of notation.

Specializing to drift scaling, with $\underset{\sim}{E} = O(\varepsilon)$, $\partial \underset{\sim}{B}/c\partial t = O(\varepsilon)$, the leading orders of Northrop's [1] equations are these (see Appendix A):

$$\dot{\underset{\sim}{x}} \equiv \underset{\sim}{V} = V_{\parallel} \hat{\underset{\sim}{b}} + \underset{\sim}{V}_D \ , \tag{3.1}$$

$$\dot{V}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m} \frac{\partial B}{\partial s} \ , \tag{3.2}$$

$$\dot{\mu} = 0, \tag{3.3}$$

with the drift velocity given by

$$\underset{\sim}{V}_D \equiv \underset{\sim}{V}_E + \underset{\sim}{V}_{\nabla B} + \underset{\sim}{V}_{\kappa} \ , \tag{3.4}$$

$$\underset{\sim}{V}_E \equiv \frac{c}{B} \underset{\sim}{E} \times \hat{\underset{\sim}{b}} \ , \tag{3.5}$$

$$\underset{\sim}{V}_{\nabla B} \equiv \frac{c\mu}{eB} \hat{\underset{\sim}{b}} \times \nabla B \ , \tag{3.6}$$

$$\underset{\sim}{V}_{k} \equiv \frac{V_{\parallel}^{2}}{\Omega} \, \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} \equiv \frac{V_{\parallel}^{2}}{\Omega} \, \hat{\underset{\sim}{b}} \times (\hat{\underset{\sim}{b}} \cdot \nabla \hat{\underset{\sim}{b}}) . \tag{3.7}$$

The notation has been explained in Sec. 2 and Appendix A.

The magnetic moment μ can be expressed to leading order in ε as

$$\mu = \frac{m}{2B} \left\langle U_{\perp}^{2} \right\rangle , \tag{3.8}$$

where B and the perpendicular direction (with respect to B, and expressed by the subscript $\perp$) are determined at the guiding center position, and $\left\langle U_{\perp}^{2} \right\rangle$ is the appropriate gyro-average over the square of the perpendicular particle velocity relative to the G.C. position. Note that to this order $V_{\parallel} = \left\langle V_{\parallel} \right\rangle$, that is, the parallel G.C. velocity equals the gyro-average of the parallel particle velocity, where, again, the "parallel direction" refers to the direction of $\underset{\sim}{B}$ at the G.C. position. Hence, the kinetic energy is defined to leading order in ε as

$$W_{K} \equiv \frac{m}{2} V_{\parallel}^{2} + \mu B . \tag{3.9}$$

Equations (3.1) through (3.7) and (3.9) form a closed, self-contained set of equations whose properties can be investigated. Let us consider first the question of the power balance equation. After some manipulation one finds $\dot{W}_{k}$ in the form

$$\dot{W}_{K} = e \underset{\sim}{E} \cdot \underset{\sim}{V} + \mu \frac{\partial B}{\partial t} - m V_{\parallel}^{2} \, \underset{\sim}{V}_{D} \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} . \tag{3.10}$$

This does not agree with the desired form [eq. (2.14)] and, moreover, does not yield energy conservation in time-independent fields. On defining the total energy

$$W \equiv W_K + e\phi \tag{3.11}$$

one finds in fact

$$\frac{dW}{dt} \equiv \dot{W} = - m v_\parallel^2 \, \underset{\sim}{v}_D \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} \not\equiv 0 . \tag{3.12}$$

A Liouville's theorem is not available either in Northrop's theory. If one uses the lowest-order phase space volume element, viz.

$$d\tau \equiv \frac{2\pi}{m} B \, d^3x \, dv_\parallel \, d\mu , \tag{3.13}$$

i.e. $J \propto B$, then one finds

$$S = \frac{c}{eB} \left[-e \underset{\sim}{E} + \mu \nabla B \right] \cdot \left(\nabla \times \underset{\sim}{B} \right) - \frac{v_\parallel^2}{\Omega} \nabla \cdot \left\{ \left[\hat{\underset{\sim}{b}} \cdot \left(\nabla \times \hat{\underset{\sim}{b}} \right) \right] \hat{\underset{\sim}{b}} \right\}$$

$$\not\equiv 0 , \tag{3.14}$$

with S defined in eq. (2.8). Hence $\dot{d\tau} \not\equiv 0$, and Liouville's theorem is violated. Since we are dealing with a leading-order theory, it would not help to consider higher-order corrections to $d\tau$ (see below).

It follows that Northrop's theory, as listed above, is <u>non-optimized,</u>

since it does not conserve energy, lacks a Liouville's theorem, and contains a power balance equation of an undesired and unphysical form. It should be mentioned that Northrop and Rome [6] have recently published higher-order corrections to Northrop's original theory without, however, considering the problem of preserving exact symmetries of particle theory.

Northrop's theory can easily be partially improved so that it satisfies energy conservation and obeys an appropriate power balance equation. On postulating (see Sec. 2)

$$\dot{W}_K \equiv \frac{dW_K}{dt} = e\,\underset{\sim}{E}\cdot\underset{\sim}{V} + \mu\frac{\partial B}{\partial t} \tag{3.15}$$

instead of eq. (3.10), with W_k still defined by eq. (3.9), the equation $\dot{\mu}=0$ and eq. (3.1) for the G.C. velocity $\underset{\sim}{V}$ may remain unaltered. One then derives the following modified expression for $\dot{V}_{\parallel}$, viz.

$$\dot{V}_{\parallel} = \frac{e}{m}E_{\parallel} - \frac{\mu}{m}\frac{\partial B}{\partial s} + V_{\parallel}\,\underset{\sim}{V}_D\cdot\frac{\partial\hat{\underset{\sim}{b}}}{\partial s}, \tag{3.16}$$

which replaces eq. (3.2). Energy conservation (in time-independent fields) is then obeyed, viz.

$$W \equiv W_K + e\Phi = \text{const.} \tag{3.17}$$

It is important to note here that the simple trick of constructing $\dot{V}_{\parallel}$ from the power balance equation together with $\dot{\mu}=0$ and the expression for $\underset{\sim}{V}$ only works when the expression for $\underset{\sim}{V}$ is of appropriate form. If, instead of eq. (3.4), an expression for $\underset{\sim}{V}_D$ is used that either lacks $\underset{\sim}{V}_{\nabla B}$ while keeping $\underset{\sim}{V}_E$ or vice versa, then, with $\dot{\mu} = 0$ the resulting expression for $\dot{v}_\parallel$ will diverge for $V_\parallel \to 0$. This does not occur when both $\underset{\sim}{V}_E$ and $\underset{\sim}{V}_{\nabla B}$ are kept because then two terms exactly cancel owing to the relation

$$e \underset{\sim}{E} \cdot \underset{\sim}{V}_{\nabla B} = \mu \nabla B \cdot \underset{\sim}{V}_E \, . \tag{3.17a}$$

This relation means that the gain of electric energy $e\phi$ caused by the velocity component $\underset{\sim}{V}_{\nabla B}$ is exactly cancelled by the loss of internal energy μB caused by the velocity component $\underset{\sim}{V}_E$. It is therefore no surprise that the conventional G.C. theories with $\dot{\mu} = 0$ and $\underset{\sim}{V} \equiv V_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{V}_E$ do not conserve G.C. energy, i.e. eq. (3.17) is then not satisfied whatever non-diverging expressions for $\dot{v}_\parallel$ are employed. If one insists on the mutilated expression

$$\dot{\underset{\sim}{x}} \equiv \underset{\sim}{V} = V_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{V}_E \, , \tag{3.1a}$$

it is necessary to abandon $\dot{\mu} = 0$ in order to save energy conservation (see Sec. 7). Clearly, energy, as an exact constant of the motion, is more important than μ, which is only an adiabatic invariant.

On using the improved $\dot{v}_\parallel$ of eq. (3.16) and the phase space volume element $d\tau$ of eq. (3.13) one now finds the Liouville quantity S to be given by

$$S = \frac{1}{m\Omega} \left\{ -eE_\parallel + \left(\mu + \frac{m v_\parallel^2}{B} \right) \frac{\partial B}{\partial s} - \right.$$

$$- m v_\parallel^2 \frac{\partial}{\partial s} \Big\} \left[\hat{\underset{\sim}{b}} \cdot (\nabla \times \hat{\underset{\sim}{b}}) \right] \neq 0 , \tag{3.18}$$

i.e. $d\tau$ is still not conserved, with $S = 0(\varepsilon)$. By choosing instead the volume element corrected for first-order terms, viz.

$$d\tau^* \equiv \frac{2\pi}{m} B^* \, d^3x \, dv_\parallel \, d\mu , \tag{3.19}$$

with

$$B^* \equiv B + \frac{m c v_\parallel}{e} b_0 , \tag{3.20}$$

$$b_0 \equiv \hat{\underset{\sim}{b}} \cdot (\nabla \times \hat{\underset{\sim}{b}}) , \tag{3.21}$$

one obtains instead

$$S^* = \frac{v_\parallel}{\Omega^*} \left\{ \frac{\partial b_0}{\partial t} + \hat{\nabla} \cdot (b_0 \underset{\sim}{v}_D) + 2 b_0 \underset{\sim}{v}_D \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} \right\} , \tag{3.22}$$

i.e. $S^* = 0(\varepsilon^2)$. Here $\hat{\nabla}$ is the nabla operator, but with $v_\parallel$ and μ kept constant when the spatial derivatives are performed. If one wants to obtain a $d\tau_1$ with $S_1 = 0$, $d\dot{\tau}_1 = 0$, so that Liouville's theorem is exactly obeyed, e.g.

$$d\tau_1 \equiv J_1 \, d^3x \, dv_\parallel \, d\mu ,$$

then the condition for J_1 is given by [see eq. (2.10)]

$$\frac{\partial J_1}{\partial t} + \sum_i \frac{\partial}{\partial \alpha_i} (\dot{\alpha}_i J_1) = 0 ,$$

with $\{\alpha_i\} = \{ \underset{\sim}{x}, v_\parallel, \mu \}$. This is tantamount to requiring solution of the drift-kinetic equation in the first place [put $Jf \to J_1$ in eq. (2.9)]. In order to avoid this and still obtain a fully optimized theory, the G.C. equations of motion must be modified (see Secs. 4 through 6).

It follows that both Northrop's theory [1] and the above partially optimized theory (Northrop corrected to obtain enery conservation) do not obey a known Liouville's theorem. Hence equilibrium distribution functions cannot be expressed as $f_o(W, \mu)$, but must be determined by integrating the pertinent drift-kinetic equations, as was explained in Sec. 2.

Notwithstanding this complication, associated, exactly compatible G.C. drift-kinetic equations follow from the above partially optimized theory together with either one of the definitions $dN = f\, d\tau$ or $dN = f^* d\tau^*$ [eq. (2.3)], viz.

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \hat{\nabla} + \dot{v}_{\parallel} \frac{\partial}{\partial v_{\parallel}} + S \right) f = 0 \,, \tag{3.23}$$

or

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \hat{\nabla} + \dot{v}_{\parallel} \frac{\partial}{\partial v_{\parallel}} + S^* \right) f^* = 0 \,, \tag{3.24}$$

where $\underset{\sim}{v}$ and $\dot{v}_{\parallel}$ must be taken from eqs. (3.1) with (3.4) through (3.7) and (3.16), S and S^* from eq. (3.18) and (3.22), and $\hat{\nabla}$ has been explained after eq. (3.22). Alternative forms of the drift-kinetic equations are

$$\frac{\partial}{\partial t}(Bf) + \hat{\nabla} \cdot (\underset{\sim}{v}\, Bf) + \frac{\partial}{\partial v_{\parallel}}(\dot{v}_{\parallel}\, Bf) = 0 \tag{3.25}$$

and

$$\frac{\partial}{\partial t}\left(B^* f^*\right) + \hat{\nabla}\cdot\left(\underset{\sim}{v}\, B^* f^*\right) + \frac{\partial}{\partial v_\parallel}\left(\dot{v}_\parallel\, B^* f^*\right) = 0 \,. \tag{3.26}$$

For both partially optimized G.C. drift-kinetic theories conservation theorems of G.C. particle number and energy follow, as explained in Sec. 2.

4. Boozer's theory generalized: an optimized theory with parallel drift

Boozer [2] constructed a G.C. drift theory valid for time-independent fields that contains an energy theorem and a Liouville's theorem. This theory can easily be generalized to apply to the case of time-dependent electromagnetic fields. The resulting optimized theory is presented here. The equations of motion are

$$\dot{\underset{\sim}{x}} \equiv \underset{\sim}{v} = v_{\parallel} \hat{\underset{\sim}{b}} + \underset{\sim}{v}_E + \underset{\sim}{v}_{\nabla B} + \frac{v_{\parallel}^2}{\Omega} \nabla \times \hat{\underset{\sim}{b}} , \tag{4.1}$$

$$\dot{v}_{\parallel} = \left(\frac{e}{m} \underset{\sim}{E} - \frac{\mu}{m} \nabla B \right) \cdot \left[\hat{\underset{\sim}{b}} + \frac{v_{\parallel}}{\Omega} \nabla \times \hat{\underset{\sim}{b}} \right] , \tag{4.2}$$

$$\dot{\mu} = 0 , \tag{4.3}$$

where $\underset{\sim}{v}_E$ and $\underset{\sim}{v}_{\nabla B}$ are again defined by eqs. (3.5) and (3.6), and $\Omega \equiv eB/mc$. A peculiar property of this theory is the presence of a "parallel drift", i.e. $\underset{\sim}{v} \cdot \hat{\underset{\sim}{b}}$ is not equal to $v_{\parallel}$, as can be seen from eq. (4.1). This is not a mathematical inconsistency, however, since the parallel drift is of higher order in ε than $v_{\parallel}$ itself (see Appendix A), and the theory only has to agree with the original particle equations of motion to leading order in ε. Physically, reservations may be had because no physical content of this drift is visible, and the dependence on the sign of electric charge (via Ω) seems strange. Kinematic contradictions do not occur because the parallel drift vanishes for $v_{\parallel} = 0$. The perpendicular component of the last term of eq. (4.1) is, of course, identical to the curvature drift $\underset{\sim}{v}_\kappa$ of eq. (3.7).

We now show that this theory is optimized because a well-behaved power balance equation, an energy conservation theorem, and a Liouville's theorem all follow as exact consequences of the above equations of motion. (The derivations will not be given because they are elementary.) With the kinetic energy defined as

$$W_K \equiv \frac{m}{2} v_{\parallel}^2 + \mu B , \tag{4.4}$$

the power balance equation is of the desired form, viz.

$$\dot{W}_K = e \underset{\sim}{E} \cdot \underset{\sim}{v} + \mu \frac{\partial B}{\partial t} , \tag{4.5}$$

and energy is conserved in time-independent fields, e.g.

$$W \equiv e\Phi + W_K = \text{const.} \tag{4.6}$$

Use of the zeroth-order phase space volume element

$$d\tau \equiv \frac{2\pi}{m} B \, d^3x \, dv_{\parallel} \, d\mu \tag{4.7}$$

leads to Liouville's theorem in the form

$$S = 0 , \tag{4.8}$$

where S has been defined in eq. (2.8). Note that the definition of W_k does not include the parallel drift velocity; but, again, the mathematical deviation is of higher order in ε and hence irrelevant. The above equations, together with the definition of particle density in phase space, $dN = f \, d\tau$, can be used to derive an associated, exactly compatible G.C. drift-kinetic equation of the form

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \hat{\nabla} + \dot{v}_{\parallel} \frac{\partial}{\partial v_{\parallel}}\right) f = 0 , \tag{4.9}$$

or, alternatively,

$$\frac{\partial}{\partial t}(Bf) + \hat{\nabla} \cdot (\underset{\sim}{v} Bf) + \frac{\partial}{\partial v_{\parallel}}(\dot{v}_{\parallel} Bf) = 0 . \tag{4.10}$$

Here $\hat{\nabla}$ is again the nabla operator, but with $v_{\parallel}$ and μ kept constant when the spatial derivatives are performed. The conservation theorems (continuity, energy) derived in Sec. 2 immediately follow from the above equations.

5. Littlejohn's theory modified: an optimized theory without parallel drift

The impressive paper by Littlejohn [4] gives a G.C. orbit theory [i.e. with $\underset{\sim}{E}_{\perp} = 0(1)$] with an energy theorem and a Liouville's theorem. Littlejohn's work emphasizes the following aspects:

a) a new method of derivation that employes Hamiltonian theory with non-canonical coordinates,

b) inclusion of higher-order correction terms,

c) the preservation of the conservation theorems mentioned, from the basic particle theory.

On the other hand, Littlejohn [4] does not give a G.C. kinetic theory.

In this paper we only consider a modified, leading-order version of Littlejohn's theory that obeys drift scaling i.e. $\underset{\sim}{E} = 0(\varepsilon)$. It is an important merit of the form of Littlejohn's results that a whole class of theories with exact energy and Liouville's theorems can be extracted from them. It will therefore not be necessary to repeat his derivation procedure (with modified scaling assumptions and modified truncation) in order to arrive at the modified theory below. It is somewhat annoying that Littlejohn [3, 4] has used special units, e.g. with c = m = e = 1, and has dropped the sign of the particle charge. We have here restored normal cgs units and also taken sign (e) into account.

Following Littlejohn [4], we introduce the quantities $\underset{\sim}{B}^*$, $\underset{\sim}{E}^*$, $\underset{\sim}{A}^*$, ϕ^*, which are functions of (t, $\underset{\sim}{x}$, $v_{\parallel}$, μ), that are to satisfy the relations

$$\hat{\nabla} \cdot \underset{\sim}{B}^* = 0 , \tag{5.1}$$

$$c\,\hat{\nabla} \times \underset{\sim}{E}^* = - \frac{\partial \underset{\sim}{B}^*}{\partial t} , \tag{5.2}$$

$$\underset{\sim}{B}^* = \hat{\nabla} \times \underset{\sim}{A}^* , \tag{5.3}$$

$$\underset{\sim}{E}^* = - \hat{\nabla} \phi^* - \frac{1}{c} \frac{\partial \underset{\sim}{A}^*}{\partial t} . \tag{5.4}$$

The symbol $\hat{\nabla}$ has been explained after eq. (4.10). The abbreviation

$$B^* \equiv \hat{\underset{\sim}{b}} \cdot \underset{\sim}{B}^* \tag{5.5}$$

is also used; note that B^* is not the magnitude of $\underset{\sim}{B}^*$. By means of these quantities we can define the following class of optimized G.C. drift equations:

$$\dot{\underset{\sim}{x}} \equiv \underset{\sim}{v} = v_{\parallel} \frac{\underset{\sim}{B}^*}{B^*} + \frac{c}{B^*} \underset{\sim}{E}^* \times \hat{\underset{\sim}{b}} + \frac{\mu}{m\Omega^*} \hat{\underset{\sim}{b}} \times \nabla B , \tag{5.6}$$

$$\dot{v}_{\parallel} = \frac{\underset{\sim}{B}^*}{B^*} \cdot \left(\frac{e}{m} \underset{\sim}{E}^* - \frac{\mu}{m} \nabla B \right) , \tag{5.7}$$

$$\dot{\mu} = 0 . \tag{5.8}$$

Here $\Omega^* = eB^*/mc$, and $v_{\parallel} = \underset{\sim}{v} \cdot \hat{\underset{\sim}{b}}$, i.e. no "parallel drift" appears. It is easy to show that eqs. (5.6) through (5.8) conserve the energy expression

$$W^* \equiv \frac{m}{2} v_{\parallel}^2 + \mu B + e\phi^* \tag{5.9}$$

in time-independent fields. The power balance equation is given by

$$\dot{W}_k = e \underset{\sim}{E}^* \cdot \underset{\sim}{v} + \mu \frac{\partial B}{\partial t} , \tag{5.10}$$

where the kinetic energy W_k is again defined by

$$W_k \equiv \frac{m}{2} v_{\parallel}^2 + \mu B . \tag{5.11}$$

When the phase space volume element is defined by

$$d\tau \equiv \frac{2\pi}{m} B^* d^3x \, dv_{\parallel} \, d\mu , \tag{5.12}$$

then a sufficient condition for the validity of the respective

Liouville's theorem, i.e. for

$$S \equiv \frac{1}{B^*} \left\{ \frac{\partial B^*}{\partial t} + \hat{\nabla} \cdot (B^* \underset{\sim}{v}) + \frac{\partial}{\partial v_{\parallel}} (B^* \dot{v}_{\parallel}) \right\} = 0 , \tag{5.13}$$

is given by the validity of

$$\frac{\partial \underset{\sim}{B}^*}{\partial v_{\parallel}} = \frac{mc}{e} \nabla \times \hat{\underset{\sim}{b}} \tag{5.14}$$

and

$$\frac{\partial \underset{\sim}{E}^*}{\partial v_{\parallel}} = - \frac{m}{e} \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} . \tag{5.15}$$

The quantity S is, of course, still defined by eq. (2.8). Littlejohn [4] does not explicitly mention these conditions of eqs. (5.14) and (5.15). He rather gives particular expressions for $\underset{\sim}{B}^*$ and $\underset{\sim}{E}^*$ that guarantee the validity of Liouville's theorem. Equations (5.14) and (5.15) possess the general solutions

$$\underset{\sim}{A}^* = \underset{\sim}{A} + \frac{mc v_{\parallel}}{e} \hat{\underset{\sim}{b}} + \delta \underset{\sim}{A}(t, \underset{\sim}{x}, \mu) , \tag{5.16}$$

$$\phi^* = \phi + \delta\phi(t, \underset{\sim}{x}, \mu) , \tag{5.17}$$

or, in terms of asterisked fields,

$$\underset{\sim}{B}^* = \underset{\sim}{B} + \frac{mc v_\parallel}{e} \nabla \times \hat{\underset{\sim}{b}} + \delta \underset{\sim}{B}(t, \underset{\sim}{x}, \mu) \tag{5.18}$$

and

$$\underset{\sim}{E}^* = \underset{\sim}{E} - \frac{m v_\parallel}{e} \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} + \delta \underset{\sim}{E}(t, \underset{\sim}{x}, \mu). \tag{5.19}$$

In the following we only consider the particular solution of eqs. (5.14) and (5.15) with $\delta \underset{\sim}{A} = \delta \Phi = \delta \underset{\sim}{B} = \delta \underset{\sim}{E} = 0$. The optimized G.C. drift equations of eqs. (5.6) through (5.8) then assume the particular form

$$\dot{\underset{\sim}{x}} \equiv \underset{\sim}{v} = v_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{v}_D^* , \tag{5.20}$$

$$\dot{v}_\parallel = \frac{e}{m} E_\parallel - \frac{\mu}{m} \frac{\partial B}{\partial s} + v_\parallel \underset{\sim}{v}_D^* \cdot \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} , \tag{5.21}$$

$$\dot{\mu} = 0 , \tag{5.22}$$

with the definitions

$$\underset{\sim}{v}_D^* \equiv \underset{\sim}{v}_E^* + \underset{\sim}{v}_{\nabla B}^* + \underset{\sim}{v}_k^* + \underset{\sim}{v}_{CD}^* , \tag{5.23}$$

$$\underset{\sim}{v}_E^* \equiv \frac{c}{B^*} \underset{\sim}{E} \times \hat{\underset{\sim}{b}} , \tag{5.24}$$

$$\underset{\sim}{v}_{\nabla B}^* \equiv \frac{\mu}{m \Omega^*} \hat{\underset{\sim}{b}} \times \nabla B , \tag{5.25}$$

$$\underset{\sim}{v}_k^* \equiv \frac{v_\parallel^2}{\Omega^*} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial s} , \tag{5.26}$$

$$\underset{\sim}{v}^*_{CD} \equiv \frac{v_\parallel}{\Omega^*} \hat{\underset{\sim}{b}} \times \frac{\partial \hat{\underset{\sim}{b}}}{\partial t} , \tag{5.27}$$

$$B^* \equiv B + \frac{mcv_\parallel}{e} \left[\hat{\underset{\sim}{b}} \cdot (\nabla \times \hat{\underset{\sim}{b}}) \right] , \tag{5.28}$$

$$\Omega^* \equiv eB^*/mc . \tag{5.29}$$

Equations (5.20) through (5.29) conserve the usual energy expression

$$W^* \equiv W \equiv \frac{m}{2} v_\parallel^2 + \mu B + e\Phi \tag{5.30}$$

in time-independent fields. The power balance equation can be formulated as

$$\dot{W}_k = e \underset{\sim}{E} \cdot \underset{\sim}{v} - \underset{\sim}{\mu} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} , \tag{5.31}$$

the kinetic energy W_k being defined by eq. (5.11); but the vectorial magnetic moment $\underset{\sim}{\mu}$ has an unusual and counter-intuitive form, viz.

$$\underset{\sim}{\mu} = -\mu \hat{\underset{\sim}{b}} + \frac{mv_\parallel}{B} \underset{\sim}{v}^*_D . \tag{5.32}$$

Still, the conservation theorem of energy in the form of eq. (2.23) follows. In addition, Liouville's theorem is, of course, satisfied, according to eqs. (5.12) through (5.15). From the above equations and the definition $dN \equiv f d\tau$ [eq. (2.3)] an associated, exactly compatible G.C. drift-kinetic equation of the form

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \hat{\nabla} + \dot{v}_\parallel \frac{\partial}{\partial v_\parallel} \right) f = 0 , \tag{5.33}$$

or, alternatively,

$$\frac{\partial}{\partial t}(B^* f) + \nabla \cdot (\underset{\sim}{v} B^* f) + \frac{\partial}{\partial v_\parallel}(\dot{v}_\parallel B^* f) = 0 \tag{5.34}$$

can be derived, where $\hat{\nabla}$ has been explained after eq. (4.10). Again the conservation theorems of G.C. particle number and energy follow. When comparing eqs. (5.33), (5.34) with eqs. (4.9), (4.10) one must, of course, remember that the definitions of $d\tau$, f, $\underset{\sim}{\mathbf{v}}$, and $\dot{v}_{\parallel}$, differ in the two cases, if only by non-leading-order terms.

6. Quasi-optimized theories with improved power balance equation

Littlejohn's theory, as presented in the previous section, has the somewhat irritating property that the power balance equation, as given by eqs. (5.31) and (5.32), has a complicated and counter-intuitive form. In this section two theories are presented that contain a power balance equation of the simpler form

$$\dot{W}_K = e\,\underset{\sim}{E}\cdot\underset{\sim}{v} + \mu\frac{\partial B}{\partial t}, \tag{6.1}$$

again with the definition $W_K \equiv (m/2)\,v_\parallel^2 + \mu B$. One pays for this by having a Liouville's theorem only in the case of time-independent fields. This suffices, however, to obtain equilibrium distribution functions by the ansatz $f_o(c_\nu)$, where the c_ν are constants of the motion. The first of the two theories is of the form

$$\dot{\underset{\sim}{x}} \equiv \underset{\sim}{v} = v_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{v}_D^*, \tag{6.2}$$

$$\dot{v}_\parallel = \frac{e}{m}E_\parallel - \frac{\mu}{m}\frac{\partial B}{\partial s} + \underset{\sim}{v}_D^*\cdot\left(v_\parallel\frac{\partial\hat{\underset{\sim}{b}}}{\partial s} + \frac{\partial\hat{\underset{\sim}{b}}}{\partial t}\right), \tag{6.3}$$

$$\dot{\mu} = 0 \tag{6.4}$$

$$\dot{W}_K = e\,\underset{\sim}{E}\cdot\underset{\sim}{v} + \mu\frac{\partial B}{\partial t}, \tag{6.5}$$

$$W_K \equiv \frac{m}{2}v_\parallel^2 + \mu B \tag{6.6}$$

$$d\tau \equiv \frac{2\pi}{m}B^*\,d^3x\,dv_\parallel\,d\mu, \tag{6.7}$$

$$S = \frac{2 v_{\parallel}}{\Omega^*} \frac{\partial \hat{b}}{\partial t} \cdot \left(\hat{b} \times \frac{\partial \hat{b}}{\partial s} \right), \tag{6.8}$$

so that $d\dot{\tau} = 0$, $S = 0$ for time-independent fields. The quantities v_D^*, B^*, Ω^* are again defined by eqs. (5.23) through (5.29).

The second of the two theories has the form

$$\dot{x} \equiv v = v_{\parallel} \hat{b} + v_E^* + v_{\nabla B}^* + v_{\kappa}^*, \tag{6.9}$$

$$\dot{v}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m} \frac{\partial B}{\partial s} + v_{\parallel} \left(v - v_{\parallel} \hat{b} \right) \cdot \frac{\partial \hat{b}}{\partial s}, \tag{6.10}$$

$$\dot{\mu} = 0, \tag{6.11}$$

$$\dot{W}_K = e E \cdot v + \mu \frac{\partial B}{\partial t}, \tag{6.12}$$

$$W_K \equiv \frac{m}{2} v_{\parallel}^2 + \mu B, \tag{6.13}$$

$$d\tau \equiv \frac{2\pi}{m} B^* \, d^3x \, dv_{\parallel} \, d\mu, \tag{6.14}$$

$$S = \frac{v_{\parallel}}{\Omega^*} \frac{\partial}{\partial t} \left[\hat{b} \cdot (\nabla \times \hat{b}) \right]. \tag{6.15}$$

Again, $d\dot{\tau} = 0$, $S = 0$ for time-independent fields. In both theories the vectorial magnetic moment is given by

$$\mu \equiv -\mu \hat{b}. \tag{6.16}$$

The form of the exactly compatible G.C. drift-kinetic equation obtained from either one of the above two sets of equations, together with the definition $dN \equiv f d\tau$ [eq. (2.3)], is the same for both sets, viz.

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \hat{\nabla} + \dot{v}_{\parallel} \frac{\partial}{\partial v_{\parallel}} + S \right) f = 0 , \tag{6.17}$$

or, alternatively,

$$\frac{\partial}{\partial t}\left(B^{*} f\right) + \hat{\nabla} \cdot \left(\underset{\sim}{v}\, B^{*} f\right) + \frac{\partial}{\partial v_{\parallel}}\left(\dot{v}_{\parallel}\, B^{*} f\right) = 0 , \tag{6.18}$$

where $\hat{\nabla}$ has been explained after eq. (4.10). Again, the conservation theorems of G.C. particle number and energy follow.

7. Internal consistency relations for guiding-center drift theories

When expressions for the guiding center velocity $\underset{\sim}{v}$, the G.C. kinetic energy W_k, and the G.C. power input $\dot{W}_k$ are given, then the expressions for both $\dot{v}_\parallel$ and $\dot{\mu}$ are essentially determined. In a leading-order theory $\dot{\mu}$ must vanish to leading order, i.e. $c\dot{\mu}/m = O(\varepsilon)$ at least; otherwise the expressions for $\underset{\sim}{v}$, W_k, $\dot{W}_k$ do not define a consistent and accurate theory (see Sec. 1). In order to derive $\dot{v}_\parallel$ and $\dot{\mu}$ we define

$$\underset{\sim}{v} = v_\parallel \hat{\underset{\sim}{b}} + \underset{\sim}{v}_D \, , \tag{7.1}$$

where $\underset{\sim}{v}_D$ is not yet fixed, and

$$W_K \equiv \frac{m}{2} v_\parallel^2 + \mu B \, , \tag{7.2}$$

$$\dot{W}_K = e \underset{\sim}{E} \cdot \underset{\sim}{v} + \mu \frac{\partial B}{\partial t} \tag{7.3}$$

(see Secs. 2, 3 and Appendix B). As explained earlier, the full G.C. velocity $\underset{\sim}{v}$ must be used in $\dot{W}_k$ in order to ensure the validity of conservation of energy in time-independent fields, viz.

$$W \equiv W_K + e\Phi = const, \tag{7.4}$$

while in W_k the term $(m/2) v_D^2 = 0(\varepsilon^2)$ can be neglected. From eqs. (7.1) through (7.3) it follows that

$$\dot{v}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m} \frac{\partial B}{\partial s}$$

$$- \frac{B}{m v_{\parallel}} \left\{ \dot{\mu} - \frac{1}{B} \underset{\sim}{v}_D \cdot \left(e \underset{\sim}{E} - \mu \nabla B \right) \right\}. \tag{7.5}$$

It is necessary to eliminate the singularity at $v_{\parallel} \rightarrow 0$. This is done by first decomposing $\underset{\sim}{v}_D$, i.e. for simplicity

$$\underset{\sim}{v}_D = \underset{\sim}{v}_{D0} + v_{\parallel} \underset{\sim}{v}_{D1} + v_{\parallel}^2 \underset{\sim}{v}_{D2} , \tag{7.6}$$

where $\underset{\sim}{v}_{D0}$, $\underset{\sim}{v}_{D1}$, and $\underset{\sim}{v}_{D2}$ are either independent of $v_{\parallel}$ or may show a non-dominant $v_{\parallel}$ dependence, e.g. in the fashion derived by Littlejohn [4] (see Sec. 5). It follows that

$$\dot{v}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m} \frac{\partial B}{\partial s}$$

$$+ \left(\underset{\sim}{v}_{D1} + v_{\parallel} \underset{\sim}{v}_{D2} \right) \cdot \left(\frac{e}{m} \underset{\sim}{E} - \frac{\mu}{m} \nabla B \right)$$

$$- \frac{B}{m v_{\parallel}} \left\{ \dot{\mu} - \frac{1}{B} \underset{\sim}{v}_{D0} \cdot \left(e \underset{\sim}{E} - \mu \nabla B \right) \right\}. \tag{7.7}$$

The simplest non-singular solution of this equation is given by

$$\dot{v}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m} \frac{\partial B}{\partial s}$$

$$+ \left(\underset{\sim}{v}_{D1} + v_{\parallel} \underset{\sim}{v}_{D2} \right) \cdot \left(\frac{e}{m} \underset{\sim}{E} - \frac{\mu}{m} \nabla B \right) , \tag{7.8}$$

with

$$\dot{\mu} = \frac{1}{B} \underset{\sim}{V}_{Do} \cdot \left(e \underset{\sim}{E} - \mu \nabla B \right) . \tag{7.9}$$

In order that $\dot{\mu} = 0$ hold, the r.h.s. of eq. (7.9) must vanish. This is trivially satisfied for $\underset{\sim}{V}_{Do} = 0$ and non-trivially for

$$\underset{\sim}{V}_{Do} = \underset{\sim}{V}_{E} + \underset{\sim}{V}_{\nabla B} \quad , \tag{7.10}$$

with $\underset{\sim}{v}_E$ and $\underset{\sim}{v}_{\nabla B}$ given by eqs. (3.5) and (3.6). The latter result rests upon the relation

$$e \underset{\sim}{E} \cdot \underset{\sim}{V}_{\nabla B} = \mu \nabla B \cdot \underset{\sim}{V}_{E} \quad . \tag{7.11}$$

Theories that use $\underset{\sim}{V}_D = \underset{\sim}{V}_E$ or $\underset{\sim}{V}_{Do} = \underset{\sim}{V}_E$ can only be made compatible with energy conservation and eq. (7.3) if $\dot{\mu} \not\equiv 0$ (at least to non-leading order in ε) is admitted. It follows from eq. (7.9) that then

$$\dot{\mu} = - \frac{\mu}{B} \underset{\sim}{V}_{E} \cdot \nabla B = \frac{m}{c} \, O(\varepsilon) \, , \tag{7.12}$$

which is compatible with a leading-order theory. Clearly, conservation of energy is more important than conservation of magnetic moment for at least two reasons:

a) Energy is an exact constant of the motion <u>for charged particles</u> in time-independent fields, while μ is only an adiabatic invariant.

b) Without W = const (and Liouville's theorem) equilibrium distribution functions cannot be expressed by $f_o(W,\mu)$ or $f_o(W)$, but must be found by integrating along characteristics (see Sec. 2).

The above analysis can be extended to the case of G.C. theories, with $\underset{\sim}{v}_D \approx \underset{\sim}{v}_E = 0(1)$. Equations (7.1) and (7.3) then remain unaltered, while eq. (7.2) must be replaced by

$$W_k \equiv \frac{m}{2} v_{\parallel}^2 + \frac{m}{2} v_E^2 + \mu B . \tag{7.13}$$

Equation (7.7) is then replaced by

$$\dot{v}_{\parallel} = \frac{e}{m} E_{\parallel} - \frac{\mu}{m}\frac{\partial B}{\partial s} - \frac{1}{2}\frac{\partial}{\partial s}(v_E^2)$$
$$- \frac{B}{m v_{\parallel}} \left\{ \dot{\mu} + \frac{\mu}{B} \underset{\sim}{v}_E \cdot \nabla B + \frac{m}{2B}\frac{\partial}{\partial t}(v_E^2) + \frac{m}{2B} \underset{\sim}{v}_E \cdot \nabla(v_E^2) \right\} . \tag{7.14}$$

The singularity at $v_{\parallel} \to 0$ can only be avoided if

$$\dot{\mu} = -\frac{\mu}{B} \underset{\sim}{v}_E \cdot \nabla B - \frac{m}{2B}\frac{\partial}{\partial t}(v_E^2)$$
$$- \frac{m}{2B} \underset{\sim}{v}_E \cdot \nabla(v_E^2) + O(v_{\parallel}) , \tag{7.15}$$

which implies that $c\dot{\mu}/m = O(1)$. This contradicts adiabatic invariance of μ to leading order. It follows that $\underset{\sim}{v}_D \approx \underset{\sim}{v}_E = 0(1)$, together with energy conservation [and eq. (7.3)], is not compatible with $c\dot{\mu}/m = O(\varepsilon)$, that is, these assumptions yield a theory that is inaccurate to leading order. It is, of course, the unphysical assumption of $\underset{\sim}{v}_D \approx \underset{\sim}{v}_E = 0(1)$ that is to blame for this failure.

On the other hand, G.C. theories with $\underset{\sim}{v}_D = \underset{\sim}{v}_E = 0(1)$ and $\dot{\mu} = 0$, but without energy conservation are not attractive for the reasons given above.

8. Concluding remarks

It is a surprising aspect of plasma theory that G.C. theories and G.C. drift theories of past decades have not provided for exact energy conservation (see Secs. 3 and 7). To appreciate this fact, let us just imagine that relativistic mechanics had been invented first, with non-relativistic mechanics derived later by expanding in $\varepsilon = V/c$. To leading order, the relativistic energy theorem would have degenerated to become $m_o c^2$ = const, i.e. a useless relation. However, we may assume that the appropriate energy theorem, as an indispensable relation, would immediately have been recovered. Even though the situation is a bit more involved in the G.C. case, it seems to remain somewhat of a mystery why conservation of energy was disregarded in this case for such a long time.

This paper presents a list of maximally consistent ("optimized") G.C. drift theories, including kinetic theories, and a theoretical framework that allows direct and exact derivation of drift-kinetic equations from G.C. drift mechanics. Earlier results of other authors [2, 3, 4] are used, but had to be either generalized, specialized, or modified. Boozer [2] only considered time-independent fields, while Littlejohn [3, 4] only investigated G.C. mechanics, but not kinetic equations (nor the associated moment equations). The present account reveals a considerable formal simplicity in that G.C. drift orbits are exact characteristics of drift-kinetic equations and equilibrium distribution functions can be exactly expressed by constants of the motion, viz. $f_o = f_o(W, \mu)$. Conservation theorems hold for single G.C. particles as well as for the system consisting of the G.C. drift plasma and its fields. Liouville's theorem can be exactly satisfied, and this in more than one way (compare Secs. 4, 5, and 6). Guiding-center drift theories have now the same formal advantages and merits as mechanics and kinetic theory of charged particles do (except for Galilei invariance).

For simplicity, only leading-order, collisionless G.C. drift theories have been considered in this paper. Existing higher-order theories have been cited (Refs. 4 and 6). Collisional drift-kinetic equations can be constructed by supplementing the above collisionless drift-kinetic theories with appropriate collision integrals that also satisfy exact conservation theorems. Of course, such theories are only applicable to plasmas (or plasma problems) where drift effects are dominant because the collision-free drift excursions are large compared with the gyro-radius.

The internal consistency relations involved in G.C. drift theories have been systematized in Sec. 7. It follows that G.C. drift theories with $\dot{\mu} \neq 0$ (to non-leading order) must be admitted if energy conservation is to be preserved when $\underset{\sim}{v}_D$ is inappropriately approximated, e.g. $\underset{\sim}{v}_D \equiv \underset{\sim}{v}_E = 0(\varepsilon)$. It is also shown there that conventional G.C. theories with $\underset{\sim}{v}_D \equiv \underset{\sim}{v}_E = 0(1)$ and $\dot{\mu} = 0$ cannot be improved to become consistent and accurate theories (see also Sec. 1).

Acknowledgments

The author sincerely thanks A. Salat and F. Sardei for carefully reading the manuscript and making valuable suggestions for improvement.

Appendix A: Drift ordering

The expansion parameter ε is defined as the ratio between the gyroradius R_g and a typical macroscopic length L, i.e. $\varepsilon = R_g/L$. The conventional drift ordering assumes that

$$\frac{1}{\Omega t} \sim \frac{c E_\perp}{V_{th} B} \sim O(\varepsilon) , \tag{A.1}$$

while

$$\frac{e E_\parallel t}{m V_{th}} \sim O(1) . \tag{A.2}$$

Here V_{th} is a typical particle velocity, $\Omega = eB/mc$, $t \sim L/V_{th}$, and the notation is otherwise standard. It should be noted that there are at least two quantities ε, i.e. ε_i and ε_e, for ions and electrons, with $\varepsilon_e \ll \varepsilon_i \ll 1$. The above orderings for $E_\perp$ and $E_\parallel$ imply that

$$\frac{e\Phi}{T} \sim O(1) \tag{A.3}$$

for a fictitious potential difference Φ if one assumes $E_\perp \sim E_\parallel \sim \Phi/L$ and $V_{th} \sim (T/m)^{1/2}$. It should also be noted that B does not enter eq. (A.3). Another consequence of eqs. (A.1), (A.2) is that

$$\frac{V_D}{V_{th}} \sim \frac{V_E}{V_{th}} \sim \frac{V_{\nabla B}}{V_{th}} \sim \frac{V_\kappa}{V_{th}} \sim O(\varepsilon) , \tag{A.4}$$

in the notation of Sec. 3. Often the physically relevant or interesting plasma times are larger than $t \sim L/V_{th}$, as can be seen from numerical examples.

For simplicity, it has been conventional in G.C. and G.C. drift theories to use dimensional representations of the G.C. and G.C. drift orderings. That is to say, some dimensional quantities are attributed an order in ε. In the present case of the drift ordering one conventionally puts

$$L \sim t \sim V_{th} \sim v_{\parallel} \sim \dot{v}_{\parallel} \sim \frac{B}{c} \sim O(1) \tag{A.5}$$

and

$$\frac{1}{\Omega} \sim \frac{m}{e} \sim \underset{\sim}{E} \sim \underset{\sim}{V}_D \sim \underset{\sim}{V}_E \sim \underset{\sim}{V}_{\nabla B} \sim \underset{\sim}{V}_K \sim O(\varepsilon). \tag{A.6}$$

It should be noted that $\underset{\sim}{B}$ alone, c alone, E/B, or V_{th}/c are "free", i.e. they are not attributed an order in ε. Often one finds that the magnetic moment μ is attributed an order, or that μ is expanded in ε. On remembering eq. (3.8) it becomes clear, however, that only the quantity $c\mu/m$ ought to be given an order, viz.

$$\frac{c\mu}{m} \sim O(1). \tag{A.7}$$

Different notations have been used by various authors. For example, Northrop [1] replaces $\frac{m}{e}$ by ε throughout and uses $E_{\parallel} = O(\varepsilon)$. Littlejohn [3, 4] replaces e by e/ε and uses the prescription $\varepsilon = 1$ for numerical evaluations. In this paper, G.C. drift theories are only considered after truncation, i.e. as closed theories. Then ε does not appear in the equations. Only occasionally, the order in of a particular quantity is indicated. This is in agreement with common usage in physics; for instance, non-relativistic particle mechanics is not usually adorned with terms $0\left[(V/c)^n\right]$ or the like, but is written as a closed theory in its own right.

Appendix B: Derivation of the G.C. power balance equation

An informal derivation of the power balance equation for a G.C. particle [to leading order in ε, see eq. (2.14)] is given here. For a charged particle the power balance equation is simply

$$\dot{W}_K \equiv \frac{dW_K}{dt} = e\,\underset{\sim}{E}\cdot\underset{\sim}{V}\,, \tag{B.1}$$

with $\underset{\sim}{V}$ the particle velocity, and $W_k \equiv (m/2)\,V^2$ the kinetic energy. On decomposing $\underset{\sim}{V} = \underset{\sim}{v} + \underset{\sim}{U}_\perp$, $\underset{\sim}{v}$ being the G.C. velocity and $\underset{\sim}{U}_\perp$ being the gyration velocity relative to the G.C. position, one obtains

$$\dot{W}_K = e\,\underset{\sim}{E}\cdot\underset{\sim}{v} + e\,\underset{\sim}{E}\cdot\underset{\sim}{U}_\perp\,. \tag{B.2}$$

Here the $\underset{\sim}{U}_\perp$ motion will be approximated by a circular one, and the term $e\,\underset{\sim}{E}\cdot\underset{\sim}{U}_\perp$ is approximated by its gyro-average, i.e. the average over the phase of the gyration, viz.

$$\langle e\,\underset{\sim}{E}\cdot\underset{\sim}{U}_\perp\rangle = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} e\,\underset{\sim}{E}\cdot\underset{\sim}{U}_\perp\,dt$$

$$= -\frac{\Omega}{2\pi}\int_0^{2\pi R_g} e\,\underset{\sim}{E}\cdot d\underset{\sim}{\ell}$$

$$= -\frac{\Omega e}{2\pi}\iint d\underset{\sim}{f}\cdot(\nabla\times\underset{\sim}{E})$$

$$\approx -\frac{\Omega e}{2} R_g^2 \; \hat{\underset{\sim}{b}} \cdot (\nabla \times \underset{\sim}{E}) . \tag{B.3}$$

On using eq. (1.3) and

$$\frac{\Omega e}{2c} R_g^2 = \frac{m}{2B} \langle U_\perp^2 \rangle = \mu \tag{B.4}$$

one obtains

$$\langle e \underset{\sim}{E} \cdot \underset{\sim}{U}_\perp \rangle = \mu \hat{\underset{\sim}{b}} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} = \mu \frac{\partial B}{\partial t} . \tag{B.5}$$

Hence, to leading order, eq. (B.2) becomes

$$\dot{W}_K = e \underset{\sim}{E} \cdot \underset{\sim}{v} + \mu \frac{\partial B}{\partial t} , \tag{B.6}$$

which is identical with eq. (2.14) if $\underset{\sim}{\mu} = -\mu \hat{\underset{\sim}{b}}$ is substituted there. It should be noted that the full G.C. velocity $\underset{\sim}{v}$, including drifts, must be used in eq. (B.6) so that exact energy conservation (in time-independent fields) follows, i.e.

$$W_K + e\Phi = \text{const} . \tag{B.7}$$

Appendix C: Derivation of the effective current density of a G.C. component

We shall show here that the expression of eq. (2.17a), viz.

$$\underset{\sim}{j}_{eff} \equiv e\underset{\sim}{\Gamma} + c\nabla \times \underset{\sim}{M} , \tag{C.1}$$

agrees to leading order in ε with the true current density of a charged-particle plasma component. That is, one is entitled to identify $\underset{\sim}{j}_{eff}$ of eq. (C.1) with the effective current density of a single G.C. component.

Let us first agree what "leading order" in ε is to denote in the present context. First of all, according to Appendix A, quantities with the dimension of an electric current density have not been attributed an order in ε. Instead, one may use current densities divided by (ne) as quantities whose order in ε is well-defined. Secondly, it is the parallel and perpendicular components (relative to the direction of $\underset{\sim}{B}$ at the G.C. position) of eq. (C.1) whose orders in ε must be separately considered. The result is that

$$e\Gamma_{\parallel}/ne \sim O(v_{\parallel}) \sim O(1) , \tag{C.2}$$

$$e\underset{\sim}{\Gamma}_{\perp}/ne \sim O(\underset{\sim}{v}_{D}) \sim O(\varepsilon) , \tag{C.3}$$

and

$$\frac{c\nabla\times\underset{\sim}{M}}{ne} \sim O\left(\frac{c\mu}{eL}\right) \sim O\left(\frac{U_{\perp}^{2}}{\Omega L}\right) \sim O(\varepsilon) , \tag{C.4}$$

in the notation of Sec. 2 and Appendices A and B.

The current density $\underset{\sim}{I}$ of one particle component of a plasma is given by

$$\underset{\sim}{I}(\underset{\sim}{X}) = e\int d\tau_V\, \underset{\sim}{V}\, F(\underset{\sim}{X}, \underset{\sim}{V}), \tag{C.5}$$

where $\underset{\sim}{X}$ and $\underset{\sim}{V}$ are the position and velocity of a particle, F is the particle distribution function, and $d\tau_V \equiv d^3V$ is the volume element of particle velocity space. We may introduce the particle phase space volume element

$$d\tau_p \equiv d^3X \; d^3V \tag{C.6}$$

and decompose $\underset{\sim}{I}$ into the respective contributions made by the G.C. velocity $\underset{\sim}{v}$ and the gyration velocity $\underset{\sim}{U}_\perp$, i.e.

$$\underset{\sim}{I} = \underset{\sim}{I}_1 + \underset{\sim}{I}_2 , \tag{C.7}$$

with $\underset{\sim}{V} = \underset{\sim}{v} + \underset{\sim}{U}_\perp$ and

$$\underset{\sim}{I}_1(\underset{\sim}{X}_0) \equiv e\int \underset{\sim}{v}(\underset{\sim}{x})\, F(\underset{\sim}{X}, \underset{\sim}{V})\, \delta(\underset{\sim}{X} - \underset{\sim}{X}_0)\, d\tau_p , \tag{C.8}$$

$$\underset{\sim}{I}_2(\underset{\sim}{X}_0) \equiv e\int \underset{\sim}{U}_\perp(\underset{\sim}{x})\, F(\underset{\sim}{X}, \underset{\sim}{V})\, \delta(\underset{\sim}{X} - \underset{\sim}{X}_0)\, d\tau_p . \tag{C.9}$$

Note that $\underset{\sim}{v}$ and $\underset{\sim}{U}_\perp$ are defined as functions of G.C. variables $(\underset{\sim}{x}, v_\parallel, \mu, \varphi)$, where φ is the azimuth of $\underset{\sim}{U}_\perp$; for brevity the notation $\underset{\sim}{v}(\underset{\sim}{x})$, $\underset{\sim}{U}_\perp(\underset{\sim}{x})$ is used in eqs. (C.8) and (C.9).

In the above equations the particle variables may be expressed by G.C. variables, using

$$\underset{\sim}{X} = \underset{\sim}{x} + \underset{\sim}{\varrho} \tag{C.10}$$

and

$$F(\underset{\sim}{X}, \underset{\sim}{V})\, d\tau_p = f(\underset{\sim}{x}, v_{\parallel}, \mu, \varphi)\, d\tau . \tag{C.11}$$

Here $\underset{\sim}{\varrho} = O(\varepsilon)$ is the vectorial gyro-radius and $d\tau$ is the G.C. phase space volume element in the form

$$d\tau \equiv d^3x \, \frac{B}{m} \, dv_{\parallel} \, d\mu \, d\varphi . \tag{C.12}$$

It follows that

$$\underset{\sim}{I}_1(\underset{\sim}{X}_0) = e \int \underset{\sim}{v} \, f \, \delta(\underset{\sim}{x} + \underset{\sim}{\varrho} - \underset{\sim}{X}_0)\, d\tau \tag{C.13}$$

and

$$\underset{\sim}{I}_2(\underset{\sim}{X}_0) = e \int \underset{\sim}{U}_{\perp} \, f \, \delta(\underset{\sim}{x} + \underset{\sim}{\varrho} - \underset{\sim}{X}_0)\, d\tau . \tag{C.14}$$

For brevity of notation we shall use in the following a Taylor expansion of the δ-function [7] in the form

$$\delta(\underset{\sim}{x} + \underset{\sim}{\varrho} - \underset{\sim}{X}_0) = \delta(\underset{\sim}{x} - \underset{\sim}{X}_0) + \underset{\sim}{\varrho} \cdot \hat{\nabla} \delta(\underset{\sim}{x} - \underset{\sim}{X}_0) + O(\varepsilon^2) , \tag{C.15}$$

where $\hat{\nabla}$ denotes the gradient taken with respect to $\underset{\sim}{x}$, and with $(v_{\parallel}, \mu, \varphi) = \text{const}$. Equation (C.13) then becomes

$$\underset{\sim}{I}_1(\underset{\sim}{X}_0) \approx e \int \underset{\sim}{v} \, f \, \delta(\underset{\sim}{x} - \underset{\sim}{X}_0)\, d\tau \tag{C.16}$$

or

$$\underset{\sim}{I}_1(\underset{\sim}{x}) \approx e \int \underset{\sim}{v} \, f \, d\tau_v \equiv e \underset{\sim}{\Gamma}(\underset{\sim}{x}) \tag{C.17}$$

to leading order in ε as defined by eqs. (C.2) and (C.3).

The transformation of the diamagnetic contribution $\underset{\sim}{I}_2$ is somewhat more involved. Equation (C.14) becomes

$$\underset{\sim}{I}_2(\underset{\sim}{X}_0) = e\int \underset{\sim}{U}_\perp f \; \delta(\underset{\sim}{x} - \underset{\sim}{X}_0)\, d\tau$$

$$+ e\int \underset{\sim}{U}_\perp f \; \underset{\sim}{\varrho} \cdot \hat{\nabla}\delta(\underset{\sim}{x} - \underset{\sim}{X}_0)\, d\tau$$

$$+ ne \cdot O(\varepsilon^2) . \qquad \text{(C.18)}$$

Here $\partial f/\partial \varphi = 0 + O(\varepsilon^2)$ may be used because the $O(\varepsilon)$ terms have been eliminated by subtracting the drift velocity $\underset{\sim}{v}$. The first term on the r.h.s. of eq. (C.18) then vanishes, yielding to leading order in ε

$$\underset{\sim}{I}_2(\underset{\sim}{X}_0) \approx e\int \underset{\sim}{U}_\perp f \; \underset{\sim}{\varrho} \cdot \hat{\nabla}\delta(\underset{\sim}{x} - \underset{\sim}{X}_0)\, d\tau . \qquad \text{(C.19)}$$

On using $\underset{\sim}{U}_\perp = U_\perp \hat{\underset{\sim}{e}}_\perp$ and

$$\underset{\sim}{\varrho} \approx \Omega^{-1}\, \underset{\sim}{U}_\perp \times \hat{\underset{\sim}{b}} + O(\varepsilon^2) \qquad \text{(C.20)}$$

this becomes

$$\underset{\sim}{I}_2(\underset{\sim}{X}_0) \approx \int \frac{e}{\Omega} U_\perp^2 f \, \hat{\underset{\sim}{e}}_\perp \left[(\hat{\underset{\sim}{e}}_\perp \times \hat{\underset{\sim}{b}}) \cdot \hat{\nabla}\delta \right] d\tau . \qquad \text{(C.21)}$$

Here one has

$$\frac{e}{\Omega} U_\perp^2 = 2c\mu . \qquad \text{(C.22)}$$

Averaging over the azimuth φ and using the definition $\underset{\sim}{\mu} \equiv -\mu \hat{\underset{\sim}{b}}$

then yields

$$\underset{\sim}{I}_2(\underset{\sim}{X}_0) \approx -c \int \left[\underset{\sim}{\mu} \times \hat{\nabla} \delta(\underset{\sim}{x} - \underset{\sim}{X}_0) \right] f \, d\tau \, . \qquad \text{(C.23)}$$

On using eq. (C.12) for $d\tau$ and performing a partial integration one further obtains

$$\underset{\sim}{I}_2(\underset{\sim}{X}_0) \approx c \iint \left[\hat{\nabla} \times \left(\frac{2\pi B}{m} f \underset{\sim}{\mu} \right) \right] \delta(\underset{\sim}{x} - \underset{\sim}{X}_0) \, d^3x \, dv_\parallel \, d\mu \, . \qquad \text{(C.24)}$$

This is identical to (substitute $\underset{\sim}{X}_0$ by $\underset{\sim}{x}$):

$$\underset{\sim}{I}_2(\underset{\sim}{x}) \approx c \, \mathrm{rot} \int \underset{\sim}{\mu} f \, d\tau_v \equiv c \nabla \times \underset{\sim}{M}(\underset{\sim}{x}) \, . \qquad \text{(C.25)}$$

It follows from eqs. (C.5), (C.7), (C.17), and (C.25) that $\underset{\sim}{j}_{eff}$ of eq. (C.1) agrees with $\underset{\sim}{I}$ of eq. (C.5) to leading order in ε as defined by eqs. (C.2) through (C.4). That is, the "effective current density" of the G.C. drift model represents, with sufficient accuracy, the true particle current density.

References

[1] T.G. Northrop, The Adiabatic Motion of Charged Particles (Interscience 1963).

[2] A.H. Boozer, Phys. Fluids 23, 904 (1980).

[3] R.G. Littlejohn, J. Math. Phys. 20, 2445 (1979).

[4] R.G. Littlejohn, Phys. Fluids 24, 1730 (1981).

[5] S. Chapman and T.G. Cowling, The Mathematical Theory of Non-Uniform Gases (Cambridge Univ. Press, 1958), p. 371.

[6] T.G. Northrop and J.A. Rome, Phys. Fluids 21, 384 (1978).

[7] I.M. Gelfand and G.E. Schilow, Verallgemeinerte Funktionen (Distributionen) (VEB Deutscher Verlag der Wissenschaften, Berlin, 1960), Vol. I.