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Extended Standard Vector Analysis
for Plasma Physics

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Abstract

Standard vector analysis in 3-dimensional space, as found in most tables and textbooks, is complemented by a number of basic formulas that seem to be largely unknown, but are important in themselves and for some plasma physics applications, as is shown by several examples.

1. INTRODUCTION

It has apparently gone unnoticed that conventional, 3-dimensional vector analysis, as presented in tables and textbooks [1-11], is incomplete in the sense that several basic relations are not listed. This paper derives and lists these missing identities, which are important in themselves and for applications in physics. This will be demonstrated by several examples originating from plasma physics. It would probably be advantageous to include these formulas in standard tables and textbooks.

Let us examine a conventional list of vector-analytical formulas [1-11]. Expressions for $\nabla \cdot (\alpha\mathbf{a})$, $\nabla \times (\alpha\mathbf{a})$, $\nabla(\alpha\mathbf{a})$, $\nabla \cdot (\mathbf{a} \times \mathbf{b})$, $\nabla \times (\mathbf{a} \times \mathbf{b})$, $\nabla(\mathbf{a} \cdot \mathbf{b})$, and sometimes $\nabla \cdot (\mathbf{a}\mathbf{b})$ are given, followed by cases of higher-order differentiation, special coordinate systems, and integral theorems. Inspection shows that the list of formulas involving two vector fields and one ∇ operator is incomplete, and that formulas involving three vector fields and one ∇ operator are absent, even though they are needed for applications, e.g. in MHD [12] and in guiding center drift theories [13]. This paper concentrates on these two types of vector identities, while cases involving higher-order differentiation, special coordinates and integral theorems will not be considered.

In vector analysis, simple methods of derivation are the use of partial differentiation on the one hand and algebra of dyadics (= second-rank tensors) and vectors on the other. In particular, any algebraic identity involving a dyadic field $\mathbf{T}$ yields an analytic identity by substituting $\mathbf{T} \rightarrow \nabla\mathbf{b}$. Many results can be written as commutation, or anticommutation, rules. This helps in systematizing tables. Looking for such rules also assists in detecting missing formulas.

The following notation is used in this paper: scalars are denoted by small Greek letters, vectors by bold-face, lower-case roman letters, and dyadics by bold-face, capital, roman letters. The nabla operator ∇ operates only on the next field symbol to its right, except when otherwise indicated by brackets.

2. BASIC FORMULAS

In this section, algebraic identities involving $\underset{\sim}{a} \cdot \underset{\sim}{T}$, $\underset{\sim}{T} \cdot \underset{\sim}{a}$, $\underset{\sim}{a} \times \underset{\sim}{T}$, and $\underset{\sim}{T} \times \underset{\sim}{a}$ ($\underset{\sim}{a}$ = 3-D vector field, $\underset{\sim}{T}$ = 3-D dyadic field) will be derived. Analogous vector-analytical identities will be obtained by substituting $\underset{\sim}{T} \rightarrow \nabla\underset{\sim}{b}$, where appropriate. The ∇ operator will operate only on the next field symbol to its right, except when otherwise indicated by brackets. Pseudo-vectors and pseudo-tensors will simply be termed "vectors" and "tensors", respectively. The reader is assumed to know how to express symbolic vector and tensor formulas in terms of Cartesian components.

In the following we shall need the definitions of "vector of a dyadic" and "transpose of a dyadic". The "transpose of a vector" is identified with the vector, i.e. row and column vectors are not distinguished. The transpose $\underset{\sim}{T}^T$ of a dyadic $\underset{\sim}{T}$ is obtained by permuting the indices of its Cartesian components. It satisfies the following identities:

$$\underset{\sim}{a} \cdot \underset{\sim}{T}^T = \underset{\sim}{T} \cdot \underset{\sim}{a} \; ; \quad \underset{\sim}{T}^T \cdot \underset{\sim}{a} = \underset{\sim}{a} \cdot \underset{\sim}{T} \, , \tag{1}$$

$$\underset{\sim}{a} \times \underset{\sim}{T}^T = - (\underset{\sim}{T} \times \underset{\sim}{a})^T \, , \tag{1a}$$

$$\underset{\sim}{T}^T \times \underset{\sim}{a} = - (\underset{\sim}{a} \times \underset{\sim}{T})^T , \tag{1b}$$

and further relations that are obtained by forming the transpose of eqs. (1a), (1b).

The "vector of a dyadic" is defined as [14]

$$\underset{\sim}{t} \equiv \text{vec}\, \underset{\sim}{T} \tag{2}$$

with the Cartesian components

$$t_k = \varepsilon_{kmn} \, T_{mn} \, , \tag{2a}$$

where the ε_{kmn} are the components of the well-known third-rank, invariant, antisymmetric ε-tensor, and the summation convention is used. (In some texts [15] differing definitions of the "vector of a dyadic" are used). Note that $\underset{\sim}{t} = 0$ for $\underset{\sim}{T}$ symmetric, and

$$\boxed{\text{vec}(\nabla\underset{\sim}{b}) = \text{curl}\, \underset{\sim}{b} \equiv \nabla \times \underset{\sim}{b} \, .} \tag{2b}$$

The "antisymmetric part of $\underset{\sim}{T}$", i.e.

$$\overset{A}{\underset{\sim}{T}} \equiv \frac{1}{2} (\underset{\sim}{T} - \underset{\sim}{T}^T) , \tag{2c}$$

where $\underset{\sim}{T}^T$ is the transpose of $\underset{\sim}{T}$, can be reconstructed from vec $\underset{\sim}{T}$ by

$$\overset{A}{\underset{\sim}{T}} = - \frac{1}{2} \underset{\sim}{1} \times \text{vec } \underset{\sim}{T} \equiv - \frac{1}{2} (\text{vec } \underset{\sim}{T}) \times \underset{\sim}{1} , \tag{3}$$

where $\underset{\sim}{1}$ is the unit dyadic. Note that the vector products of the unit dyadic with any vector $\underset{\sim}{a}$ obey

$$\underset{\sim}{1} \times \underset{\sim}{a} \equiv \underset{\sim}{a} \times \underset{\sim}{1} , \tag{3a}$$

contrary to the commutation rule for the vector product of two vectors. An operator that transforms any vector $\underset{\sim}{a}$ into a dyadic may be defined, viz.

$$\underset{\sim}{A} \equiv \text{dyad } \underset{\sim}{a} , \tag{3b}$$

such that

$$\text{vec (dyad } \underset{\sim}{a}) = \underset{\sim}{a} \tag{3c}$$

and

$$\text{dyad (vec } \overset{A}{\underset{\sim}{T}}) = \overset{A}{\underset{\sim}{T}} . \tag{3d}$$

One simply has to put [see eq.(3)]:

$$\text{dyad } \underset{\sim}{a} \equiv - \frac{1}{2} \underset{\sim}{1} \times \underset{\sim}{a} \equiv - \frac{1}{2} \underset{\sim}{a} \times \underset{\sim}{1} . \tag{3e}$$

After these preliminaries, let us list some vector and tensor formulas that follow from simple considerations and can be used to obtain several vector-analytical identities. They are easily verified by writing everything in Cartesian components. We shall first obtain identities for vec $(\underset{\sim}{a}\underset{\sim}{b})$, vec $(\underset{\sim}{a} \times \underset{\sim}{T})$, and vec $(\underset{\sim}{T} \times \underset{\sim}{a})$, $\underset{\sim}{a}\underset{\sim}{b}$ being the dyadic product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Firstly,

$$\text{vec } (\underset{\sim}{a}\underset{\sim}{b}) \equiv \frac{1}{2} \text{vec } (\underset{\sim}{a}\underset{\sim}{b} - \underset{\sim}{b}\underset{\sim}{a}) = \underset{\sim}{a} \times \underset{\sim}{b} . \tag{4}$$

It follows from eqs.(1c), (2), and (4) that

$$\underset{\sim}{1} \times (\underset{\sim}{a} \times \underset{\sim}{b}) = \underset{\sim}{b}\underset{\sim}{a} - \underset{\sim}{a}\underset{\sim}{b} . \tag{4a}$$

On applying the divergence operator $\nabla \cdot$ to the two sides of eq.(4a) and observing that

$$\nabla \cdot (\underset{\sim}{1} \times \underset{\sim}{c}) \equiv \nabla \times \underset{\sim}{c} \tag{4b}$$

one obtains

$$\boxed{\nabla \times (\underset{\sim}{a} \times \underset{\sim}{b}) = \nabla \cdot (\underset{\sim}{b}\underset{\sim}{a} - \underset{\sim}{a}\underset{\sim}{b})} \tag{4c}$$

as a basic identity to be used in Sec.4. The "divergence of a dyadic", viz. $\nabla \cdot \underset{\sim}{T}$, is defined here in the natural way, i.e. with components

$$(\nabla \cdot \underset{\sim}{T})_k \equiv \frac{\partial T_{ik}}{\partial x_i} . \tag{4d}$$

Consequently, the divergence of a dyadic product is defined by the components

$$\left[\nabla \cdot (\underset{\sim}{a}\underset{\sim}{b})\right]_k \equiv \frac{\partial}{\partial x_i} (a_i b_k) . \tag{4e}$$

Some authors [16] use a deviating definition that violates the systematics of symbolic notation and should therefore be avoided.

For vec $(\underset{\sim}{a} \times \underset{\sim}{T})$, etc. the following identities are obtained:

$$\text{vec } (\underset{\sim}{a} \times \underset{\sim}{T}) = \underset{\sim}{T} \cdot \underset{\sim}{a} - (\text{trace } \underset{\sim}{T})\, \underset{\sim}{a} , \tag{5}$$

$$\text{vec } (\underset{\sim}{T} \times \underset{\sim}{a}) = \underset{\sim}{a} \cdot \underset{\sim}{T} - (\text{trace } \underset{\sim}{T})\, \underset{\sim}{a} , \tag{5a}$$

whence

$$\text{vec } (\underset{\sim}{a} \times \underset{\sim}{T} - \underset{\sim}{T} \times \underset{\sim}{a}) = \underset{\sim}{T} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{T} \tag{5b}$$

[compare eq.(6b)]. Of course

$$\text{trace } \underset{\sim}{T} \equiv T_{ii} , \tag{5c}$$

and

$$\boxed{\text{trace } (\nabla \underset{\sim}{b}) = \nabla \cdot \underset{\sim}{b} \equiv \text{div } \underset{\sim}{b} .} \tag{5d}$$

Rather than substituting $\underset{\sim}{T} \rightarrow \nabla\underset{\sim}{b}$ in eqs.(5) to (5b), these equations will be used in Sec.3 as auxiliary formulas. Equations (5) and (5a) can also be put in an alternative form by applying the "dyad" operator of eq.(3e) to both sides. The result consists of two auxiliary relations, viz.

$$\underset{\sim}{a} \times \underset{\sim}{T} - (\underset{\sim}{a} \times \underset{\sim}{T})^{T} = \underset{\sim}{1} \times [(\text{trace } \underset{\sim}{T})\, \underset{\sim}{a} - \underset{\sim}{T} \cdot \underset{\sim}{a}]$$
$$\equiv [(\text{trace } \underset{\sim}{T})\, \underset{\sim}{a} - \underset{\sim}{T} \cdot \underset{\sim}{a}] \times \underset{\sim}{1} \qquad (5e)$$

and

$$\underset{\sim}{T} \times \underset{\sim}{a} - (\underset{\sim}{T} \times \underset{\sim}{a})^{T} = \underset{\sim}{1} \times [(\text{trace } \underset{\sim}{T})\, \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{T}]$$
$$\equiv [(\text{trace } \underset{\sim}{T})\, \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{T}] \times \underset{\sim}{1}\ . \qquad (5f)$$

By substituting $\underset{\sim}{T} \rightarrow \nabla\underset{\sim}{b}$ one obtains two vector-analytical identities that will be used below and in Sec.3, viz.

$$\boxed{\begin{aligned}\underset{\sim}{a} \times \nabla\underset{\sim}{b} - (\underset{\sim}{a} \times \nabla\underset{\sim}{b})^{T} &= \underset{\sim}{1} \times [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \nabla\underset{\sim}{b} \cdot \underset{\sim}{a}] \\ &\equiv [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \nabla\underset{\sim}{b} \cdot \underset{\sim}{a}] \times \underset{\sim}{1}\end{aligned}} \qquad (5g)$$

and

$$\boxed{\begin{aligned}\nabla\underset{\sim}{b} \times \underset{\sim}{a} - (\nabla\underset{\sim}{b} \times \underset{\sim}{a})^{T} &= \underset{\sim}{1} \times [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b}] \\ &\equiv [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b}] \times \underset{\sim}{1}\end{aligned}} \qquad (5h)$$

It should be remembered that the ∇ operator operates only on $\underset{\sim}{b}$ in eqs.(5g) and (5h).

An important identity can be derived by considering the expression

$$2\underset{\sim}{a} \cdot \overset{A}{\underset{\sim}{T}} \equiv \underset{\sim}{a} \cdot (\underset{\sim}{T} - \underset{\sim}{T}^{T}) \equiv \underset{\sim}{a} \cdot \underset{\sim}{T} - \underset{\sim}{T} \cdot \underset{\sim}{a}\ , \qquad (6)$$

for which the alternative expression

$$2\underset{\sim}{a} \cdot \overset{A}{\underset{\sim}{T}} = - \underset{\sim}{a} \cdot (\underset{\sim}{1} \times \text{vec } \underset{\sim}{T}) = - \underset{\sim}{a} \times \text{vec } \underset{\sim}{T} \qquad (6a)$$

obtains. Comparing eqs.(6) and (6a) yields the <u>commutation rule</u> for tensor-vector scalar products, viz.

$$\underset{\sim}{T} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{T} = \underset{\sim}{a} \times \mathrm{vec}\, \underset{\sim}{T} \tag{6b}$$

or, on substituting $\underset{\sim}{T} \rightarrow \nabla\underset{\sim}{b}$,

$$\boxed{\nabla\underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b} = \underset{\sim}{a} \times (\nabla \times \underset{\sim}{b}) \ .} \tag{6c}$$

This relation is usually not recognized as a basic identity; when found in a textbook [7], it is then mostly in a non-standard notation and/or as an auxiliary equation used during a mathematical derivation. The book by Shkarovsky et al. [11] is an exception. It may be instructive to compare eq.(6c) with the relations

$$\boxed{\nabla\, (\underset{\sim}{a} \cdot \underset{\sim}{b}) = \nabla\underset{\sim}{a} \cdot \underset{\sim}{b} + \nabla\underset{\sim}{b} \cdot \underset{\sim}{a}} \tag{6d}$$

and

$$\boxed{\nabla \cdot (\underset{\sim}{a}\underset{\sim}{b}) = (\nabla \cdot \underset{\sim}{a})\, \underset{\sim}{b} + \underset{\sim}{a} \cdot \nabla\underset{\sim}{b}} \tag{6e}$$

that follow from <u>partial differentiation</u> (and some algebra).

A further relation resembling an <u>anticommutation rule</u> can be derived for tensor-vector cross products. Consider

$$2\underset{\sim}{a} \times \overset{A}{\underset{\sim}{T}} \equiv \underset{\sim}{a} \times (\underset{\sim}{T} - \underset{\sim}{T}^{T}) \equiv \underset{\sim}{a} \times \underset{\sim}{T} + (\underset{\sim}{T} \times \underset{\sim}{a})^{T} \ . \tag{7}$$

Here the alternative expression

$$2\underset{\sim}{a} \times \overset{A}{\underset{\sim}{T}} = - \underset{\sim}{a} \times (\underset{\sim}{t} \times \underset{\sim}{1}) = (\underset{\sim}{a} \cdot \underset{\sim}{t})\, \underset{\sim}{1} - \underset{\sim}{t}\underset{\sim}{a} \tag{7a}$$

exists, with $\underset{\sim}{t} \equiv \mathrm{vec}\, \underset{\sim}{T} \equiv \mathrm{vec}\, \overset{A}{\underset{\sim}{T}}$. Comparing eqs.(7) and (7a) yields

$$\underset{\sim}{a} \times \underset{\sim}{T} + (\underset{\sim}{T} \times \underset{\sim}{a})^{T} = [\underset{\sim}{a} \cdot \mathrm{vec}\, \underset{\sim}{T}]\, \underset{\sim}{1} - (\mathrm{vec}\, \underset{\sim}{T})\, \underset{\sim}{a} \tag{7b}$$

and

$$\underset{\sim}{T} \times \underset{\sim}{a} + (\underset{\sim}{a} \times \underset{\sim}{T})^{T} = [\underset{\sim}{a} \cdot \mathrm{vec}\, \underset{\sim}{T}]\, \underset{\sim}{1} - \underset{\sim}{a}\, (\mathrm{vec}\, \underset{\sim}{T}) \ . \tag{7c}$$

On substituting $\underset{\sim}{T} \to \nabla\underset{\sim}{b}$ one obtains two auxiliary relations, to be used in Sec.3 and immediately below, viz.

$$\boxed{\underset{\sim}{a} \times \nabla\underset{\sim}{b} + (\nabla\underset{\sim}{b} \times \underset{\sim}{a})^T = [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\, \underset{\sim}{1} - (\nabla \times \underset{\sim}{b})\, \underset{\sim}{a}} \tag{7d}$$

and

$$\boxed{\nabla\underset{\sim}{b} \times \underset{\sim}{a} + (\underset{\sim}{a} \times \nabla\underset{\sim}{b})^T = [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\, \underset{\sim}{1} - \underset{\sim}{a}\, (\nabla \times \underset{\sim}{b})\ .} \tag{7e}$$

By combining eqs.(5g) and (7e) one obtains an <u>anticommutation rule</u> for the cross products:

$$\boxed{\begin{aligned} \underset{\sim}{a} \times \nabla\underset{\sim}{b} + \nabla\underset{\sim}{b} \times \underset{\sim}{a} &= \underset{\sim}{1} \times [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \nabla\underset{\sim}{b} \cdot \underset{\sim}{a}] \\ &\quad + [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\, \underset{\sim}{1} - \underset{\sim}{a}\, (\nabla \times \underset{\sim}{b})\ . \end{aligned}} \tag{8}$$

On the other hand, combination of eqs.(5h) and (7d) yields the alternative expression

$$\boxed{\begin{aligned} \underset{\sim}{a} \times \nabla\underset{\sim}{b} + \nabla\underset{\sim}{b} \times \underset{\sim}{a} &= \underset{\sim}{1} \times [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b}] \\ &\quad + [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\, \underset{\sim}{1} - (\nabla \times \underset{\sim}{b})\, \underset{\sim}{a}\ . \end{aligned}} \tag{8a}$$

Forming the difference between eqs.(8) and (8a) yields

$$\underset{\sim}{1} \times [\nabla\underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b}] = (\nabla \times \underset{\sim}{b})\, \underset{\sim}{a} - \underset{\sim}{a}\, (\nabla \times \underset{\sim}{b})\ , \tag{8b}$$

which can be transformed to recover eq.(6c). Equations (8) to (8b) may, of course, be generalized by substituting $\nabla\underset{\sim}{b} \to \underset{\sim}{T}$, $\nabla \times \underset{\sim}{b} \to \text{vec}\ \underset{\sim}{T}$, $\nabla \cdot \underset{\sim}{b} \to \text{trace}\ \underset{\sim}{T}$. The above identities may be compared with the relations

$$\boxed{\nabla(\underset{\sim}{a} \times \underset{\sim}{b}) = \nabla\underset{\sim}{a} \times \underset{\sim}{b} - \nabla\underset{\sim}{b} \times \underset{\sim}{a}\ ,} \tag{8c}$$

$$\boxed{\nabla \times (\underset{\sim}{a}\underset{\sim}{b}) = (\nabla \times \underset{\sim}{a})\, \underset{\sim}{b} - \underset{\sim}{a} \times \nabla\underset{\sim}{b}\ ,} \tag{8d}$$

$$\boxed{\nabla \cdot (\underset{\sim}{a} \times \underset{\sim}{b}) = \underset{\sim}{b} \cdot (\nabla \times \underset{\sim}{a}) - \underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})\ ,} \tag{8e}$$

which derive from partial differentiation (and some algebra).

For completeness the following remark is added. The right-hand sides of eqs.(5g), (5h), (8), (8a) could be written in an alternative form owing to the identities

$$\nabla \underset{\sim}{b} \cdot \underset{\sim}{a} - (\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} = (\underset{\sim}{a} \times \nabla) \times \underset{\sim}{b} \tag{9}$$

and

$$\underset{\sim}{a} \cdot \nabla \underset{\sim}{b} - (\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} = (\underset{\sim}{a} \times \nabla) \times \underset{\sim}{b} - \underset{\sim}{a} \times (\nabla \times \underset{\sim}{b}) \ . \tag{9a}$$

This application of the operator $(\underset{\sim}{a} \times \nabla)$ does not seem to be particularly useful, however; hence this notation [17] will not be used here. Some authors also derive

$$(\underset{\sim}{a} \times \nabla) \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b}) \ , \tag{9b}$$

a relation that does not seem very useful either.

3. THREE-VECTOR FORMULAS

The formulas below, all of algebraic origin, are again derived by representing $\nabla\underset{\sim}{b}$ as a general dyadic $\underset{\sim}{T}$ and later letting $\underset{\sim}{T} \to \nabla\underset{\sim}{b}$. The following identities derive from eqs. (6b) and (6c):

$$\underset{\sim}{a} \cdot \underset{\sim}{T} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{T} \cdot \underset{\sim}{a} = (\underset{\sim}{a} \times \underset{\sim}{c}) \cdot \mathrm{vec}\ \underset{\sim}{T} \tag{10}$$

and

$$\boxed{\underset{\sim}{a} \cdot \nabla\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \nabla\underset{\sim}{b} \cdot \underset{\sim}{a} = (\underset{\sim}{a} \times \underset{\sim}{c}) \cdot (\nabla \times \underset{\sim}{b})\ .} \tag{10a}$$

Equation (10a) is also found in ref.7, p.126. By repeated alternating application of eqs.(6c) and (6d) one obtains moreover

$$\boxed{\begin{aligned} 2\underset{\sim}{a} \cdot \nabla\underset{\sim}{b} \cdot \underset{\sim}{c} &= \underset{\sim}{a} \cdot \nabla (\underset{\sim}{b} \cdot \underset{\sim}{c}) + \underset{\sim}{c} \cdot \nabla (\underset{\sim}{b} \cdot \underset{\sim}{a}) - \underset{\sim}{b} \cdot \nabla (\underset{\sim}{a} \cdot \underset{\sim}{c}) \\ &+ (\underset{\sim}{b} \times \underset{\sim}{c}) \cdot (\nabla \times \underset{\sim}{a}) + (\underset{\sim}{b} \times \underset{\sim}{a}) \cdot (\nabla \times \underset{\sim}{c}) + (\underset{\sim}{a} \times \underset{\sim}{c}) \cdot (\nabla \times \underset{\sim}{b})\ . \end{aligned}} \tag{10b}$$

Of course, eq.(10b) is only useful in special cases where the r.h.s simplifies. Such a case (from guiding center drift theory [13]) is briefly discussed in Sec.5.

The rest of this section will be devoted to the following list of expressions:

$$(\underset{\sim}{a} \times \underset{\sim}{c}) \cdot \underset{\sim}{T} \equiv \underset{\sim}{a} \cdot (\underset{\sim}{c} \times \underset{\sim}{T}) \equiv - \underset{\sim}{c} \cdot (\underset{\sim}{a} \times \underset{\sim}{T})\ , \tag{11}$$

$$\underset{\sim}{T} \cdot (\underset{\sim}{a} \times \underset{\sim}{c}) \equiv (\underset{\sim}{T} \times \underset{\sim}{a}) \cdot \underset{\sim}{c} \equiv - (\underset{\sim}{T} \times \underset{\sim}{c}) \cdot \underset{\sim}{a}\ , \tag{11a}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{T} \times \underset{\sim}{c} \equiv (\underset{\sim}{a} \cdot \underset{\sim}{T}) \times \underset{\sim}{c} \equiv \underset{\sim}{a} \cdot (\underset{\sim}{T} \times \underset{\sim}{c}) \equiv - \underset{\sim}{c} \times (\underset{\sim}{a} \cdot \underset{\sim}{T})\ , \tag{11b}$$

$$\underset{\sim}{a} \times \underset{\sim}{T} \cdot \underset{\sim}{c} \equiv \underset{\sim}{a} \times (\underset{\sim}{T} \cdot \underset{\sim}{c}) \equiv (\underset{\sim}{a} \times \underset{\sim}{T}) \cdot \underset{\sim}{c} \equiv - (\underset{\sim}{T} \cdot \underset{\sim}{c}) \times \underset{\sim}{a}\ , \tag{11c}$$

and further ones originating from permuting the vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$. The reader might substitute $\underset{\sim}{T} \to \nabla\underset{\sim}{b}$ for himself in order to visualize the corresponding vector-analytical expressions. Several commutation rules will be derived that relate various of these expressions, leaving out, of course, trivial modifications of earlier two-vector formulas. One may first derive the identity

$$\mathbf{a} \cdot \mathbf{T} \times \mathbf{c} + \mathbf{c} \times \mathbf{T} \cdot \mathbf{a} = \mathbf{c} \times (\mathbf{a} \times \mathrm{vec}\,\mathbf{T}) \,, \tag{12}$$

either by forming the vector product of eq.(6b) and $\mathbf{c}$ or by forming the scalar product of eq.(7b) or (7c) and $\mathbf{c}$ (and then permuting $\mathbf{a}$ and $\mathbf{c}$). Substituting $\mathbf{T} \to \nabla\mathbf{b}$ then yields

$$\boxed{\mathbf{a} \cdot \nabla\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \nabla\mathbf{b} \cdot \mathbf{a} = \mathbf{c} \times [\mathbf{a} \times (\nabla \times \mathbf{b})] \,.} \tag{12a}$$

Further important formulas are

$$\mathbf{a} \cdot \mathbf{T} \times \mathbf{c} - \mathbf{c} \cdot \mathbf{T} \times \mathbf{a} = [(\mathrm{trace}\,\mathbf{T})\,\mathbf{1} - \mathbf{T}] \cdot (\mathbf{a} \times \mathbf{c}) \tag{12b}$$

and

$$\mathbf{a} \times \mathbf{T} \cdot \mathbf{c} - \mathbf{c} \times \mathbf{T} \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{c}) \cdot [(\mathrm{trace}\,\mathbf{T})\,\mathbf{1} - \mathbf{T}] \,. \tag{12c}$$

These formulas can be derived either by using eqs.(6b) and (5) or (5a) or by scalar multiplication of eqs.(5e), (5f) by $\mathbf{c}$. Again substituting $\mathbf{T} \to \nabla\mathbf{b}$ yields the following vector-analytical identities:

$$\boxed{\mathbf{a} \cdot \nabla\mathbf{b} \times \mathbf{c} - \mathbf{c} \cdot \nabla\mathbf{b} \times \mathbf{a} = [(\nabla \cdot \mathbf{b})\,\mathbf{1} - \nabla\mathbf{b}] \cdot (\mathbf{a} \times \mathbf{c})} \tag{12d}$$

and

$$\boxed{\mathbf{a} \times \nabla\mathbf{b} \cdot \mathbf{c} - \mathbf{c} \times \nabla\mathbf{b} \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{c}) \cdot [(\nabla \cdot \mathbf{b})\,\mathbf{1} - \nabla\mathbf{b}] \,.} \tag{12e}$$

These two identities can also be obtained by scalar multiplication of eqs.(5g), (5h) by $\mathbf{c}$. This completes the list of three-vector relations. For the sake of brevity, three-vector formulas involving partial differentiation have not been considered. Section 4 gives <u>a systematic list of vector-analytical identities</u>.

4. SYSTEMATIC LIST OF SOME VECTOR-ANALYTICAL IDENTITIES

The identities listed here contain two or three different vector fields and only first-order derivatives. Scalar fields, second- or higher-rank tensor fields, and integral theorems are not considered; they should be looked up elsewhere [1-11]. Part of the identities listed have been derived in Secs.2 and 3; the others are well-known [1-11]. The reader is reminded that the ∇ operator only operates on the next field quantity to its right, except where otherwise indicated by brackets.

The first five formulas follow from partial differentiation (and some algebra), viz.

$$\nabla(\underset{\sim}{a} \cdot \underset{\sim}{b}) = \nabla\underset{\sim}{a} \cdot \underset{\sim}{b} + \nabla\underset{\sim}{b} \cdot \underset{\sim}{a} \;, \tag{L1}$$

$$\nabla(\underset{\sim}{a} \times \underset{\sim}{b}) = \nabla\underset{\sim}{a} \times \underset{\sim}{b} - \nabla\underset{\sim}{b} \times \underset{\sim}{a} \;, \tag{L2}$$

$$\nabla \cdot (\underset{\sim}{a}\,\underset{\sim}{b}) = (\nabla \cdot \underset{\sim}{a})\,\underset{\sim}{b} + \underset{\sim}{a} \cdot \nabla\underset{\sim}{b} \equiv (\nabla \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \nabla)\,\underset{\sim}{b} \;, \tag{L3}$$

$$\nabla \times (\underset{\sim}{a}\,\underset{\sim}{b}) = (\nabla \times \underset{\sim}{a})\,\underset{\sim}{b} - \underset{\sim}{a} \times \nabla\underset{\sim}{b} \;, \tag{L4}$$

$$\nabla \cdot (\underset{\sim}{a} \times \underset{\sim}{b}) = \underset{\sim}{b} \cdot (\nabla \times \underset{\sim}{a}) - \underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b}) \;. \tag{L5}$$

Here the divergence of a dyadic product is defined by the components

$$[\nabla \cdot (\underset{\sim}{a}\,\underset{\sim}{b})]_k = \frac{\partial}{\partial x_i}(a_i b_k) \;, \tag{L3a}$$

and the curl of a dyadic product has the components

$$[\nabla \times (\underset{\sim}{a}\,\underset{\sim}{b})]_{ik} = \varepsilon_{imn} \frac{\partial}{\partial x_m}(a_n b_k) \;. \tag{L4a}$$

The next six identities are purely algebraic in origin, viz.

$$\nabla \cdot (\underset{\sim}{b}\,\underset{\sim}{a} - \underset{\sim}{a}\,\underset{\sim}{b}) = \nabla \times (\underset{\sim}{a} \times \underset{\sim}{b}) \;, \tag{L6}$$

$$\nabla\underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b} = \underset{\sim}{a} \times (\nabla \times \underset{\sim}{b}) \;, \tag{L7}$$

$$\nabla\underset{\sim}{b} \times \underset{\sim}{a} + (\underset{\sim}{a} \times \nabla\underset{\sim}{b})^T = [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\,\underset{\sim}{1} - \underset{\sim}{a}\,(\nabla \times \underset{\sim}{b}) \;, \tag{L8}$$

$$\underset{\sim}{a} \times \nabla\underset{\sim}{b} + (\nabla\underset{\sim}{b} \times \underset{\sim}{a})^T = [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\, \underset{\sim}{1} - (\nabla \times \underset{\sim}{b})\, \underset{\sim}{a} \,, \tag{L9}$$

$$\underset{\sim}{a} \times \nabla\underset{\sim}{b} - (\underset{\sim}{a} \times \nabla\underset{\sim}{b})^T = \underset{\sim}{1} \times [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \nabla\underset{\sim}{b} \cdot \underset{\sim}{a}] \,, \tag{L10}$$

$$\nabla\underset{\sim}{b} \times \underset{\sim}{a} - (\nabla\underset{\sim}{b} \times \underset{\sim}{a})^T = [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b}] \times \underset{\sim}{1} \,. \tag{L11}$$

Equations (L6) to (L11) were derived in Sec.2. Of these, eqs.(L8) to (L11) are useful as auxiliary formulas (see Sec.3). When using eqs.(L10), (L11), one should remember that eq.(2a) holds for the vector products of the unit dyadic with a vector.

The conventional formulas for $\nabla (\underset{\sim}{a} \cdot \underset{\sim}{b})$ and $\nabla \times (\underset{\sim}{a} \times \underset{\sim}{b})$ are obtained by combining eqs.(L1) and (L7), viz.

$$\nabla(\underset{\sim}{a} \cdot \underset{\sim}{b}) = \underset{\sim}{a} \cdot \nabla\underset{\sim}{b} + \underset{\sim}{b} \cdot \nabla\underset{\sim}{a} + \underset{\sim}{a} \times (\nabla \times \underset{\sim}{b}) + \underset{\sim}{b} \times (\nabla \times \underset{\sim}{a}) \,, \tag{L12}$$

and eqs.(L3) and (L6), viz.

$$\begin{aligned} \nabla \times (\underset{\sim}{a} \times \underset{\sim}{b}) &= \underset{\sim}{b} \cdot \nabla\underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b} + \underset{\sim}{a}\, (\nabla \cdot \underset{\sim}{b}) - \underset{\sim}{b}\, (\nabla \cdot \underset{\sim}{a}) \\ &\equiv (\underset{\sim}{b} \cdot \nabla + \nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - (\underset{\sim}{a} \cdot \nabla + \nabla \cdot \underset{\sim}{a})\, \underset{\sim}{b} \,. \end{aligned} \tag{L13}$$

Combination of eqs.(L8) and (L10) yields

$$\begin{aligned} \underset{\sim}{a} \times \nabla\underset{\sim}{b} + \nabla\underset{\sim}{b} \times \underset{\sim}{a} &= \underset{\sim}{1} \times [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \nabla\underset{\sim}{b} \cdot \underset{\sim}{a}] \\ &\quad + [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\, \underset{\sim}{1} - \underset{\sim}{a}\, (\nabla \times \underset{\sim}{b}) \,, \end{aligned} \tag{L14}$$

while combining eqs.(L9) and (L11) yields the alternative expression

$$\begin{aligned} \underset{\sim}{a} \times \nabla\underset{\sim}{b} + \nabla\underset{\sim}{b} \times \underset{\sim}{a} &= \underset{\sim}{1} \times [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla\underset{\sim}{b}] \\ &\quad + [\underset{\sim}{a} \cdot (\nabla \times \underset{\sim}{b})]\, \underset{\sim}{1} - (\nabla \times \underset{\sim}{b})\, \underset{\sim}{a} \,, \end{aligned} \tag{L15}$$

where eq.(2a) should again be remembered.

The following three-vector identities, all of algebraic origin, were derived in Sec.3:

$$\underset{\sim}{a} \cdot \nabla\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \nabla\underset{\sim}{b} \cdot \underset{\sim}{a} = (\underset{\sim}{a} \times \underset{\sim}{c}) \cdot (\nabla \times \underset{\sim}{b}) \,, \tag{L16}$$

$$\underset{\sim}{a} \cdot \nabla\underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{c} \times \nabla\underset{\sim}{b} \cdot \underset{\sim}{a} = \underset{\sim}{c} \times [\underset{\sim}{a} \times (\nabla \times \underset{\sim}{b})] \,, \tag{L17}$$

$$\underset{\sim}{a} \cdot \nabla\underset{\sim}{b} \times \underset{\sim}{c} - \underset{\sim}{c} \cdot \nabla\underset{\sim}{b} \times \underset{\sim}{a} = [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{1} - \nabla\underset{\sim}{b}] \cdot (\underset{\sim}{a} \times \underset{\sim}{c}) \,, \tag{L18}$$

$$\underset{\sim}{a} \times \nabla\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{c} \times \nabla\underset{\sim}{b} \cdot \underset{\sim}{a} = (\underset{\sim}{a} \times \underset{\sim}{c}) \cdot [(\nabla \cdot \underset{\sim}{b})\, \underset{\sim}{1} - \nabla\underset{\sim}{b}] \,. \tag{L19}$$

Lortz [23] has observed that eqs.(L17) through (L19) can all be derived by appropriately multiplying the fifth rank tensor

$$L_{jk\ell mn} \equiv \delta_{jk}\,\varepsilon_{\ell mn} - \delta_{j\ell}\,\varepsilon_{mnk} + \delta_{jm}\,\varepsilon_{nk\ell} - \delta_{jn}\,\varepsilon_{k\ell m} \equiv 0$$

with the vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ and the dyadic $\nabla\underset{\sim}{b}$. This null tensor can be derived from the well-known null tensor of fourth rank, viz.

$$N_{jk\ell m} \equiv \varepsilon_{jkp}\,\varepsilon_{\ell mp} - \delta_{j\ell}\,\delta_{km} + \delta_{jm}\,\delta_{k\ell} \equiv 0 \;,$$

by an appropriate multiplication with a further ε-tensor.

The identity

$$2\underset{\sim}{a} \cdot \nabla\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \nabla\,(\underset{\sim}{b} \cdot \underset{\sim}{c}) + \underset{\sim}{c} \cdot \nabla\,(\underset{\sim}{b} \cdot \underset{\sim}{a}) - \underset{\sim}{b} \cdot \nabla\,(\underset{\sim}{a} \cdot \underset{\sim}{c})$$

$$+ (\underset{\sim}{b} \times \underset{\sim}{c}) \cdot (\nabla \times \underset{\sim}{a}) + (\underset{\sim}{b} \times \underset{\sim}{a}) \cdot (\nabla \times \underset{\sim}{c}) + (\underset{\sim}{a} \times \underset{\sim}{c}) \cdot (\nabla \times \underset{\sim}{b}) \qquad \text{(L20)}$$

is also listed; it will be used in Sec.5.

5. PLASMA PHYSICS APPLICATIONS

Some applications to plasma physics (MHD [12] and guiding center drift theory [13]) are presented in order to show the use of several of the above vector identities.

In MHD and guiding center drift theory velocity fields $\underset{\sim}{v}(\underset{\sim}{x}, t)$ are often applied in a special way [18]. When a plasma in a slowly time-varying, spatially inhomogeneous, electromagnetic field is investigated, it may be advantageous to consider velocity fields $\underset{\sim}{v}$ that "conserve magnetic field lines". This means that all points coinciding with a field line at a certain time will also coincide with one field line at later times, while they are moving according to the field $\underset{\sim}{v}$. It is even more useful to consider velocity fields $\underset{\sim}{v}$ that also conserve the magnetic flux through an arbitrary surface (or any surface element) that (locally) moves according to the field $\underset{\sim}{v}$. The conditions that $\underset{\sim}{v}$ must satisfy in order to be line conserving and/or flux conserving are most easily derived by employing some of the new formulas of Sec.4. When an appropriate $\underset{\sim}{v}(\underset{\sim}{x}, t)$ has been determined one may speak of "moving magnetic field lines" or, if $\underset{\sim}{v}$ can be identified with a material velocity of the plasma, of "frozen-in field lines".

We start by deriving the (total) time derivatives of moving line, surface, and volume elements. For a line element $d\underset{\sim}{\ell}$ one has

$$\boxed{d\dot{\underset{\sim}{\ell}} = d\underset{\sim}{\ell} \cdot \nabla\underset{\sim}{v} \ ,} \tag{13}$$

as can be seen by writing $d\underset{\sim}{\ell} = \underset{\sim}{x}_2 - \underset{\sim}{x}_1$ and using a Taylor expansion of $\underset{\sim}{v}$. Computation of $d\dot{\underset{\sim}{f}}$ is more involved. Let us define

$$d\underset{\sim}{f} \equiv d\underset{\sim}{\ell}_1 \times d\underset{\sim}{\ell}_2 \ . \tag{13a}$$

It then follows that

$$\begin{aligned} d\dot{\underset{\sim}{f}} &= d\underset{\sim}{\ell}_1 \cdot \nabla\underset{\sim}{v} \times d\underset{\sim}{\ell}_2 + d\underset{\sim}{\ell}_1 \times (d\underset{\sim}{\ell}_2 \cdot \nabla\underset{\sim}{v}) \\ &\equiv d\underset{\sim}{\ell}_1 \cdot \nabla\underset{\sim}{v} \times d\underset{\sim}{\ell}_2 - d\underset{\sim}{\ell}_2 \cdot \nabla\underset{\sim}{v} \times d\underset{\sim}{\ell}_1 \end{aligned} \tag{13b}$$

However, $d\dot{\underset{\sim}{f}}$ is required in the form

$$d\dot{\underset{\sim}{f}} = \underset{\sim}{T} \cdot d\underset{\sim}{f} \ , \tag{13c}$$

with $\underset{\sim}{T}$ an appropriate dyadic. This transformation can be performed by means of eq.(L18) yielding

$$\boxed{d\dot{\underset{\sim}{f}} = [(\nabla \cdot \underset{\sim}{v})\, \underset{\sim}{1} - \nabla\underset{\sim}{v}] \cdot d\underset{\sim}{f} \equiv (\nabla \cdot \underset{\sim}{v})\, d\underset{\sim}{f} - \nabla\underset{\sim}{v} \cdot d\underset{\sim}{f}\ .} \tag{13d}$$

Finally, by defining

$$d\tau \equiv d\underset{\sim}{f} \cdot d\underset{\sim}{\ell}_3 \tag{13e}$$

one easily derives the well-known result

$$\boxed{d\dot{\tau} = (\nabla \cdot \underset{\sim}{v})\, d\tau\ .} \tag{13f}$$

Let us now derive the condition to be satisfied by $\underset{\sim}{v}\,(\underset{\sim}{x}, t)$ in order to conserve the magnetic flux through an arbitrary surface F that moves with $\underset{\sim}{v}$. The flux is defined as

$$\psi \equiv \int_F \underset{\sim}{B} \cdot d\underset{\sim}{f}\ , \tag{14}$$

with $\underset{\sim}{B}$ the magnetic field, $\nabla \cdot \underset{\sim}{B} \equiv 0$. It follows that

$$\frac{d\psi}{dt} = \int_F \left(\frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{v} \cdot \nabla\underset{\sim}{B}\right) \cdot d\underset{\sim}{f} + \int_F \underset{\sim}{B} \cdot d\dot{\underset{\sim}{f}}\ . \tag{14a}$$

On inserting $d\dot{\underset{\sim}{f}}$ from eq.(13d) this transforms to

$$\frac{d\psi}{dt} = \int_F \left[\frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{v} \cdot \nabla\underset{\sim}{B} + \underset{\sim}{B}\,(\nabla \cdot \underset{\sim}{v}) - \underset{\sim}{B} \cdot \nabla\underset{\sim}{v}\right] \cdot d\underset{\sim}{f}\ . \tag{14b}$$

On applying eq.(L13) to transform the integrand one obtains

$$\frac{d\psi}{dt} = \int_F \left[\frac{\partial \underset{\sim}{B}}{\partial t} - \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B})\right] \cdot d\underset{\sim}{f}\ . \tag{14c}$$

The velocity field conserves magnetic flux through an arbitrary F [i.e. $d\psi/dt \equiv 0$] if the integrand vanishes identically, viz.

$$\nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \equiv \frac{\partial \underset{\sim}{B}}{\partial t}\ , \tag{14d}$$

which is a condition for $\underset{\sim}{v}\,(\underset{\sim}{x}, t)$.

Comparing this with the induction law

$$\frac{\partial \underset{\sim}{B}}{\partial t} = - c \nabla \times \underset{\sim}{E} \tag{14e}$$

yields an alternative condition for flux conserving velocity fields, viz.

$$\nabla \times [\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B}] = 0 . \tag{14f}$$

For given fields $\underset{\sim}{B}$ ($\underset{\sim}{x}$, t), $\underset{\sim}{E}$ ($\underset{\sim}{x}$, t) this may be solved [18] to yield a whole class of permissible velocity fields. It should be noted, however, that the special ansatz

$$\underset{\sim}{v} = \underset{\sim}{v}_E \equiv \frac{c}{B^2} \underset{\sim}{E} \times \underset{\sim}{B} , \tag{14g}$$

where $\underset{\sim}{v}_E$ is the well-known $\underset{\sim}{E} \times \underset{\sim}{B}$ drift velocity of charged plasma particles, is only compatible with eqs.(14d) and (14f) if

$$(\nabla \times \underset{\sim}{E}_{\parallel}) \equiv 0 , \tag{14h}$$

which is a severe restriction of admissible electromagnetic fields. Here $\underset{\sim}{E}_{\parallel}$ is the vector component of $\underset{\sim}{E}$ parallel to $\underset{\sim}{B}$, i.e. $\underset{\sim}{E}_{\parallel} = (\underset{\sim}{E} \cdot \hat{\underset{\sim}{b}}) \hat{\underset{\sim}{b}}$, with $\hat{\underset{\sim}{b}} = \underset{\sim}{B}/B$.

Next, the condition on $\underset{\sim}{v}$ ($\underset{\sim}{x}$, t) for <u>conservation of magnetic field lines</u> will be briefly derived. Let us consider a field of line elements $d\underset{\sim}{\ell}$ parallel to $\underset{\sim}{B}$ at a certain time t_o, i.e.

$$\underset{\sim}{B} \times d\underset{\sim}{\ell} \equiv 0 \tag{15}$$

at t_o for all $\underset{\sim}{x}$. In order that eq.(15) hold at the later time t_o + dt, when the line elements have been moved according to $\underset{\sim}{v}$ [eq.(13)], the requirement on $\underset{\sim}{v}$ ($\underset{\sim}{x}$, t) is given by

$$\frac{d}{dt} (\underset{\sim}{B} \times d\underset{\sim}{\ell}) = 0 \tag{15a}$$

or

$$\left(\frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{B}\right) \times d\underset{\sim}{\ell} + \underset{\sim}{B} \times (d\underset{\sim}{\ell} \cdot \nabla \underset{\sim}{v}) = 0 . \tag{15b}$$

By using eqs.(L18) and (15) this assumes the form

$$\left(\frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{B} - \underset{\sim}{B} \cdot \nabla \underset{\sim}{v}\right) \times d\underset{\sim}{\ell} = 0 \ , \tag{15c}$$

which can be transformed by means of eqs.(L13), (15) and $\nabla \cdot \underset{\sim}{B} = 0$, yielding

$$\left[\frac{\partial \underset{\sim}{B}}{\partial t} - \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B})\right] \times d\underset{\sim}{\ell} = 0 \ . \tag{15d}$$

This condition on $\underset{\sim}{v}$ says that the vector component perpendicular to $\underset{\sim}{B}$ of the square bracket must vanish. Combination with eq.(14e) yields

$$[\nabla \times (\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B})] \times d\underset{\sim}{\ell} = 0 \ . \tag{15e}$$

Comparison with eqs.(14d) and (14f) shows that the conditions for line conservation are less severe than those for flux conservation. The result agrees with that obtained by Newcomb [18]. We have directly proved only conservation of parallelism of line elements to field lines. However, usually the field of line elements will uniquely determine the magnetic field lines. Then global conservation of field lines is also secured by eqs.(15d), (15e). This is seen by explicitly doing a global analysis where "magnetic coordinates" [19] α and β are employed to identify magnetic field lines. Equations (15d), (15e) are so recovered. Note that the logic of this derivation is <u>clarified</u> by <u>avoiding</u> the usual supplementary condition $\nabla\alpha \times \nabla\beta = \underset{\sim}{B}$; i.e. α and β ought <u>not</u> to be specialized to become "flux coordinates".

The last example is taken from guiding center drift theory [13]. There vector expressions are derived for the magnetic moment of a particle gyrating in an inhomogeneous magnetic field [20]. These expressions contain several terms, of which we consider only the following one:

$$\delta\mu_1 \equiv - \frac{m^2 c v_{\shortparallel}}{2eB^2} (\hat{\underset{\sim}{b}} \times \underset{\sim}{v}_{\perp}) \cdot \nabla\hat{\underset{\sim}{b}} \cdot \underset{\sim}{v}_{\perp} \ . \tag{16}$$

Here $\hat{\underset{\sim}{b}}$ is the unit vector in the direction of the $\underset{\sim}{B}$ field, $\underset{\sim}{v}\,(t)$ is the particle velocity, $\underset{\sim}{v}_{\shortparallel}$ and $\underset{\sim}{v}_{\perp}$ are the components of $\underset{\sim}{v}$ parallel and perpendicular to the direction of $\underset{\sim}{B}$, taken at the position of the "guiding center" [13] of the gyrating particle, viz.

$$\underset{\sim}{v}_{\shortparallel} \equiv \underset{\sim}{v} \cdot \hat{\underset{\sim}{b}} \ , \tag{16a}$$

$$\underset{\sim}{v}_{\perp} \equiv \underset{\sim}{v} - (\underset{\sim}{v} \cdot \hat{\underset{\sim}{b}})\, \hat{\underset{\sim}{b}} \equiv - \hat{\underset{\sim}{b}} \times (\hat{\underset{\sim}{b}} \times \underset{\sim}{v}) \ . \tag{16b}$$

One is interested in averaging $\delta\mu_1$ over the azimuthal angle of $\underset{\sim}{v}_\perp$. This can be done either by expressing the r.h.s. of eq.(16) by Cartesian components or by using eq.(L20), which simplifies considerably here because $\hat{\underset{\sim}{b}}$, $\underset{\sim}{v}_\perp$, and $\underset{\sim}{w} \equiv \hat{\underset{\sim}{b}} \times \underset{\sim}{v}_\perp$ are mutually perpendicular. Hence

$$2\, \underset{\sim}{w} \cdot \nabla\hat{\underset{\sim}{b}} \cdot \underset{\sim}{v}_\perp = \underset{\sim}{w} \cdot (\nabla \times \underset{\sim}{w}) - \underset{\sim}{v}_\perp \cdot (\nabla \times \underset{\sim}{v}_\perp) - v_\perp^2\, \hat{\underset{\sim}{b}} \cdot (\nabla \times \hat{\underset{\sim}{b}}) \;. \tag{16c}$$

Because the azimuthal averages of $\underset{\sim}{w} \cdot (\nabla \times \underset{\sim}{w})$ and $\underset{\sim}{v}_\perp \cdot (\nabla \times \underset{\sim}{v}_\perp)$ are equal, the final result reads

$$2 \langle \underset{\sim}{w} \cdot \nabla\hat{\underset{\sim}{b}} \cdot \underset{\sim}{v}_\perp \rangle = - \langle v_\perp^2 \rangle\, \hat{\underset{\sim}{b}} \cdot (\nabla \times \hat{\underset{\sim}{b}}) \;, \tag{16d}$$

where the pointed brackets indicate the azimuthal average. This completes our short list of simple examples from plasma physics.

6. CONCLUDING REMARKS

When tables of vector formulas are compared with tables of integrals, derivatives, or series, it is surprising to see how comparatively fragmentary the vector tables are. It is hoped that the present paper may contribute somewhat towards a more complete documentation of elementary vector analysis.

Problems of notation may have played their role in impeding the development of a more complete set of vector-analytical formulas. We think it useful to discuss briefly a few pertinent points. Obviously, one or two principles are needed to arrive at a rational and consistent notation on which most authors could agree. Firstly, a "principle of permanence" ought to be observed in questions of notation. As an example, some authors [7] have the ∇ operator operate on all field quantities to its right without indicating this by brackets. Such a notation should be avoided, however, because it contradicts the usual rules of differential calculus. The convention of this paper is recommended, namely that the ∇ operator should only operate on the field quantity immediately to its right, unless otherwise indicated by brackets. Brackets are properly used for indicating the sequence in which operations are to be performed, not to denote operations. Other uses only occur in special instances (where confusion cannot arise), e.g. to denote averages, expectation values, matrix elements, matrices, binomial coefficients, Christoffel symbols, etc. The use of brackets (rather than dots and crosses) to denote scalar and vector products [22] leads to unnecessary difficulties in vector analysis and should therefore be avoided. If possible, a notation should at the same time be clear, economical, flexibel, descriptive, and mnemonic. For instance, it seems better to write $\underset{\sim}{a} \cdot \nabla \underset{\sim}{b}$ than $(\underset{\sim}{a}\nabla)\ \underset{\sim}{b}$ or $(\underset{\sim}{a} \cdot \nabla)\ \underset{\sim}{b}$. Inspection shows that the notation $\underset{\sim}{a} \cdot \nabla \underset{\sim}{b}$ has a unique meaning, uses a minimum of symbols, is analogous to its component representation, and is easier to use as a part of more involved expressions.

Some authors [21] have proposed abandoning symbolic vector notation altogether and using only components. However, the obvious advantages of symbolic notation, viz. economy, descriptiveness, heuristic value, and mnemonic effect, may bestow preference on symbolic notation whenever it is applicable.

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