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Theoretical Scaling Law for Ohmically Heated Tokamaks

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Abstract

The electrostatic drift instability arising from the reduction of shear damping, due to toroidal effects, is assumed to be the basic source of the anomalous electron transport in tokamaks. The Maxwellian population of electrons constitutes a medium whose adiabatic nonlinear reaction to the instability (described in terms of an effective dielectric constant of the medium) determines the stationary electrostatic fluctuation level in marginally unstable situations. The existence of a random electrostatic potential implies a fluctuating current of the Maxwellian electrons which creates a random magnetic field and a stocasticization of the magnetic configuration. The application of recent results allows the calculation of the related radial electron transport. It is found that the confinement time under stationary ohmic conditions scales as $\sim T_i^{-1/2}$ and is proportional roughly to the cube of the geometric dimensions. Moreover, it is deduced that the loop voltage is approximately the same for all tokamaks, irrespective of temperature and density and to a large extent, also of geometrical conditions. These results are characteristic of the ohmic stationary regime and can hardly be extrapolated to other heating regimes.

1. Introduction

Nonlinear Dielectric Constant in a Reactive Medium

In our simplified model the plasma is considered as a medium of individual fluctuating particles on which a smeared out collective (Vlasov) electrostatic configuration is superimposed. Then any (in general non-collective) charge fluctuation around an electrically neutral equilibrium will take the general form

$$\sigma_1 = \sigma(\omega, \varphi) + \sigma_{eff} \tag{1.1}$$

Here σ represents the collective Vlasov configuration, associated with a given real frequency ω; σ_{eff} describes the real additional charge (in addition to σ and representing then effectively the departure from the Vlasov configuration described by σ) excited by the individual particles of the medium, fluctuating in the presence of the collective field. Thus one can formally introduce a dielectric constant ε which connects the total charge σ_1 with the additional charge σ_{eff} through the relation $\sigma_1 = \sigma_{eff}/\varepsilon$. Hence ε describes the reaction of the medium of free particles to the collective field and must be such as to satisfy the relation

$$\sigma_1 - \sigma(\omega, \varphi) \equiv \varepsilon\, \sigma_1 \tag{1.2}$$

After introducing the Poisson equation $\sigma_1 = -\Delta\varphi$ (σ_1 is the total charge density multiplied by 4π) the equation above can be considered as a definition of the dielectric constant ε in terms of the potential φ . We shall only consider situations with one undamped collective mode with a single frequency, so that dispersive or dissipative effects are ignored. More precisely we assume that a reference frame exists in which $\sigma(\omega, \varphi)$ is static, thus representing a true collective equilibrium. This is an essential requirement for applying the methods of statistical mechanics in order to calculate the fluctuations σ_{eff} around the static equilibrium σ by means of a suitable canonical average (Minardi, 1979; see also Section 2 of this paper). We also assume that the collective part σ can be expressed up to second order in φ in the following form

$$\sigma(\omega,\varphi) = -\frac{1}{\lambda_1^2}\varphi - \frac{1}{2\lambda_2^2}\varphi^2 \tag{1.3}$$

where the reference potential must be so chosen as to satisfy the condition $\langle\sigma\rangle = 0$ ($\langle\ \rangle$ denotes the average over a wavelength), namely such that

$$\langle\varphi\rangle = -\left(\frac{\lambda_1}{\lambda_2}\right)^2\langle\varphi^2\rangle \tag{1.4}$$

Eq. (1.2) then becomes

$$-\Delta\varphi + \frac{1}{\lambda_1^2}\varphi + \frac{1}{2\lambda_2^2}\varphi^2 = -\varepsilon\,\Delta\varphi$$

or after Fourier transform

$$\left(k^2+\frac{1}{\lambda_1^2}\right)\varphi_{\vec{k}} + \frac{1}{2\lambda_2^2}\sum_{\vec{k}'}\varphi_{\vec{k}-\vec{k}'}\varphi_{k'} = \sum_{\vec{k}'}\varepsilon_{\vec{k}-\vec{k}'}\,k'^2\varphi_{\vec{k}'} \tag{1.5}$$

where ε is represented in Fourier space by the matrix $\varepsilon_{\vec{k},\vec{k}'} = \varepsilon_1\delta_{\vec{k}-\vec{k}'} + \varepsilon_{2,\vec{k}-\vec{k}'}$ (ε_1 is the part resulting from the linearization in φ). So one obtains from (1.5)

$$\varepsilon_1 = 1 + \frac{1}{\lambda_1^2 k^2} \tag{1.6}$$

$$\frac{1}{2\lambda_2^2}\frac{1}{V^{1/2}}\sum_{\vec{k}'}\varphi_{\vec{k}'}\varphi_{\vec{k}-\vec{k}'} = \sum_{k'}\varepsilon_{2,\vec{k}-\vec{k}'}\,k'^2\varphi_{\vec{k}'}$$

In the case of a Maxwellian equilibrium ($\sigma \sim n \exp e\varphi/T_e$), λ_1 is equal to the Debye length λ_D and ε_1 , which is positive, describes the reaction of the thermal background to the presence of the mode k. But situations frequently exist in which λ_1^2 is negative and correspondingly a real value ω_m, k_m of ω, k can exist such that ε_1 vanishes (in general λ_1 will depend on ω and k); moreover, in the neighbourhood of k_m the linear part of the dielectric constant can become negative

$$\left(\frac{\partial \varepsilon_1}{\partial k}\right)_{\omega_m, k_m} \Delta k < 0 \tag{1.7}$$

The corresponding modes $k = k_m + \Delta k$ are inherently unstable linearly (Minardi 1974, 1979), the mode $k_m = |\lambda_1|^{-1}$ being the marginally unstable mode. However, in a situation in which a population of particles constituting a thermal background exists, one may have a reaction of this population to the collective growth, which modifies nonlinearly the dielectric constant in such a way as to damp the instability by counterbalancing, in the second order of φ, the negative value of the linear part ε_1. The system may then stabilize in a new inhomogeneous equilibrium with a finite amplitude φ, nonlinearly neighbouring to the originally unstable equilibrium and which corresponds approximately again to a situation with a vanishing dielectric constant, as is described by (1.5) with $\varepsilon = 0$:

$$\Delta \varphi = -\frac{1}{\lambda_1^2}\varphi - \frac{1}{2\lambda_2^2}\varphi^2 \tag{1.8}$$

An example of this situation occurs in the case of the marginal modes of the collisionless drift instability in the slab model and in the absence of shear (Krall and Rosenbluth, 1965). In this case the electrons are essentially Maxwellian, responding adiabatically to the collective field, and provide the required thermal background. The linear dielectric constant becomes negative in the unstable neighbourhood of the marginal point, defined by $k = k_m + \Delta k$ with $\Delta k > 0$. The nonlinear response is positive with the coefficient given by the expression (Minardi, 1979):

$$\lambda_2^{-2} \equiv -\left(\frac{d^2\sigma}{d\varphi^2}\right)_m = \frac{4\pi e^3 n}{T_e^2} \tag{1.9}$$

Here λ_2^{-2} is determined only by the Maxwellian electrons (with charge-e), the ion response being negligible at second order with respect to the electron response, provided that the phase velocity of the mode along the magnetic field is much larger than the thermal velocity of the ions and much lower than that of the electrons.

The reactive process considered above, which leads to the saturation of the instability, can be naturally interpreted in terms of the statistical thermodynamic description of Vlasov equilibria in interaction with a thermal background which was developed recently (Minardi, 1979; for a more complete description of the statistical formalism see Minardi (1981), Section 2). Some aspects of the statistical procedure will be indicated in the next Section. The instability corresponds to an equilibrium with minimum entropy while the saturation is associated with a neighbouring configuration with maximum entropy. This configuration agrees qualitatively, near the marginal point, with that described by the solution of the nonlinear equation (1.8). The statistical method predicts the following mean square average for φ (see Section 2)

$$\overline{\left(\frac{e\varphi}{T_e}\right)^2} = \frac{T_e}{2\pi e^2 n} k^2 \frac{\Delta k}{k_m} \tag{1.10}$$

This equality shows that $\overline{\varphi^2}$ decreases with increasing density, according to the intuition that the reaction of the electron medium should be more effective at higher electron density. The same scaling is obtained from the relation between wave-number and amplitude resulting from the solution of the nonlinear equation (1.8) in the neighbourhood of the marginal point (see Minardi, 1979, Appendix), namely for $\Delta k/k < 2k^2\lambda_0^2$. As we know, this solution describes a pure Vlasov equilibrium, with vanishing dielectric constant. However, the stabilizing reactive process of the medium, as described by the thermodynamic formalism, is not necessarily related to the rigorous vanishing of ε. Indeed, as can be seen from the equation (1.5) for ε, one obtains in general a non-vanishing dielectric constant. Accordingly, outside the immediate neighbourhood of the marginal point (for $\Delta k/k > 2k^2\lambda_0^2$), the behaviour (1.10) of $\overline{\varphi^2}$ as a function of k, predicted by the thermodynamic method, differs from that of the nonlinear solution (1.5) with $\varepsilon = 0$. Then the non collective contribution (described by the term with $\varepsilon \neq 0$) of the reactive medium to the saturation level cannot be entirely neglected outside the marginal point and although the thermodynamic method cannot be applied too far from this point (the reactive process can be overwhelmed by a too strong collective instability) it can nevertheless include situations more comprehensive than purely Vlasov equilibria. In fact, outside the marginal equilibrium, one can have a collective configuration which is not an equilibrium in the sense of Vlasov, because it interacts with the free particles of the thermal background, but which is still in thermodynamic equilibrium with the fluctuating background.

Although in the slab model the instability above is stabilized by shear, the result (1.10) remains physically significant in the light of recent results (Connor et al., 1979; Chen et al., 1980; Johner et al., 1980; Hesketh, 1980), which show that the toroidal effects in tokamaks can create localized modes not subject to shear damping and which are then unstable, basically reinstating in this respect the shearless situation of the slab model. In the present paper the equation (1.10) is rederived on the basis of our statistical procedure, in the case of the toroidal drift instability mentioned above and applied to calculate the fluctuation level near the marginal point. The random character of the electrostatic potential φ which is implicit in our statistical treatment, is automatically reflected in the random character of that part of the electron current which depends on the fluctuating φ . Thus this current creates a random magnetic field and, in view of the high m- number of the instability, this results in a stochasticization of the magnetic configuration. One is then in position to apply the recently developed theory of the electron transport in a braiding magnetic field (Rosenbluth et al., 1966; Zaslavsky et al., 1972; Rechester et al., 1978; Krommes et al., 1978).

The resulting electron thermal conductivity in the collisonless diffusion limit depends in general on the detailed structure of the instability near the marginal point. However, in the case of ohmically heated tokamaks in a steady state, one very simple and general property emerges: the loop voltage in the ohmic steady state of all tokamaks is independent of the temperature and of the magnetic field and only depends, although not strongly, on Z and on the geometry of the machine. The calculated loop voltage is $\sim 0.5\, Z^{2/3}$ Volt. The magnitude and the scaling of the electron thermal conductivity with respect to density, temperature and geometry agree reasonably with the observations.

2. Stabilization Associated with Maximum Entropy

We introduce here, for the reader's convenience, some basic relations derived from our statistical treatment of a collective plasma equilibrium in interaction with a thermal background. Let the configuration be described by an electrostatic potential φ and a so called information variable $\tilde{\sigma}$, whose definition implies the physical characterization of the system under study. In our case $\tilde{\sigma}$ is related to the charge density and will be defined later.

In the statistical formalism a pure Vlasov equilibrium corresponds to absence of correlations between the fluctuations of the part of $\tilde{\sigma}$ pertaining to the thermal background and the collective potential of this equilibrium. This implies that the electrostatic interaction energy between the collective configuration and the background is vanishing. On the other hand one can define in the formalism a parameter $\tilde{\tau}$ which plays the same role of a temperature and which is related to the fluctuations of the background by the relation

$$\tilde{\tau} = \frac{\lambda_1^2\, \overline{\Delta\sigma_{in}^2}\, \Delta V}{4\pi} \tag{2.1}$$

where $\overline{\Delta\sigma_{in}^2}$ is the mean square average of the fluctuations of $\tilde{\sigma}$ in a cell with volume ΔV, excited by the individual particles of the medium. Considering variations of the collective quantities such that $\tilde{\tau}$ is kept fixed ("isothermal" variations) one finds that the interaction energy is expressed by the relation

$$\Phi_{int} = -\frac{1}{8\pi}\int_V \left(\tilde{\sigma} + \lambda_1^{-2}\varphi\right)\left(\varphi - \varphi_0\right) dV \tag{2.2}$$

where the reference potential φ_0 is taken equal to the space average $\langle\varphi\rangle$ satisfying (1.4). In the linear approximation, identifying $\tilde{\sigma}$ with the total charge density $\sigma_1 = -\Delta\varphi$ and remembering (1.6), Φ_{int} is equal to the opposite in sign of the total electrostatic energy of the plasma (including the medium):

$$\Phi_{int} = -\frac{1}{8\pi}\int_V \varepsilon_1 \sigma_1 (\varphi - \varphi_0)\, dV \tag{2.3}$$

Φ_{int} vanishes at the marginal equilibrium $\varepsilon_1 = 0$. Thus outside the marginal collective equilibrium, Φ_{int} describes an energy transfer from the background to the plasma related to the "isothermal" fluctuations around a pure Vlasov (collective) equilibrium with zero interaction energy.

In general, taking into account the nonlinear terms and remembering (1.2) and (1.3) the total electrostatic energy of the plasma is given by the relation

$$\frac{1}{8\pi}\int_V \varepsilon \sigma_1 (\varphi - \varphi_0) dV = \frac{1}{8\pi}\int_V (\sigma_1 - \sigma)(\varphi - \varphi_0) dV = \frac{1}{8\pi}\int_V \left(\sigma_1 + \lambda_1^{-2}\varphi + \lambda_2^{-2}\frac{\varphi^2}{2}\right)(\varphi - \varphi_0) dV \tag{2.4}$$

According to the physical interpretation of the interaction energy (2.2) given above in the linear case, we now identify, also in the nonlinear case, the total electrostatic energy (2.4) with $-\Phi_{int}$. Comparison between (2.2) and (2.4) shows that this is possible provided that the definition of the information variable $\tilde{\sigma}$ is extended to the nonlinear case as follows:

$$\tilde{\sigma} \equiv \sigma_1 + \lambda_2^{-2}\frac{\varphi^2}{2} \tag{2.5}$$

The entropy of the system, considered as a statistical assembly of cells with volume ΔV, characterized by the information variable $\tilde{\sigma}$, is expressed by the equality (Minardi, 1979, 1981):

$$S(\overline{\Delta\sigma_{in}^2}, \gamma) = -\frac{1}{2\Delta V\, \overline{\Delta\sigma_{in}^2}}\int (\tilde{\sigma} - \tilde{\sigma}_0)^2 dV - \frac{1}{8\pi\tilde{\gamma}}\int (\tilde{\sigma} - \tilde{\sigma}_0)(\varphi - \varphi_0) dV \tag{2.6}$$

where $\tilde{\sigma}_0 = \langle\tilde{\sigma}\rangle$ is the space average of $\tilde{\sigma}$. Using the equalities (2.1) and (2.5), S takes the form

$$S(\overline{\Delta\sigma_{in}^2}) = -\frac{1}{2\Delta V\, \overline{\Delta\sigma_{in}^2}}\int \left(\sigma_1^2 + \lambda_1^{-2}\sigma_1\varphi + \frac{1}{4}\lambda_2^{-2}(\varphi^2 - \langle\varphi^2\rangle)^2\right) dV \tag{2.7}$$

where $\sigma_1 = -\Delta\varphi$. For illustration sake let us assume that the system is perturbed by a single mode, say $\varphi = \varphi_0 + \varphi_1 \cos \vec{k}\cdot\vec{x} + \varphi_2 \sin \vec{k}\cdot\vec{x}$ (φ_0 is $O(\varphi^2)$ and can henceforth be neglected). After substitution into (2.7) one finds, in the case $\varepsilon_1 < 0$, that S has two extremum, a minimum for $\varphi_1 = \varphi_2 = 0$, which corresponds to the unstable homogeneous equilibrium and a maximum for $\varphi_1^2 + \varphi_2^2 = 4\lambda_2^4 k^4 |\varepsilon_1|$, which corresponds to the saturated amplitude of the originally unstable mode. Summing up with respect to the two signs of $\vec{k}$ we obtain the saturation level already given in (1.10) in terms of the Fourier transform $\varphi_{\vec{k}}$:

$$\frac{1}{V}\frac{e^2|\varphi_k|^2}{T_e^2} = \frac{1}{2}\frac{e^2}{T_e^2}\left(\varphi_1^2+\varphi_2^2\right) = \frac{2e^2}{T_e^2}\lambda_2^4 k^4 |\varepsilon_1| \approx \frac{T_e}{2\pi e^2 n} k^2 \frac{\Delta k}{k_m} \tag{2.8}$$

In the case $\varepsilon_1 > 0$ at the contrary, S has only one maximum for $\varphi_k = 0$ and the system is thermodynamically stable.

The probability distribution of the fluctuation amplitude φ_k is given by the basic relation $P(\varphi_k) \sim \exp S(\varphi_k)$ of statistical mechanics. After averaging (1.5) over the distribution P and taking into account the quasi-neutrality condition (1.4) (which fixes $\varphi_{\vec{k}}$ for $k = 0$) one finds that

$$\sum_{\vec{k}'} \overline{\varepsilon_{\vec{k}-\vec{k}'}\varphi_{\vec{k}'}}\, k'^2 = \overline{\varepsilon\sigma} = 0 \tag{2.9}$$

for every k. This result shows that, consistently with our procedure above, the system fluctuates around the pure collective equilibrium associated with a vanishing dielectric constant.

3. Linear Dielectric Constant of Toroidal Drift Modes

We start from the perturbed charge density related to long wavelength ($k a_i \ll 1$) drift waves in a large aspect ratio torus (Hastie et al., 1979)

$$\sigma = \frac{4\pi e^2 n}{T_e}\left(\tau + \frac{\omega_*}{\omega}\right)\left\{a_i^2\frac{\partial^2}{\partial x^2} - k^2 a_i^2 - \left(\frac{\omega_*\varepsilon_c}{\omega k a_i}\right)^2\left(\frac{\partial}{\partial\theta} + iksx\right)^2 + \frac{\omega_*-\omega}{\omega_*+\tau\omega} - 2\varepsilon_n\frac{\omega_*}{\omega}\left(\cos\theta + i\sin\theta\frac{\partial}{k\partial x}\right)\right\}\varphi \tag{3.1}$$

where $a_i^2 = T_i M_i c^2 / q_i^2 B^2$, $k = Nq/r$ (N, toroidal wave number) $q = rB/RB_\theta$, $\varepsilon_n = q\,\varepsilon_c = r_n / R\tau$, $\tau = T_e/T_i$, $r_n^{-1} = n^{-1}dn/dr$, $s = (r/q)dq/dr$, $\omega_* = kT_e/eBr_n$, $x = r - r_0$ with $Nq(r_0) = M$ (integer) and θ is the poloidal angle measured from the outside of the torus.

Introducing the transformation (Connor et al., 1978)

$$\varphi(\theta, x) = \sum_m \exp(-im\theta) \int_{-\infty}^{+\infty} \exp\left[im\eta - iksx(\eta - \eta_0)\right] F(x, \eta)\, d\eta \tag{3.2}$$

where η_0 is an arbitrary phase, one obtains the following expression of σ in the

F representation:

$$\sigma_F = \frac{4\pi e^2 n}{T_e}\left(\tau + \frac{\omega_*}{\omega}\right)\left\{ -k^2 a_i^2 \left(s\eta + \frac{i}{k}\frac{\partial}{\partial x}\right)^2 - \left(\frac{\omega_* \varepsilon_c}{\omega k a_i}\right)^2 \frac{\partial^2}{\partial \eta^2} + \frac{\omega_* - \omega}{\omega_* + \tau\omega} - k^2 a_i^2 - \right.$$

$$\left. - 2\varepsilon_n \frac{\omega_*}{\omega}\left(\cos(\eta + \eta_0) + s\eta \sin(\eta + \eta_0) + \frac{i}{k}\sin(\eta + \eta_0)\frac{\partial}{\partial x}\right)\right\} F \tag{3.3}$$

Limiting our considerations to zero order in the parameter $h \equiv k^{-1}\partial/\partial x$

(where $\partial/\partial x$ acts on F) one obtains (Hastie et al., 1979)

$$\sigma_F = \Omega(\omega, k) F$$

where

$$\Omega(\omega, k) = \frac{4\pi e^2 n}{T_e}\left(\tau + \frac{\omega_*}{\omega}\right)\left(\frac{\omega_* \varepsilon_c}{\omega k a_i}\right)^2 \left\{ -\frac{d^2}{d\eta^2} - \lambda(\omega, k) + U(\eta, \omega, k)\right\} \tag{3.4}$$

$$\lambda(\omega, k) = \left(\frac{\omega k a_i}{\omega_* \varepsilon_c}\right)^2 \left(k^2 a_i^2 - \frac{\omega_* - \omega}{\omega_* + \tau\omega}\right)$$

$$U(\eta, \omega, k) = -\beta^2\left[\eta^2 + 2\alpha\left(\cos(\eta + \eta_0) + s\eta \sin(\eta + \eta_0)\right)\right] \tag{3.5}$$

$$\beta \equiv \frac{\omega k^2 a_i^2 s}{\omega_* \varepsilon_c} \qquad \alpha \equiv \frac{\omega_* \varepsilon_n}{k^2 a_i^2 s^2 \omega}$$

We are interested to express σ_F outside the marginal collective equilibrium, remaining in its neighbourhood, but neglecting all damping effects. In this case F is an eigenfunction with eigenvalue λ_m of the equation

$$\frac{d^2F}{d\eta^2} + \left[\lambda_m - U(\eta, k_m, \omega_m)\right] F = 0 \tag{3.6}$$

where the Landau damping term, which should appear in the r.h.s. (Hesketh, 1980) is omitted; ω_m, k_m are real marginal values of ω, k determined by $\lambda_m = \lambda(\omega_m, k_m)$. Then one has from (3.4) $\sigma_F(\omega_m, k_m) = \Omega(\omega_m, k_m)F = 0$. Now let $k = k_m + \Delta k$ correspond to an arbitrary non collective fluctuation outside the marginal Vlasov equilibrium while ω_m and F, which specify the collective equilibrium, are fixed (a change of F outside the marginal equilibrium described by (3.6) could only be due to damping). The increment Δk is associated with a change $\Delta(\Omega F)$ of σ given by the expression

$$\begin{aligned}
\sigma(\omega_m, k = k_m + \Delta k) &= \Delta(\Omega F) = (\Delta\Omega)F = \\
&= \frac{4\pi e^2 n}{T_e}\left(\tau + \frac{\omega_*}{\omega_m}\right)\left(\frac{\omega_* \varepsilon_c}{\omega_m k a_i}\right)^2 \left\{\lambda_m - \lambda(\omega_m, k) + U(\eta, \omega_m, k) - U(\eta, \omega_m, k_m)\right\} F = \\
&= \frac{4\pi e^2 n}{T_e}\left(\tau + \frac{\omega_*}{\omega_m}\right)\left\{\frac{\omega_*(k) - \omega_m}{\omega_*(k) + \tau\omega_m} - 2\varepsilon_n \frac{\omega_*(k) - \omega_*(k_m)}{\omega_m}\left[\cos(\eta + \eta_0) + s\eta \sin(\eta + \eta_0)\right] + \right. \\
&\qquad \left. + (k_m^2 - k^2)(1 + s^2\eta^2)a_i^2\right\} F
\end{aligned} \tag{3.7}$$

Transforming back into the φ space, one has the following expression of the charge density fluctuation (non collective) in the neighbourhood of the marginal collective equilibrium with the marginal frequency ω_m

$$\sigma(\omega_m, k) = \frac{4\pi e^2 n}{T_e}\left(\tau + \frac{\omega_*(k)}{\omega_m}\right)\left\{\frac{\omega_*(k)-\omega_*(k_m)}{\omega_*(k)+\tau\omega_m} - 2\varepsilon_n \frac{\omega_*(k)-\omega_*(k_m)}{\omega_m}\left(\cos\theta + s\sin\theta\right)\frac{i}{k}\frac{\partial}{\partial x}\right) + \left(k_m^2 - k^2\right)\left(1 - \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right)a_i^2\right\}\varphi \tag{3.8}$$

The first term in the parenthesis is the same (noting that $\omega_m \simeq \omega_*(k_m)$) as the leading term in the slab model (Minardi, 1979). The second term represents the toroidal correction and the last term is a contribution of second order in $k a_i$.

The linear dielectric constant is now obtained from the definition (1.2) which can be written as follows

$$\varepsilon_1 = 1 - \frac{\sigma(\omega_m, k)}{\sigma_1} = 1 + \frac{\sigma(\omega_m, k)}{\frac{\partial^2\varphi}{\partial x^2} - k^2\varphi} \simeq -\frac{\sigma(\omega_m, k)}{k^2(s^2+1)\varphi} \tag{3.9}$$

where in the last transition we have used $\partial\varphi/\partial x \sim -iks\varphi$ and applied the limit $k^2\lambda_D^2 \ll 1$ (λ_D is the Debye length). Retaining the leading term only, one finally obtains from (3.8):

$$\varepsilon_1 = -\frac{4\pi e^2 n}{T_e}\frac{\omega_*(k)+\tau\omega_m}{\omega_m k^2(s^2+1)}\, y(k) + O(k^2 a_i^2) \tag{3.10}$$

where $$y(k) \equiv \frac{k - k_m}{k_m}\left[\frac{k_m}{k+\tau k_m} - 2\varepsilon_n(\cos\theta + s\sin\theta)\right] + O(k^2a_i^2)$$

One has $y(k) \lesssim \frac{k-k_m}{k_m}$ provided that $\varepsilon_n \lesssim \frac{1}{2}$. Here k_m is the lower marginal point of k , as (3.10) only holds in the limit $k^2 a_i^2 \to 0$. It can be shown (Hesketh, 1980), that the system is unstable for $k > k_m$ and eventually an upper marginal point k_M exists (the point k_M was proven to exist for small values of r_n/R ($\simeq 0.05$)(Hesketh, private communication)); in general we expect stabilization of drift waves for $k a_i \gtrsim 1$.

The exact $y(k)$ is then expected to be of the form

$$y(k) = \frac{k - k_m}{k_m} \; g(k, \varepsilon_n, s, q, \tau, \vartheta) \tag{3.11}$$

with $y(k) > 0$ for $k_m < k < k_M$. The dielectric constant ε_1 is negative in the unstable region so that the saturation mechanism outlined above can be applied.

4. Saturated Toroidal Modes and the Stochastic Magnetic Field

The equation (2.8) expresses the mean square value of the random variable φ in terms of the linear dielectric constant ε_1 , and is directly applicable to the present situation. One has that

$$\overline{\varphi^2} = \frac{1}{V} \overline{|\varphi_k|^2} = 2\lambda_2^4 \, k^4 (s^2+1)^2 |\varepsilon_1| = \frac{T_e^3}{2\pi e^4 n} \, k^2 (s^2+1)(1+\tau)\, y(k) \tag{4.1}$$

where ε_1 is given by (3.10). One can also write

$$\frac{e^2 \overline{|\varphi|^2}}{T_e^2} = \frac{T_e}{2\pi e^2 n} \, k^2 (s^2+1)(1+\tau)\, y(k) \lesssim \frac{T_e}{\pi e^2 n} \, k^2 (s^2+1) \frac{k - k_m}{k_m} \tag{4.2}$$

where $(k - k_m)/k_m \ll 1$ and τ was assumed of order unity. In order to include all k one must perform an appropriate summation with respect to the unstable k in the range $k_m < k < k_M$.

Since the electrons behave adiabatically, the electron axial current in first order is $j_1 = j\, e\varphi / T_e$. Clearly j_1 is a random variable because φ is random. The axial current j_1 creates a random radial magnetic field $B_r = 4\pi j\, e\varphi / ikT_e$ whose mean quadratic value will be

$$\overline{B_r^2} = \frac{16\pi^2 j^2}{k^2} \frac{e^2\overline{|\varphi|^2}}{T_e^2} = \frac{8\pi j^2(s^2+1)\,T_e}{e^2 n}(r+1)\,y(k) \tag{4.3}$$

Let us neglect all effects related to the current profile. Noting that $j \sim B/2\pi Rq$, where q is the safety factor, one obtains

$$b^2 \equiv \frac{\overline{B_r^2}}{B^2} = \frac{2(s^2+1)T_e}{\pi e^2 n} \frac{r+1}{R^2 q^2}\, y(k) \tag{4.4}$$

In order to see whether the fluctuations of B_r give rise to stocasticization of the magnetic field, we examine the stocasticity condition (Krommes et al., 1978)

$$\frac{w}{\Delta} > 1 \tag{4.5}$$

where Δ and the island width w are given by the relations

$$\Delta \equiv \frac{\Delta r}{\tilde{N}} \qquad\qquad w = 4\left(2 L_s b k^{-1}\right)^{1/2} \tag{4.6}$$

Here $L_s = Rq/s$ is the shear length, $k = M/r$, Δr is the radial extension of the mode around the resonant surface r_0 and $\tilde{N}$ is the number of excited resonances

$$\tilde{N} \equiv \sum_M \sum_{k_\parallel} = \Delta M\, Rq \int_{\Delta k_\parallel} dk_\parallel \tag{4.7}$$

where ΔM is the number of the excited poloidal modes; one can put $\sum_{k_\parallel} \rightarrow Rq \int dk_\parallel$

because the $k_{\parallel}$ modes are supposed to be closely spaced and the connection length in the parallel direction is $2\pi R q$; $\Delta k_{\parallel}$ is the spreading of $k_{\parallel}(r) = (M - Nq)/qR$ along the extension Δr of the mode

$$\Delta k_{\parallel} = -\frac{M}{Rq^2}\frac{dq}{dr_0}\Delta r \tag{4.8}$$

The stocasticity condition then becomes

$$b > \left| \frac{2}{32 q R \vartheta M \Delta M} \right| \tag{4.9}$$

Since $b \gtrsim 10^{-4}$, this condition is well satisfied in practical cases for $|\Delta M| \sim |M| \gtrsim 50.$

5. The Radial Transport Coefficient for the Electrons

The electrons attached to the braiding magnetic lines are subject to an anomalous radial thermal conductivity whose form was recently investigated in a number of papers (Kadomtsev et al., 1978; Rechester et al., 1978; Krommes et al., 1978). We shall use the form given by Krommes et al. (1978) in the collisionless diffusion limit ($\nu t < 0$, ν collision frequency)

$$\chi_e = v_{th\,e}\,\pi q^2 R^2 \sum_M \int dk_{\parallel}\, b^2 \tag{5.1}$$

Here the M summation is performed with respect to the poloidal modes on the resonant surface with $M = q(r_0) N$ and $k_{\parallel} = 0$. The $k_{\parallel}$ integration takes into account that the mode extends radially of an amount $\Delta r = 2\delta r$ centered at the resonant surface.

We shall assume for simplicity that the poloidal and radial extent of the mode are comparable, namely $sk \sim \delta r^{-1} \sim sM/r$. Hastie et al. (1979) have constructed with the WKB method explicit solutions whose radial extent is $\Delta r \approx (3 r r_m/2)^{2/3}(s\,M)^{-2/3}$. For simplicity we take $\Delta r \sim 2r/|M|s$. This is a good approximation for general cases. So one has

$$\chi_e = v_{the}\,\pi q^2 R^2 \Sigma_M \int_{\Delta r} \left|\frac{dk_\parallel}{dr}\right| b^2 dr = v_{the}\,\pi q R \Sigma_M |M| s \frac{\Delta r}{r} b^2 =$$

$$= 2 v_{the}\,\pi q R \Sigma_M b^2 = 4 v_{the} \frac{(s^2+1)(\tau+1)T_e}{e^2 n R q} \Sigma_M y(k) \tag{5.2}$$

Here

$$\Sigma_{\pm M}\, y = 2 \sum_{M>0} y = \frac{2q(r_0)}{r_0 k_m} \sum_{N_m}^{N_m+\Delta N} {}_N (N - N_m)\, g\left(ka_i = \frac{q(r_0)N}{r_0} a_i, \varepsilon_n, s, q, \tau, \theta\right) =$$

$$= \frac{2q(r_0)}{r_0 k_m} f(\Delta N, \varepsilon_n, s, q, \tau, \theta) \tag{5.3}$$

where $k_m = q(r_0) N_m / r_0$

$$f(\Delta N, \varepsilon_n, s, q, \tau, \theta) = \sum_{N_m}^{N_m+\Delta N} {}_N (N-N_m)\, g\left(\frac{q(r_0)N a_i}{r_0}, \varepsilon_n, s, q, \tau, \theta\right) \tag{5.4}$$

ΔN is the number of toroidal unstable modes, which, in a marginal situation, satisfies the inequality $\Delta N / N_m \ll 1$. In order of magnitude one has $g \simeq 1/2$ so that

$$f(\Delta N) \approx \frac{1}{2} \sum_{N_m}^{N_m+\Delta N} (N - N_m) = \frac{\Delta N}{4}(\Delta N + 1) \tag{5.5}$$

The number of excited modes ΔN depends on q and on the parameters $\varepsilon_n, s, \tau, \theta$. While q and τ are not varying essentially from one machine to another, the other parameters do not depend directly on temperature and density, but, in the case of ε_n and s, only on their gradients. However, ΔN and $f(\Delta N)$ depend indirectly on T and n because the parameters can become dependent on these quantities through the transport equations. The present theory rests on the assumption that the linear structure of the instability is such that, when coupled with the transport equation, the profiles adjust themselves in order that the system remains in a marginal situation and in a quasi-steady state. This adjustment is reflected in the form of the dependence of $f(\Delta N)$ on T and n and is related to the detailed structure of the instability and of the transport. One cannot exclude, however, that as a consequence of some general property of the equilibrium equations, the global plasma behaviour becomes insensitive to the incertitude on the knowledge of $f(\Delta N, n, T)$. We shall see in the next section that this is indeed the case for the ohmic steady state.

Recalling (4.3) the electron thermal conductivity takes the form

$$\chi_e = 8\left(\frac{2M_i}{m_e}\right)^{1/2} \frac{c}{e^3} \frac{(s^2+1)(\tau+1)}{R_m a_i} \frac{T_e^{3/2} T_i^{1/2}}{B} \frac{f(\Delta N, n, T)}{n R q} \qquad (5.6)$$

Expressing T in keV (T_k), B in Tesla (B_T) and the lengths in meters, (5.6) becomes

$$\chi_e = 3.4\ 10^{17} \frac{(s^2+1)(\tau+1)}{R_m a_i} \frac{T_{ek}^{3/2} T_{ik}^{1/2}}{B_T} A_i^{1/2} \frac{f(\Delta N, n, T)}{n R q} \qquad (5.7)$$

where A_i is the mass number.

6. The Ohmic Steady State

In steady conditions the equation for the energy balance of the electrons takes the form

$$E_{\parallel} j = -\frac{1}{r}\frac{d}{dr}\left(r n \chi_e \frac{dT_e}{dr}\right) \tag{6.1}$$

or approximately

$$\frac{V}{2\pi R} j \approx 2.6\ 10^{41} \frac{(s^2+1)}{k_m a_i} \frac{T_e^3}{B} \frac{f(\Delta N)}{R\, r\, L_T^2} A_i^{1/2} \tag{6.2}$$

where $T_e = T_i$, $V = E_{\parallel} 2\pi R$ is the loop voltage and L_T is the characteristic length of the temperature gradient. We now use the Ohm law

$$\eta j = \frac{V}{2\pi R} \tag{6.3}$$

with

$$\eta = 6.14\ 10^{-12} Z\, T_e^{-3/2} \lg \Lambda \qquad (T_e \text{ in erg})$$

and the relation $j = B/2\pi R q$ to obtain

$$\frac{T_e^3}{B} = \frac{1}{V^2 q} (6.14)^2\, 10^{-24}\, 2\pi R Z^2 (\lg \Lambda)^2 j \tag{6.4}$$

Substitution into equation (6.2) gives

$$V = \left[3.8\ 10^{20} Z^2 \frac{R}{r} \frac{1}{L_T^2} f(\Delta N) \frac{(s^2+1)(\lg \Lambda)^2 A_i^{1/2}}{q\, k_m a_i} \right]^{1/3} \tag{6.5}$$

We see that, as a consequence of the cubic root, the dependence of the loop voltage V on the geometric and physical parameters is rather weak. Moreover, as already observed, $f(\Delta N)$ and $k_m a_i$ do not depend directly on T and n, but only on their derivatives and on the safety factor q, whose value is about the same for the different machines. We conclude that under ohmic stationary conditions associated with given profiles, all tokamaks tend to assume the same loop voltage, independently of temperature and density and to a large extent, also independently of the geometry. This agrees with the experimental results (Pieroni et al., 1979; Coppi et al., 1979, for an analysis of the experimental results see Düchs et al., 1980).

One can grossly evaluate the order of magnitude of ΔN and $f(\Delta N)$ from the relation $\Delta N = (k_M a_i - k_m a_i)\, r_0 / q a_i$, by taking $k_M a_i \simeq 1$ and, say 0.5 for $k_m a_i$.
Then $\Delta N \approx r_0 / 2 q a_i \gtrsim 20$ and from (5.5) one has $f(\Delta N) \gtrsim 100.$
The typical value for V obtained from (6.5) for characteristic tokamak parameters is $V \sim 0.5\, Z^{2/3}$ Volt. The typical value for the average temperature, evaluated from (6.4), is proportional to $(B Z^{1/3}/q)^{2/3}$ and is about 0.7 keV $(B = 5T,\ q \approx R/r = 3)$.
We can now reexpress the thermal conductivity and the confinement time by inserting (6.4) into (5.7). We obtain

$$\tau_E \equiv \frac{L_T^2}{4\chi_e} = 0.17\, 10^{-18} \frac{(q\, k_m a_i)^{2/3}}{\tau^{1/6}\left[(1+s^2)(1+\tau) f(\Delta N)\right]^{2/3} (\lg \Lambda)^{1/3}} \frac{n}{T_{ik}^{1/2}} \frac{(R L_T)^{4/3}\, r^{2/3}}{(A_i Z)^{1/3}} \approx$$

$$\approx 1.15\, 10^{-21} (q\, k_m a_i)^{2/3} \frac{n}{T_{ik}^{1/2}} \frac{(R L_T)^{4/3}\, r^{2/3}}{(A_i Z)^{1/3}} \qquad (\text{kev, meters}) \tag{6.6}$$

for $\lg \Lambda = 17$ and $s \approx \tau \approx 1$.

This scaling is in fair agreement with recent experimental observations (Leonov et al., 1980). A weak dependence on the parameters $\varepsilon_n, s, q,$ etc. could arise through the term $(q\, k_m a_i)^{2/3}$ whose form is related to the detailed structure of the marginal instability. In the case studied by Hesketh (1980) one has $k_m a_i \sim q^{-1}$ for ε_n small (Hesketh, private communication) so that τ_E becomes q -independent in this case.

7. Conclusion

In our picture, the drift universal mode, destabilized by toroidal effects, is nonlinearly stabilized by the adiabatic reaction of the background of Maxwellian electrons to the linear instability. This reactive process can be described in terms of a dielectric constant of the medium whose nonlinear positive part describing the nonlinear reaction of the thermal electrons counterbalances the inherently unstable negative linear part. This process results in the existence of an electrostatic potential steadily fluctuating with a mean square amplitude inversely proportional to the density of the reactive medium of electrons. The anomalous electron transport is related to a stocasticization of the magnetic field created by the adiabatically fluctuating axial electron current $j_1 \sim j e\varphi/T_e$. It should be noted that the stabilization mechanism involves the nonlinear interaction of a collective Vlasov mode with a thermal background and then it goes beyond the validity of a pure Vlasov model. From the point of view of a fundamental theory this interaction implies that the two-particle correlations cannot be neglected and that in the nonlinear domain the plasma cannot be treated as quasi-neutral. For this reason we do not expect that a scaling law derived on this basis could agree with the Connor-Taylor (1977) constraints.

One of the basic assumptions of our theory is that the system is able to adjust itself automatically to a marginally stable situation and to a steady state. In the case of an ohmically heated plasma this leads to the simple property that the loop voltage is essentially the same for all tokamaks. The scaling of the confinement time as $n\, T_i^{-1/2}$ is also a consequence of this assumption. In the ohmic case the existence of a marginal steady state is rather independent of the detailed structure of the linear instability and the above assumption looks reasonable. It is not clear that the same holds in the case of other heating regimes. In the latter cases the detailed structure of the instability may play a major role. We conclude that our results on the confinement time cannot be extended immediately to these regimes and that, before making any extrapolation, it is better to wait for the experimental results of the machines now under construction.

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