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This work was carried out under the terms of the
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*Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem
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Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.*

- 2 -

this section

TEARING MODE STABILITY IN 1D AND 2D

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Abstract

A stability code for tearing modes in 1D and 2D straight equilibria in the tokamak scaling has been developed. It finds the lowest eigenvalues of a Hermitian problem which is obtained analytically by a reduction of the full problem. The main advantage is the powerful handling of equilibria with several resonant surfaces and displaying poloidal and radial mode couplings. The code has been successfully tested by comparing it with explicitly known analytical results for external kinks.

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Introduction

Since the disruptive instability, which often terminates tokamak discharges, is likely to be caused by destruction of the magnetic surfaces, it is desirable to understand the tearing mode behaviour of a plasma. A consistent treatment of the linear tearing mode stability requires 1) resistive toroidal equilibria, 2) computation of complex eigenvalues.

As a first step in such a program we consider straight geometry. Even in this case the complex eigenvalue problem is difficult, and so we restrict ourselves to the lowest order in the tokamak scaling for which a reduced Hermitian problem can be derived [1]. The question of stability can be decided from this Hermitian eigenvalue problem, for which powerful numerical techniques [2,3] exist. This energy method holds for configurations with arbitrary cross-sections. The perturbation is of multihelical character linearly coupling single helicities at resonant surfaces through the two-dimensionality of the equilibrium. A general 2D stability code has been developed which combines the energy method and the finite-element representation of the perturbation. It is emphasized that the principal value of the energy integral at the singularities, as introduced in [1], is evaluated analytically and thereby kept out of the computation.

In this paper we describe the method for general 2D equilibria and give results for circular (1D) and elliptical (2D) cross-sections. For elliptical cross-sections good agreement (better than 1%) with analytical results for the ideal external kink instabilities of Ref. [4] is found.

The paper is arranged as follows: Section II formulates the problem and describes the method of solution. The results are presented in Sec. III for 1D equilibria with two singular surfaces and for 2D equilibria with elliptical cross-sections. The results and convergence

properties are finally discussed in Sec. IV.

II. Method

We consider resistive systems in a static equilibrium. The equilibrium magnetic field \underline{B} and current density \underline{J} are given by

$$\begin{aligned} \underline{B} &= \nabla z \times \nabla \psi + B_z \nabla z \\ \underline{J} &= j(\psi) \nabla z = \frac{1}{\mu_0} \nabla \times \underline{B} \end{aligned} \quad (1)$$

where ∇z is the unit vector along the plasma column with length L . The poloidal flux ψ satisfies the equilibrium equation

$$\frac{1}{\mu_0} \nabla^2 \psi = j(\psi) = - \frac{dp}{d\psi} \quad (2)$$

with an arbitrary pressure profile $p(\psi)$. Moreover, B_z and the electric field

$$\underline{E} = \eta_0 \underline{J} = \eta_0(\psi) j(\psi) \nabla z \quad (3)$$

where η_0 is the resistivity, are constants in the equilibrium. General two-dimensional equilibria are evaluated numerically with the Garching equilibrium code [5], which is modified for straight geometry. At present only configurations with up-down symmetry are being considered. The question of stability is decided by minimizing the functional [1]: $\delta L = \delta W - \lambda \delta K$, where

$$\begin{aligned} \delta W &= \int d\tau \left\{ \frac{1}{\mu_0} |\nabla_{\perp} A|^2 + A^* \frac{dj(\psi)}{d\psi} (A - \tilde{A}) \right\} \\ \delta K &= \int d\tau |A|^2 \end{aligned} \quad (4)$$

The sign of λ governs stability, A is the z component of the vector potential of the perturbed field, and \tilde{A} is the weighted

surface average of A, which is taken as zero at the boundary.

In the case of a Fourier expansion for A in a flux coordinate system ψ, θ, z

$$A(\psi, \theta, z) = \sum_m a_m(\psi) e^{i(m\theta + n 2\pi z/L)} \quad (5)$$

\tilde{A} is given by

$$A - \tilde{A} = \sum_m \frac{m}{nq(\psi) + m} a_m(\psi) e^{i(m\theta + n 2\pi z/L)} \quad (6)$$

where the surface quantity q denotes the safety factor. The extremum of δW is evaluated by means of the Galerkin form, where the $a_m(\psi)$ are expanded into linear elements $a_m = \sum_{k=1}^M \alpha_k e_k(\psi)$ (see Ref. [3]), leading to a matrix eigenvalue problem.

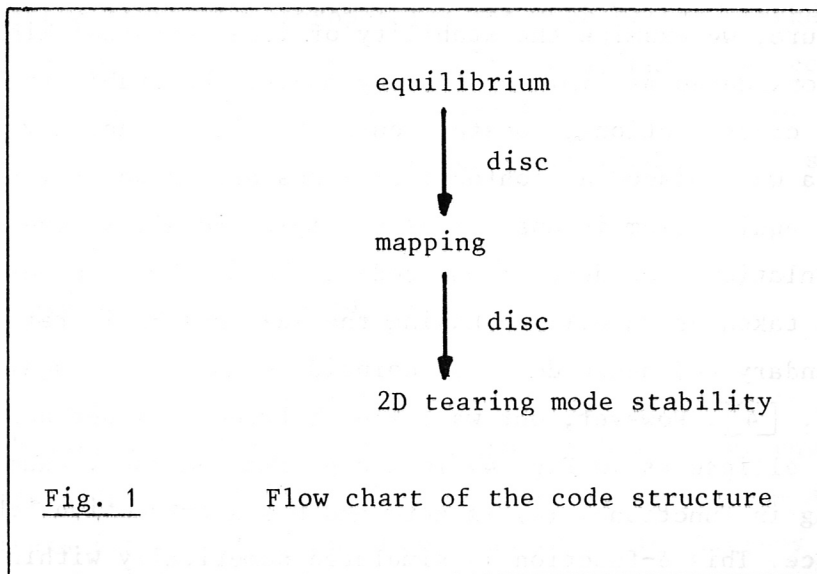
In order to simulate a plasma-vacuum-wall system within this theory, one need only make the resistivity very large and hence the current very small in the "vacuum" region. The Jacobian of the flux coordinate system is chosen as $\sqrt{g} = q(\psi)/B_z(\psi) = q(\psi)/B_z$, which ensures that the magnetic field lines are straight. A radial coordinate s , similar to a radius, is introduced as $\psi = s^2 \psi_s$, where ψ_s is the value of the flux function at the plasma surface.

The potential energy and kinetic-like energy matrices read as:

$$\begin{aligned} \delta W_{m_1 k_1; m_2 k_2} = & 2\pi \delta_{m_1 m_2} P \int_0^1 ds q(s) \frac{m \, d\tilde{g}/ds}{nq+m} e_{m_1 k_1}(s) \cdot e_{m_2 k_2}(s) \\ & + 2 \left[-m \int_0^1 ds q(s) \left(\frac{d}{ds} e_{m_1 k_1}(s) \right) e_{m_2 k_2}(s) \right. \\ & \quad \left. + m' \int_0^1 ds q(s) e_{m_1 k_1}(s) \left(\frac{d}{ds} e_{m_2 k_2}(s) \right) \right] \int_0^{2\pi} d\theta \, g^{4\theta} \sin(m-m')\theta \\ & + 2 \int_0^1 ds \left(\frac{d\psi}{ds} \right)^{-1} q(s) \left(\frac{d}{ds} e_{m_1 k_1}(s) \right) \left(\frac{d}{ds} e_{m_2 k_2}(s) \right) \int_0^{2\pi} d\theta \, g^{4\theta} \cos(m-m')\theta \\ & + 2 m m' \int_0^1 ds \frac{d\psi}{ds} q(s) e_{m_1 k_1}(s) e_{m_2 k_2}(s) \int_0^{2\pi} d\theta \, g^{6\theta} \cos(m-m')\theta \end{aligned}$$

$$\delta K_{m'h'; mh} = 2\pi \int_0^1 ds \frac{dy}{ds} q(s) e_{m'h'}(s) e_{mh}(s)$$

For L Fourier components and M radial finite elements the dimension of the matrices $D = L \cdot M$ becomes very large. But the matrices can be ordered so that they have a band structure with a band width of $2L$, for which fast solvers, in particular for the smallest eigenvalue, exist (see, for example, [6]). For a given numerical equilibrium the mapping into flux coordinates is performed once using the ERATO [2] algorithm and stored on disc. The data for all mesh sizes in the stability code are obtained by interpolation. This is demonstrated by the following flow chart:



III. Results

a) Circular cross-section (1D)

Our tearing mode stability code has been applied to various profiles, including hollow currents for which several singular

surfaces occur in the plasma [3]. A recent application [7] to a profile obtained from the Düchs transport code concerning Intor studies has been performed. The typical profiles have two singular surfaces, as shown in Fig. 2. The region with $j \equiv 0 (s \geq 1.0)$ in the figure corresponds to a vacuum. For this equilibrium there are unstable modes with wave numbers $n = 1, m = 2$; $n = 2, m = 3$ and $n = 3, m = 5$. The eigenfunction for the free surface $n = 1, m = 2$ mode is shown in Fig. 3a. The dotted line, indicating the resonant surface, is very close to the plasma boundary. By placing the wall at $s = 1$ the $n = 1, m = 2$ mode can be stabilized, but not the other tearing mode instabilities. Figure 3b) displays the $n = 2, m = 3$ mode.

b) Elliptical cross-section (2D)

The first studies with our code are aimed at reproducing known results. Since analytical results for the tearing mode are not available in the literature, we examine the stability of ideal external kinks, which can be considered as limiting tearing modes. The stability limits for elliptical cross-sections, constant current density, and a vacuum surrounded by a wall placed at confocal ellipses are known from Ref [4]. The elliptical equilibrium is analytically simple and allows the mapping and the interpolation procedure of our code to be checked. In our study the wall is taken as an ellipse having the same half-axis ratio as the plasma boundary and hence does not coincide with the confocal ellipse of Ref. [4]. However, our wall lies between an inner and an outer confocal ellipse as in Fig. 4. In the plasma and the vacuum $\frac{dj}{ds}$ intervening in functional (4) is zero and has a δ -function behaviour at the interface. This δ -function is simulated numerically within one mesh cell across the boundary. Figure 5 shows the results for an ellipse with a half-axis ratio of 2. It can be proved that the $m = 2$ mode is unstable for $nq_0 \leq nq_c \leq nq_1 \leq nq \leq 2.0$, where nq_1 and nq_0 denote the marginal values from the theory [4] for the inner and outer confocal ellipses, and nq_c the marginal value obtained with the code. The quantity SAP denotes the value of s at the plasma vacuum boundary, and $s = 1.0$ denotes the wall. The numerically obtained limit nq_c lies between nq_0 and nq_1 and agrees well with their average values, as can be expected.

Especially for a distant wall the broadening of the unstable domain for nq to values smaller than 1 is established. The larger marginal point $nq_c = 2.0$ is easily reproduced. However, it is found that for a large vacuum region many radial finite elements are needed to give the stability limit. This is shown in Fig. 6, where $SAP = 0.25$. The eigenvalue λ is plotted versus nq for different radial mesh sizes, from $M = 125$ to 2000 and for a fixed number of Fourier components $L = 9$. These data allow extrapolation to an infinite number of mesh points, showing a poor dependence of λ on N as $1/N$.

It is concluded that the chosen numerical method with regular, linear finite elements does not represent well the marginal point. The term $|\nabla_{\perp} A|^2$ in the potential energy does not balance sufficiently the second term, which is proportional to $|A|^2$. Better finite elements will be used in the future. The convergence with respect to the number of Fourier components has the expected exponential-type behaviour. It is emphasized that the convergence properties get much better if the wall is nearer the plasma surface. In this case the gradient of A becomes more pronounced and the "pollution" is smaller. This argument also holds for the typical tearing mode with its steep gradients at singular surfaces !

IV. Discussion and conclusion

The stability code presented in this paper is able to answer the question of "tearing-like" stability concerning 2D straight equilibria in the tokamak ordering. The reduction to a Hermitian problem and the use of fast solvers for lowest eigenvalues lead to great accuracy in determining the marginal points. This accuracy overcompensates, in our opinion, the inability to compute growth rates, which in fact are affected by all the missing physics too difficult to consider numerically.

The main achievement is the accurate computation of modes with several singularities, especially for 2D equilibria. The code is successfully tested by comparing it with the particular case of an elliptical plasma for which explicit analytical results are known. This test shows remarkable

accuracy, as shown in more details in the paper. Some cases need, however, more than a thousand radial mesh points, which suggests the use of other finite elements, possibly hybrid elements.

A great challenge confronting the numerical determination of stability of dissipative plasmas is the handling of non-Hermitian problems which arise as soon as, for example, the tokamak scaling is abandoned. The only hope now is to solve 1D problems. Fast solvers for the lowest real part of eigenvalues are missing.

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Figure Captions

- Fig. 1 Flow chart of the code structure.
- Fig. 2 Safety factor q and current density j profiles of a 1D equilibrium with vacuum ($s \geq 1.0$).
- Fig. 3 Unstable eigenfunction A for a 1D equilibrium with
a) $n = 1, m = 2$ and free boundary
b) $n = 2, m = 3$ and fixed boundary.
- Fig. 4 Plasma with elliptical cross-section ($b/a = 2.0$) surrounded by a wall with the same ellipticity and with the corresponding inner and outer confocal ellipses.
- Fig. 5 Domain of unstable $n = 1, m = 2$ ideal external kinks, where SAP denotes the value of s at the plasma vacuum boundary.
- Fig. 6 Convergence studies for the $n = 1, m = 2$ ideal external kink with $SAP = 0.25$. The eigenvalue λ is plotted versus nq for a fixed number of Fourier components ($L = 9, m = -6, -4, -2, 0, 2, 4, 6, 8, 10$) and for different radial mesh sizes ($M = 125$ to infinity).

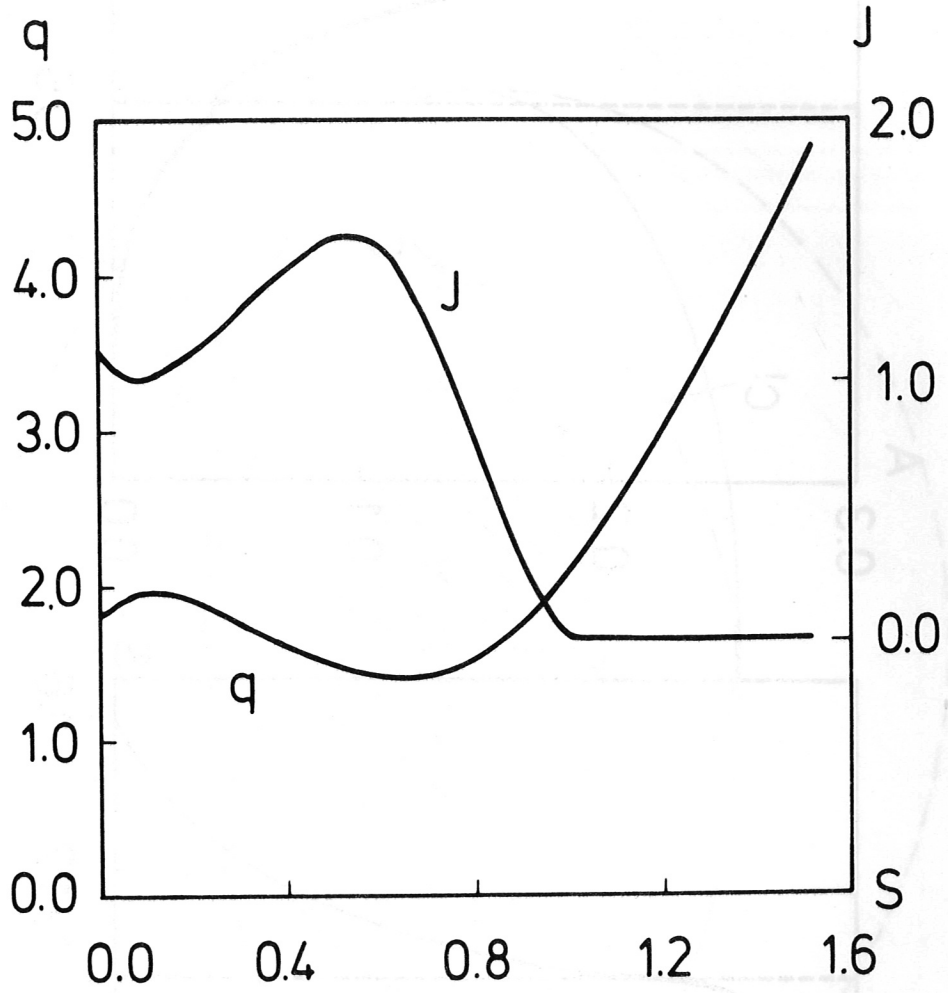


Fig. 2

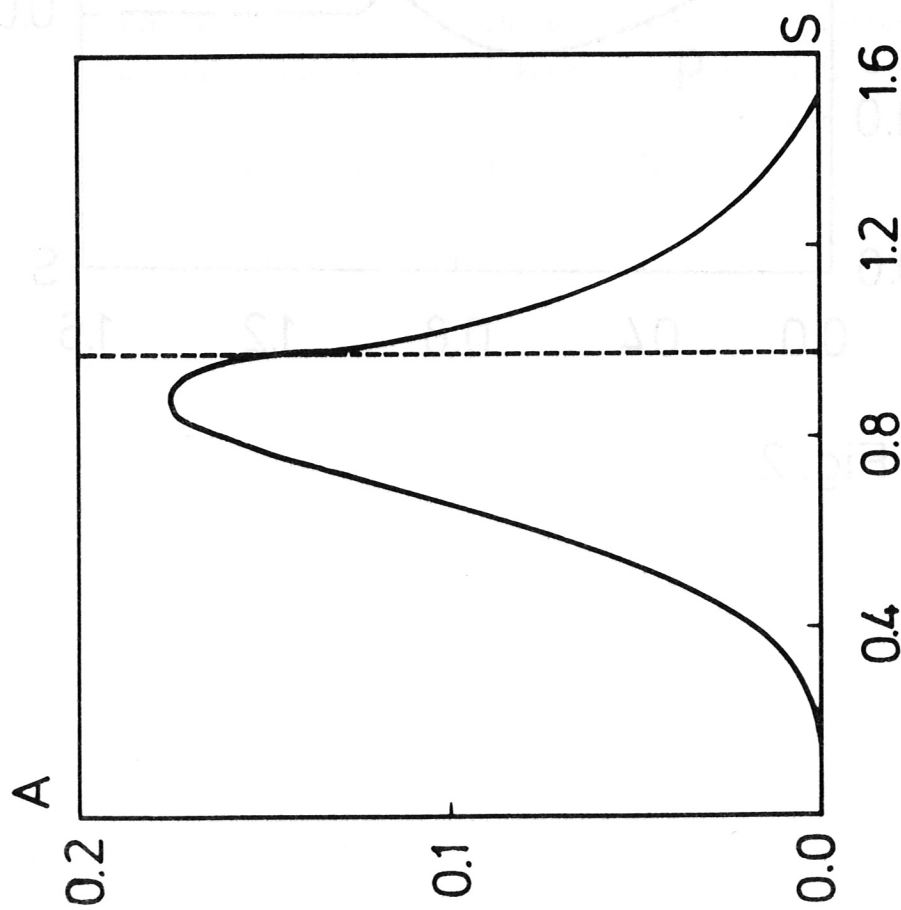
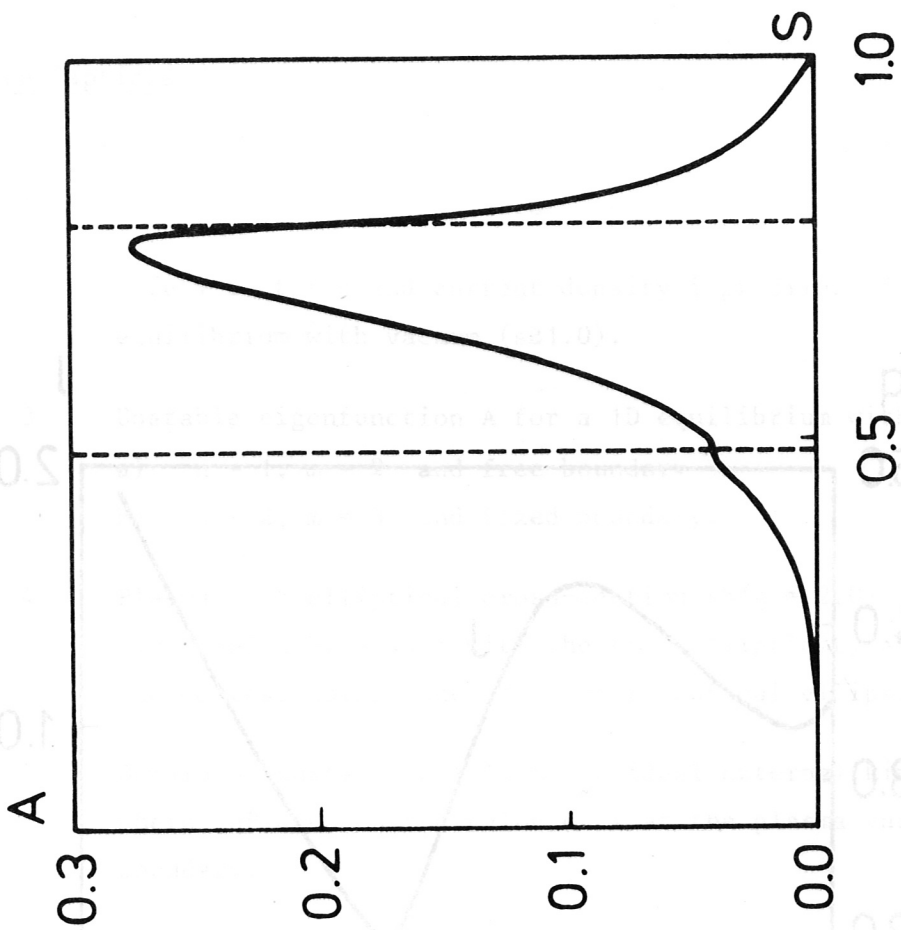


Fig. 3a

Fig. 3b

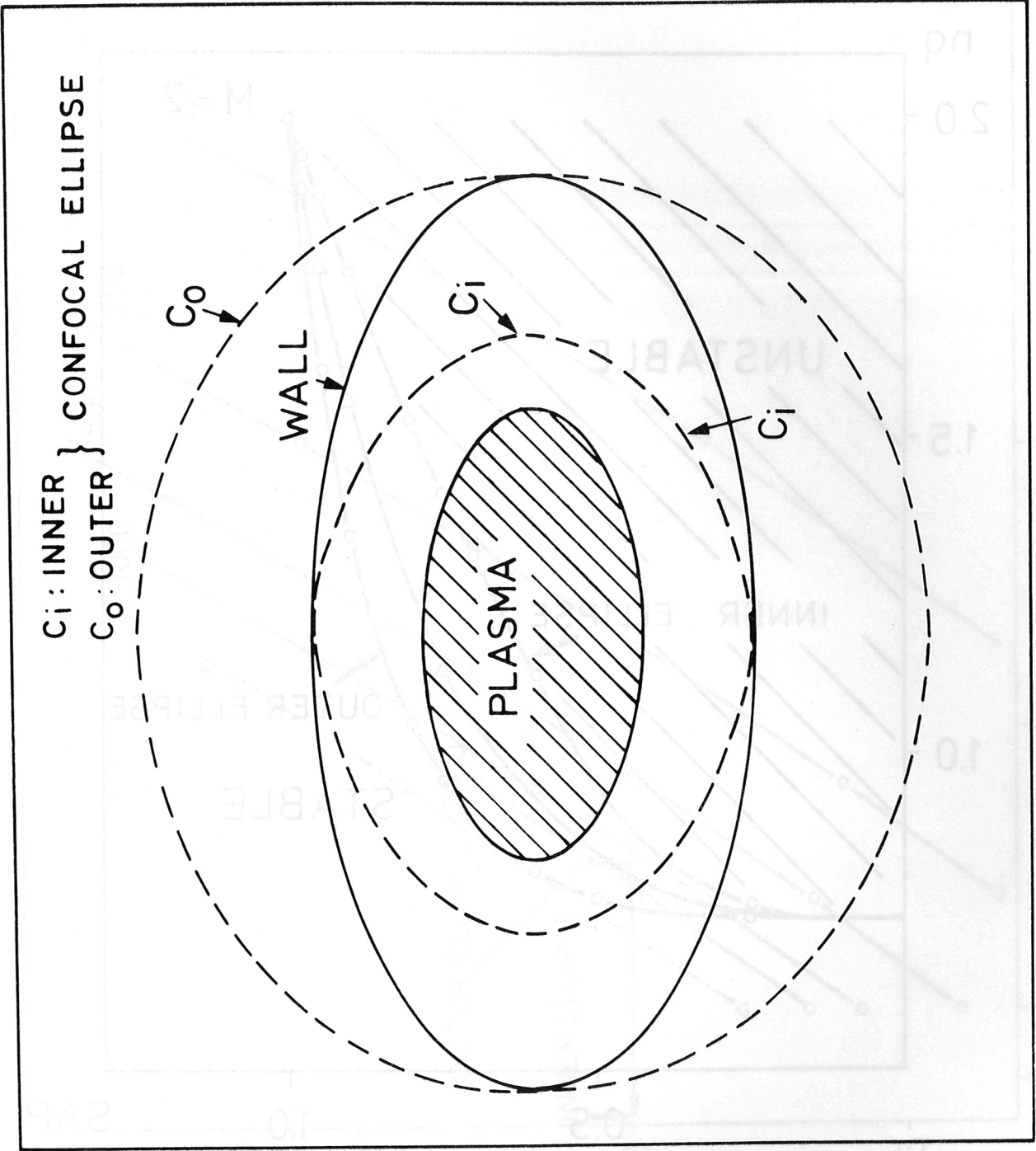


Fig. 4

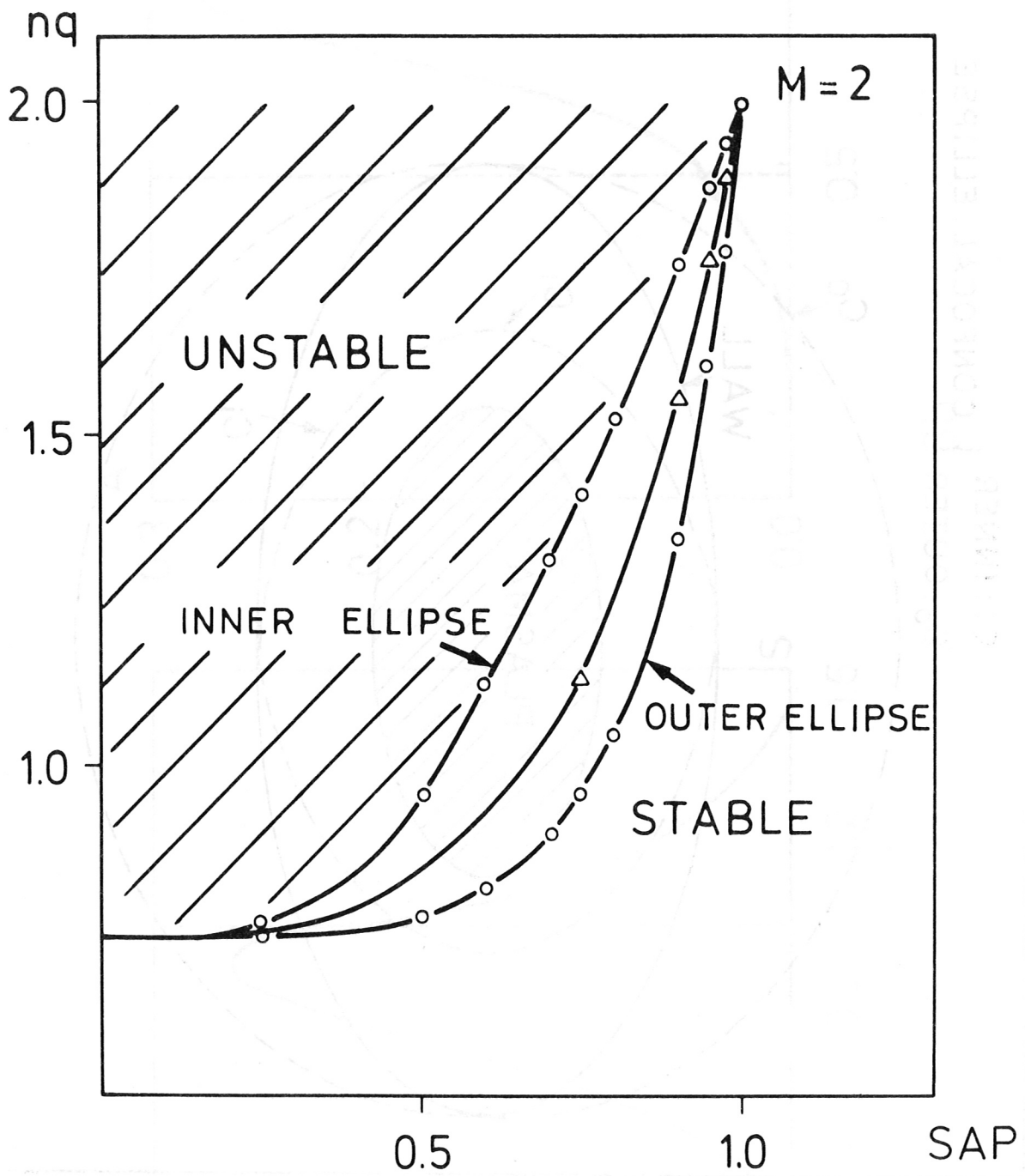


Fig. 5

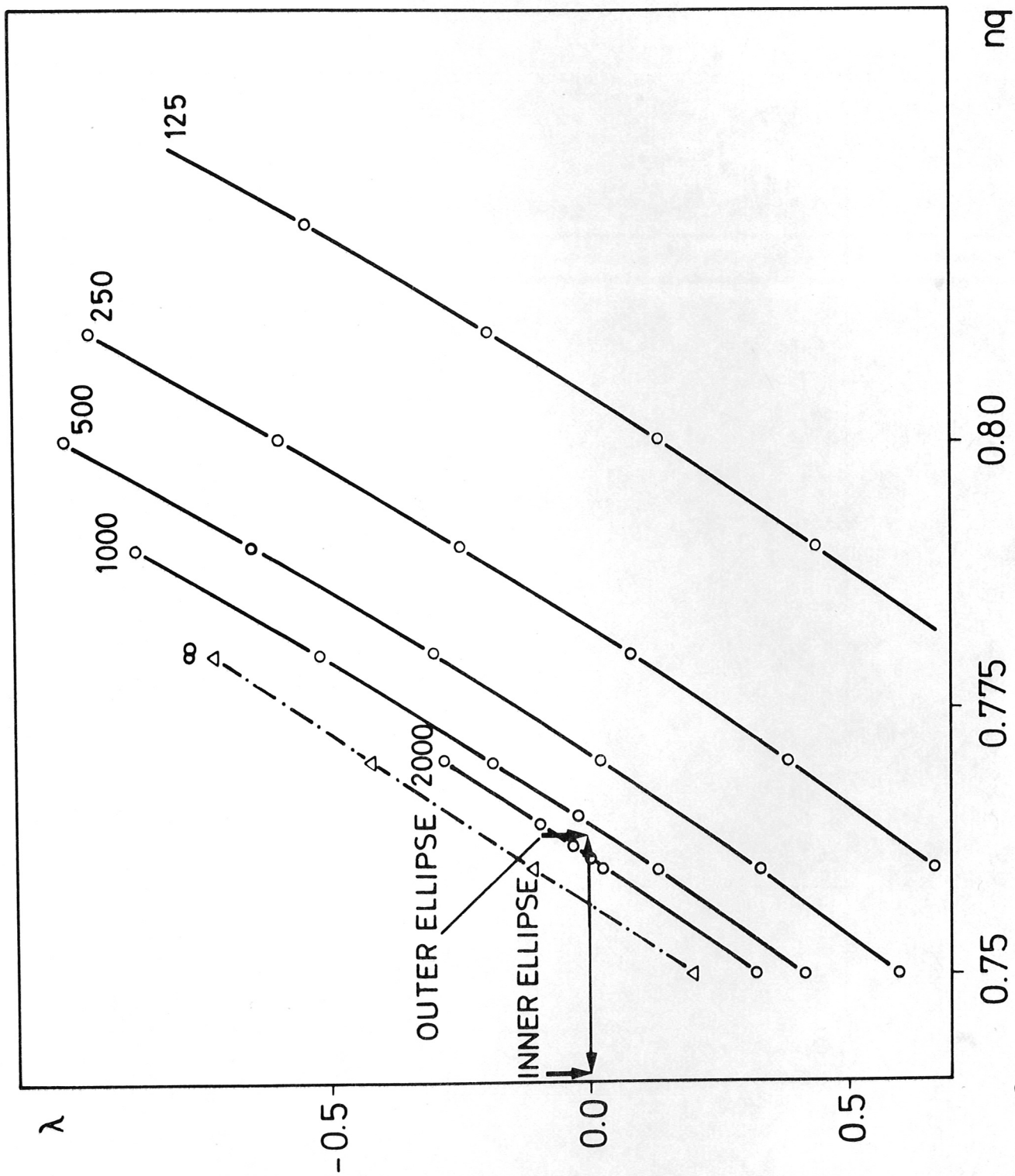


Fig. 6