Dr. Rühs

INFINITESIMAL ROTATION:

ON THE CONVERSE PROBLEM

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ABSTRACT

The problem of finding a vector of constant length verifying

$$\frac{d\vec{v}}{ds} = \omega(s)\times\vec{v}$$

has been reduced by introducing the referential built on the eigenvectors of the singular operator.

$$\Omega \equiv \vec{\omega}\times \;;$$

This procedure represents a natural introduction of complex vectorial spaces in three dimensional geometry.

INTRODUCTION

The infinitesimal rotation of a three dimensional real vector with constant length is described through the equation

$$\frac{d\vec{V}}{ds} = \omega x \vec{V} \qquad V(s),\ \omega(s)$$

in the case of one parameter dependent vector field; (the Serret-Frenet equations for example is an example of the resolvant matrix system associated with $\vec{V}$). The converse problem of finding a set of independent vectors $\vec{V}$, when $\vec{\omega}$ is given, plays an important role in mechanics and electrodynamics. The usual way to solve this problem by serie expansions in operator space (e^{Ω} operatorial definition) brings difficulties in the discussion of concepts related to the solutions of our equation. In order to avoid these obstacles, a direct method to reduce the problem, based on the use of the eigen-vectors of the $\Omega \equiv \vec{\omega}\,x$ operator, has been worked out, that inlights the role of complex vectorial spaces and their geometrical significance.

Reduction:

$$\frac{d\vec{V}}{ds} = \omega(s) \times \vec{V} \qquad (1)$$

We shall suppose that V is an unit vector $|V| = 1$ which does not restrict the problem. We have to solve a linear system of equations with non constant parameters that we may also write in matricial form, with the usual representation of vectors as column matrices and their transposed $\bar{v}$ as row matrices.

$$\vec{V} \rightarrow v \equiv \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \qquad \bar{v} \equiv \left(v^1 \; v^2 \; v^3 \right)$$

($\dot{v}$) denoting the d/ds derivation

$$\dot{v} = \omega v \qquad (1')$$

$$\dot{\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & \omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

The asymmetric operator Ω is singular ($|\Omega| = 0$) but it indeed possesses three eigenvectors.

	eigenvalues
$e_0 = \vec{n}$	$f = 0$
$e_+ = \frac{1}{\sqrt{2}} \; \vert \vec{n} \times \vec{a} + j\, \vec{n} \times (\vec{n} \times \vec{a}) \vert$	$f = -j$
$e_- = \frac{1}{\sqrt{2}} \; \vec{n} \times \vec{a} - j\, \vec{n} \times (\vec{n} \times \vec{a}) \vert$	$f = +j$

(2)

with $\vec{\omega} = \omega\vec{n}$ $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$ and $\vec{n}^2 = 1$

$\vec{a}$ being any vector, real or complex. In order to have a set of unitary vectors e_o , e_+ , e_- , we shall take $\hat{a} = \vec{a}\vec{a}^* = 1$

The vectors $\vec{l} = \vec{n} \times \vec{a}$

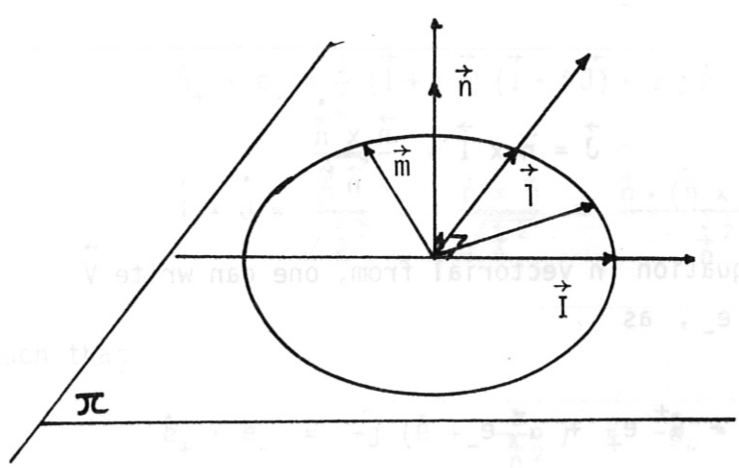
$$\vec{m} = \vec{n} \times (\vec{n} \times \vec{a}) = \vec{n} \times \vec{l}$$

are then defined to a rotation around $\vec{n}$: introducing the unit vectors.

$$\vec{I} = \frac{\dot{\vec{n}}}{\sqrt{\dot{\vec{n}}^2}}$$

$$\vec{J} = \vec{n} \times \vec{I} = \frac{\vec{n} \times \dot{\vec{n}}}{\sqrt{\dot{\vec{n}}}}$$

situated in the same plane as $\vec{l}$ and $\vec{m}$



we may write then

$$e_+ = \frac{1}{\sqrt{2}} (\vec{l} + j\vec{m}) = \frac{1}{\sqrt{2}}(\vec{I} + j\vec{J})\, e^{-j\Theta}$$

$$e_- = \frac{1}{\sqrt{2}} (\vec{l} - j\vec{m}) = \frac{1}{\sqrt{2}}(\vec{I} - j\vec{J})\, e^{+j\Theta} \qquad (3)$$

<u>Remark:</u> These relations are true even if the vector $\vec{n}$ has a non constant direction.

The indeterminacy about e_+ and e_- is then represented by the arbitraryness of $\underline{\Theta}$. One can resume what has been said as follows.

$$e_o = \vec{n} \qquad \vec{\omega} \times \vec{n} = 0$$

$$e_+ = \frac{1}{\sqrt{2}} (\vec{I} + j\vec{J})\, e^{j\Theta} \qquad \vec{\omega} \times e_+ = -je_+$$

$$e_- = \frac{1}{\sqrt{2}} (\vec{I} - j\vec{J})\, e^{-j\Theta} \qquad \vec{\omega} \times e_- = je_-$$

$e_o \cdot e_o = 1$ $\qquad$ $\cdot$ designing the euclidian scalar product

$e_+ \cdot e_+ = 0$ $\qquad$ $\underline{e_+ \cdot e_- = e_- \cdot e_+ = 1}$

$e_- \cdot e_- = 0$

$$\vec{I} = \frac{\dot{\vec{n}}}{\sqrt{\dot{\vec{n}}^2}} \qquad \vec{J} = \vec{n} \times \vec{I} = \frac{\vec{n} \times \dot{\vec{n}}}{\sqrt{\dot{\vec{n}}^2}}$$

Coming back to our equation in vectorial from, one can write $\vec{V}$ in Term of e_o, e_+, e_-, as

$$\vec{V} = \alpha^o e_o + \alpha^+ e_+ + \alpha^- e_-$$

$$|\vec{V}| = 1 \rightarrow \alpha_o + 2\alpha^+ \alpha^- = 1 \qquad (4)$$

The equation (1) becomes:

$$\dot{\alpha}^{o} e_{o} + \dot{\alpha}^{+} e_{+} + \alpha^{-} \dot{e}_{-} + \alpha^{+} \dot{e}_{+} + \alpha^{-} \dot{e}_{-} + j\omega (\alpha^{+} e^{+} - \alpha^{-} e^{-}) = 0$$

Taking the euclidian scalar product of this equation successively with e_o , e_+ , e_- one obtain three equations for α^{o} , α^{+} , α^{-}

$$\dot{\alpha}^{o} + (\dot{e}_{+} \cdot e_{o})\alpha^{+} + (\dot{e}_{-} \cdot e_{o})\alpha = 0$$

$$\dot{\alpha} + (\dot{e}_{o} \cdot e_{-})\alpha^{o} + (\dot{e}_{+} \cdot e_{-})\alpha^{o} + j\omega\alpha^{+} = 0$$

$$\dot{\alpha} + (\dot{e}_{o}\, e_{+})\alpha^{o} + (\dot{e}_{-}\, e_{+})\alpha^{-} - j\omega\alpha^{-} = 0 \qquad (5)$$

having noticed that $\dot{e}_{-} \cdot e_{-} = \dot{e}_{+} \cdot e_{+} = \dot{e}_{o} \cdot e_{o} = 0$

evaluating $\dot{e}_{o} \cdot e_{-}$, $\dot{e}_{o} \cdot e_{+}$, $\dot{e}_{+} \cdot e_{-} = -\dot{e}_{-} \cdot e_{+}$

$$\dot{e}_{o} \cdot e_{+} = \frac{\dot{\vec{n}} \cdot e^{2-j\Theta}}{\sqrt{2\dot{\vec{n}}^{2}}} = \frac{\sqrt{\dot{\vec{n}}^{2}}}{2} e^{-j\Theta} = \nu_{+} = \hat{\nu} e^{-j\Theta}$$

$$\dot{e}_{o}\, e_{-} = \frac{n^{2}}{2} e^{+j\Theta} = \nu_{-} = \hat{\nu} e^{+j\Theta}$$

$$\dot{e}_{+} \cdot e_{-} = \frac{1}{2}(\vec{I} + j\vec{J})(\vec{I} - j\vec{J}) - 2j\dot{\Theta} = -j(\dot{\Theta} + \dot{\vec{I}} \cdot \vec{J})$$

$$\dot{\vec{I}} \cdot \vec{J} = \frac{\ddot{\vec{n}}}{\sqrt{\dot{\vec{n}}^{2}}} \cdot \frac{\vec{n} \times \dot{\vec{n}}}{\sqrt{\dot{\vec{n}}^{2}}} = \frac{\vec{n} \cdot (\dot{\vec{n}} \times \ddot{\vec{n}})}{\dot{\vec{n}}^{2}} = \frac{\partial}{\dot{\vec{n}}^{2}}$$

such that

$$\dot{e}_{+} \cdot e_{-} = -j\left(\dot{\Theta} + \frac{\partial}{\dot{\vec{n}}^{2}}\right) = -\dot{e}_{-} \cdot e_{+}$$

(5) gives

$$\dot{\alpha}^{o} - \nu_{+}\alpha^{+} - \nu_{-}\alpha^{-} = 0$$

$$\dot{\alpha}^{+} + j\mu\alpha^{+} + \nu_{-}\alpha^{o} = 0$$

$$\dot{\alpha}^{-} - j\mu\alpha^{-} + \nu_{+}\alpha^{o} = 0$$

$$\mu = \omega - \frac{(\vec{n}\,,\dot{\vec{n}}\,,\ddot{\vec{n}})}{\dot{\vec{n}}^{2}} - \dot{\Theta}$$

$$= \omega' - \dot{\Theta} \qquad (6)$$

or in matrical form

$$\frac{d}{dt}\begin{pmatrix}\alpha^{o}\\ \alpha^{+}\\ \alpha^{-}\end{pmatrix} = \begin{pmatrix}0 & \nu_{+} & \nu_{-}\\ -\nu_{-} & -j\mu & 0\\ -\nu_{+} & 0 & j\mu\end{pmatrix}\begin{pmatrix}\alpha^{o}\\ \alpha^{+}\\ \alpha^{-}\end{pmatrix} \qquad (6')$$

One remarks that this linear matrix is still antihermitian due to the fact that what has been operated is equivalently written in matrix form as (with $v = s\alpha$);

$$\dot{\alpha} = (\dot{s}^{+} s + j\tilde{\omega})\alpha \quad \text{where} \quad s \equiv \begin{pmatrix} e_1^{o} & e_1^{+} & e_1^{-}\\ e_2^{o} & e_2^{+} & e_2^{-}\\ e_3^{o} & e_3^{+} & e_3^{-}\end{pmatrix}$$

$$\tilde{\omega} = \begin{pmatrix}0 & 0 & 0\\ 0 & j\omega & 0\\ 0 & 0 & -j\omega\end{pmatrix}$$

The free choice of Θ should be used in order to simplify (6)

1) Taking first $\Theta = 0$

$$\nu \equiv \hat{\nu}$$

$$\mu = \omega - \frac{\partial}{\vec{n}^2}$$

(6) is then reduced to

$$(6'') \quad \begin{pmatrix} \alpha^0 \\ \alpha^+ \\ \alpha^- \end{pmatrix} = \nu \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \alpha^0 \\ \alpha^+ \\ \alpha^- \end{pmatrix} - j\mu \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{K} \begin{pmatrix} \alpha^0 \\ \alpha^+ \\ \alpha^- \end{pmatrix}$$

Introducing the resolvant matrix Q defined as (Ref. 1)

$$\begin{pmatrix} \alpha^0 \\ \alpha^+ \\ \alpha^- \end{pmatrix} \equiv \rho \begin{pmatrix} \alpha_0^0 \\ \alpha_0^+ \\ \alpha_0^- \end{pmatrix} \rightarrow \text{initial values} \tag{7}$$

one get

$$\dot{Q} = \nu\,(A)Q - j\mu\,\grave{K}\,Q$$

$$\dot{Q}\,Q^{-1} = \nu A - j\mu\,\grave{K} \tag{8}$$

Introducing the matrix s_0 of the eigenvectors of A one may write A as:

$$\nu A = j\nu\sqrt{2} \underbrace{\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \\ -1/\sqrt{2} & -1/2 & 1/2 \end{pmatrix}}_{s_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/2 & -1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}}_{s_0^{-1}}$$

s_o being a constant matrix one may transform our equations as
$A = - \dot{s}_1 s_1^{-1}$.

This procedure cuts short the general method described on Ref. 2.

$$s_1 = s_0 \begin{pmatrix} 1 & & \\ & e^{+\int\nu'} & \\ & & e^{-j\int\nu'} \end{pmatrix} \quad \text{with} \quad \nu' = \sqrt{2} \cdot \nu$$

one has then:

$$\dot{Q}\, Q^{-1} + \dot{s}_1 s_1^{-1} = -j\,\mu K \tag{9}$$

operating two new changes of matrix

$$s_2 = \begin{pmatrix} 1 & & \\ & e^{j\int\mu/2} & \\ & & e^{-j\int\mu/2} \end{pmatrix} s_1 \equiv \begin{pmatrix} 1 & & \\ & e^{j\int\mu/2} & \\ & & e^{-j\int\mu/2} \end{pmatrix} s_0 \begin{pmatrix} 1 & & \\ & e^{j\int\nu'} & \\ & & e^{-j\int\nu'} \end{pmatrix}$$

and $$Q \equiv \begin{pmatrix} 1 & & \\ & e^{j\int\mu/2} & \\ & & e^{-j\int\mu/2} \end{pmatrix} Q_2$$

(9) is reduced to:

$$\boxed{\dot{Q}_2 Q_2^{-1} = -\dot{s}_2 s_2^{-1}} \tag{10}$$

We have to find a new referential whose Cartan matrix is opposite to the known $\dot{s}_2 s_2^{-1}$

This problem is then the key for the solution and is discussed in (Ref. 2).

<u>Remark:</u> The original matrix being antihermitian

$$s_2^{-1} \equiv s_2^{+} \quad \text{and} \quad Q_2^{-1} \equiv Q_2^{+}$$

such that (10) is equivalent to

$$\dot{Q}_2 Q_2^{+} \equiv -\dot{s}_2 s_2^{+} \tag{10'}$$

$\dot{Q}_2 Q_2^{+}$ being also antihermitian

2) A second choice for Θ, namely

$$\dot{\Theta} = \omega - \frac{\partial}{\dot{n}^2} \quad \text{which gives} \quad \mu \equiv 0$$

reduces (6) to

$$\begin{aligned} \dot{\alpha}^{o} &= \nu_{+} \alpha^{+} + \nu_{+} \alpha^{-} \\ \dot{\alpha}^{+} &= -\nu_{-} \alpha^{o} & \qquad \alpha^{+} &= -\int \nu_{-} \alpha^{o} \\ \dot{\alpha}^{-} &= -\nu_{+} \alpha^{o} & \rightarrow \qquad \alpha^{-} &= -\int \nu_{+} \alpha^{o} \end{aligned}$$

We have for α_o the following integro differential equation:

$$\boxed{\dot{\alpha}^{o} + \nu_{+} \int \nu_{-} \alpha^{o} + \nu_{-} \int \nu_{+} \alpha^{o} = 0} \tag{11}$$

ν_{+} and ν_{-} being complex conjugate

(11) may be transformed as

$$\alpha^0 \alpha^0 + \alpha^0 \nu_+ \int \nu_- \alpha^0 + \nu_- \alpha^0 \int \nu_1 \alpha^0 = 0$$

$$\frac{1}{2} (\alpha^{0 2}) + (\int \alpha^0 \nu_+ \times \int \alpha^0 \nu_-) = 0$$

$$\boxed{\alpha^{0 2} + 2 \int \alpha^0 \nu_+ \times \int \alpha^0 \nu_- = K} \qquad (12)$$

Whether (10) or (12) is easier to solve at last approximatively, depends surely on ν_+.

We may remark, as conclusion, that the solution of the second order parametric equation $y'' \equiv k^2(x)y$ leads to a similar integro differential equation so that our problem has a very strong affinity with the latest Ref. 2.

Indeed the form of (11) suggests the introduction of two auxiliary functions λ, γ such that

$$\int \nu_+ \alpha^0 = (\lambda \dot{\alpha}^0 + \gamma) \qquad (13)$$

by derivation one get

$$\nu_+ \alpha^0 = \lambda \ddot{\alpha}^0 \quad \dot{\lambda} \dot{\alpha}^0 + \dot{\gamma} \qquad (13')$$

and from (11)

$$\dot{\alpha}^0 + (\nu_- \lambda + \nu_+ \lambda^*) \dot{\alpha}^0 + \nu_- \gamma + \nu_+ \gamma^* = 0$$

which may be solved by taking

$$\nu_- \gamma + \nu_+ \gamma^* = 0$$
$$\nu_- \lambda + \nu_+ \lambda^* = -1 \qquad (15)$$

writing $\gamma = \hat{\gamma}\, e^{j\chi}$

$\lambda = \hat{\lambda}\, e^{j\psi}$

and having $\nu_+ = \hat{\nu}\, e^{-j\Theta}$

(15) gives

$\cos(\Theta + \chi) = 0 \quad \rightarrow \quad \boxed{\chi = \frac{\pi}{2} - \Theta}$

$\hat{\nu}\hat{\lambda} \cos(\Theta + \chi) = -1 \quad \rightarrow \quad \boxed{\begin{array}{l} \psi = \pi - \Theta \\ \lambda = \frac{1}{\hat{\nu}} \end{array}}$

$\hat{\gamma}$ being arbitrary we shall take $\hat{\gamma} = \underline{1}$

so that (13') gives

$$(\lambda \dot{\alpha}^{o}) - \nu_+ \alpha^{o} = 0$$

$$\boxed{\frac{d}{ds}\left(\lambda \frac{d}{ds} \alpha^{o}\right) - \nu_+ \alpha^{o} = 0}$$

Our problem is then reduced to the solution of a Sturm-Liouville equation; that is a third way to approach it.

Ref. 1 Pease. Methods of Matrix Algebra

Ref. 2 P. Javel. On Linear Parametric Differential Systems.
IPP Bericht. In preparation