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ON THE ANOMALOUS RADIATION OF ELECTRONS IN
A NON-MAXWELLIAN PLASMA

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(in English)

Summary

The radiation emitted by the electrons of a plasma with more suprathermal electrons than a Maxwellian one has been evaluated. The emitted lines attain in the plasma $N_{\perp}$ times the black-body radiation at the temperature of the "hot" electrons for some harmonic number up to a limit number m_o.
The energy is absorbed mostly by the bulk of the electrons, owing to the magnetic field gradient in a toroidal configuration. This "equipartition" effect can be larger than the effect due to the collisions. Finally, since the coupling at the upper-hybrid is not possible at higher harmonic numbers in "fusion" plasmas, a model has been proposed for the coupling of the emitted waves which are almost electrostatic, with the waves which go adiabatically into vacuum.

Introduction

Suprathermal emission at the harmonics of the electron cyclotron frequency was already observed at the beginning of fusion research; the first report is dated 1961 /1/. Some proposed explanations are reported in Chapter 7 of Ref. 2. Probably the most obvious was to assume the presence of a group of suprathermal electrons and evaluate the radiation emitted by them in the plasma, which was described as a medium with dielectric properties determined by the distribution function of the thermal electrons. (A similar approach had previously been successfully used /3/ to account for anomalous emission at the harmonics of the ion cyclotron emission in OGRA.) An open problem in the proposed model is the coupling of the preferentially emitted electrostatic waves with waves which can propagate into vacuum. Also unsatisfactory is the fact that the intensity of the emitted lines comes out to be proportional to the number of suprathermal electrons, without an evident upper limit; moreover, only the perpendicular velocity distribution seems to play a role.

More recently suprathermal emission has also been observed in Cleo and the FT tokamak. In both cases the anomalous radiation is attributed to electrons described by a distribution function with a temperature much larger than that of the plasma bulk. In the first paper the author follows a paper by Engelmann /4/ (see also /5/), where the absorption is evaluated from the dispersion relation, on the assumption that $k_\perp^2 \rho^2 << m^{1/2}$ (this condition following from the expansion of the functions $e^{-\mu} I_m(\mu)$ in powers of $\mu \equiv k_\perp^2 \rho^2/2$). This restriction is however a weak point in the theory, since one would expect the emission to be at

its greatest for $\lambda \sim \rho/m$ i.e. for $k_\perp^2 \rho^2 \sim m^2$. The emission then follows from the Rayleigh-Jeans law. The FT results /6/ are tentatively explained with the Schwinger formula for the radiation of a particle in vacuum (reference is made to a paper by Rosenbluth /7/).
The two models do not yield an upper limit to the intensity of the emitted lines.

In the following the plasma radiation is evaluated as the average over all the electrons of the radiation of a test electron. The distribution function of the electrons is taken to be the sum of two Maxwellians, the one with lower temperature describing the plasma bulk, as usual. The description is non-relativistic; it is valid as long as $N_{\parallel} > \sqrt{m}\, v_t/c$, where m is the harmonic number and v_t the thermal velocity (see, for example, /5/). Our problem is to determine under what conditions the radiation lines in the neighbourhood of the harmonics of the electron gyro-frequency have intensity exceeding that of a black-body at the temperature of the bulk of the plasma. Several recent papers deal with the problem of the radiation from a non-Maxwellian plasma (see for example /8 - 10/) but with different aims; for example /10/ is limited to the first two harmonics, in the small Larmor radius approximation, but within the framework of the weakly relativistic treatment. In Sect. 2 it is shown for a Maxwellian plasma how the Rayleigh-Jeans law follows from the assumption of "small" absorption. The proof has some interest for its own sake, because of its simplicity; its aim, however, is to facilitate the proof in Sec. 3, that the intensity of the radiation in the plasma ranges between the intensities of the black-body radiation at the temperatures T_o and T_1. The level T_1 is attained when the plasma is optically thick and the radiation is preferentially absorbed by the suprathermal electrons.

In Sec. 4 we determine the frequency intervals where the level T_1 is effectively attained. In this section we use approximations to solve the dispersion relation, namely $k_{\parallel}\rho << 1$ and $\omega_p^2 \lesssim \Omega^2$. The functions $e^{-\mu} I_m(\mu)$ are described, when necessary, by formulae valid also for $\mu \sim m^2$, i.e. for $\lambda \sim \rho/m$.

Only a small fraction of the emitted radiation will leave the plasma; the rest is absorbed by the same electron population which emitted it, when the plasma is homogeneous. When the plasma is not homogeneous, and particularly in a toroidal magnetic field, the radiation from the suprathermal electrons ($T=T_1$) will be absorbed by the thermal electrons ($T=T_o$), giving rise to an "equipartition effect". This process is described in sect. 5, where, moreover, a rough comparison with the equipartition effect due to the collisions is presented.

Section 6 presents a model for the coupling of electrostatic waves with waves which can propagate into vacuum when $\omega_p^2 \lesssim \Omega^2$. The transmission coefficient is found to be proportional to the density gradient and to the gradient of the magnetic field, as expected.

The conclusions are summarized in Sec. 7.

Section 2

Let Maxwell equations be written as

$$d_{\alpha\beta}\,\mathcal{E}_\beta = \frac{4\pi i}{\omega}\, j_\alpha$$

with

$$d_{\alpha\beta} \equiv \frac{k^2c^2}{\omega^2}\left(\delta_{\alpha\beta} - k_\alpha k_\beta / k^2\right) - \varepsilon_{\alpha\beta}$$

It then follows that

$$\mathcal{E}_\alpha = \frac{D_{\alpha\beta}}{D}\, j_\beta\, \frac{4\pi i}{\omega} \qquad \left(D \equiv \| d_{\alpha\beta} \|\right),$$

where $D_{\alpha\beta}$ is the co-factor of $d_{\alpha\beta}$.
With these notations the power radiated by an electron per unit angular frequency interval in the interval ($k_{\parallel}$, $k_{\parallel} + dk_{\parallel}$) in a uniform plasma with magnetic field B_z is /2/

$$- \frac{dE}{dt} = \frac{e^2}{\pi\omega} \, dk_{\parallel} \, \mathrm{Im} \int \sum_{m,\alpha,\beta} \frac{D_{\alpha\beta}}{D} \, .$$

$$. \prod_{\alpha,\beta}^{m} \delta(\omega - m\Omega - k_{\parallel} v_{\parallel}) \, k_{\perp} \, dk_{\perp} \qquad (1)$$

where

$$\prod_{\alpha,\beta}^{m} \equiv \begin{pmatrix} m^2\Omega^2 J_m^2/k_{\perp}^2 & i m\Omega v_{\perp} J_m J_m'/k_{\perp} & m\Omega v_{\parallel} J_m^2/k_{\perp} \\ -i m\Omega v_{\perp} J_m J_m'/k_{\perp} & v_{\perp}^2 J_m'^2 & -i v_{\parallel} v_{\perp} J_m J_m' \\ m\Omega v_{\parallel} J_m^2/k_{\perp} & i v_{\parallel} v_{\perp} J_m J_m' & v_{\parallel}^2 J_m^2 \end{pmatrix}$$

The argument of the Bessel functions is $k_{\perp}v_{\perp}/\Omega$; $v_{\parallel}$ and $v_{\perp}$ are the components of the test electron velocity; Ω is the electron cyclotron frequency. The plasma electron velocity distribution is assumed to be Maxwellian with thermal speed v_t:

$$f = \frac{n_e}{\pi^{3/2} v_t^3} \exp\{-(v_{\parallel}^2 + v_{\perp}^2)/v_t^2\} \, .$$

By averaging the energy emitted by a test electron over the distribution function f one gets

$$\int f \, \Pi^{m}_{\alpha,\beta} \, \delta(\omega - m\Omega - k_{\parallel} v_{\parallel}) \, d\underline{v} =$$

$$= \frac{-2 n_e}{\pi^{1/2} k_{\parallel} v_t} \int_0^\infty x e^{-x^2} \left[e^{-v_{\parallel}^2} \Pi^{m}_{\alpha,\beta} \right]_{v_{\parallel} = (\omega - m\Omega)/k_{\parallel} v_t} dx .$$

Note now that the components of the conductivity tensor $\sigma_{\alpha\beta}$ are

$$\sigma_{\alpha\beta} = -i \frac{\omega_p^2}{\pi k_{\parallel} v_t} \int_0^\infty x e^{-x^2} \sum_m S^m_{\alpha\beta} \, dx ,$$

, where

$$S^m_{\alpha\beta} \equiv \begin{pmatrix} \frac{m^2}{k_\perp^2 \rho^2} J_m^2 Z_m & \frac{i m}{k_\perp \rho} x J_m J'_m Z_m & \frac{m}{2 k_\perp \rho} J_m^2 Z'_m \\ -\frac{i m}{k_\perp \rho} x J_m J'_m Z_m & x^2 J'^2_m Z_m & \frac{i x}{2} J_m J'_m Z'_m \\ \frac{m}{2 k_\perp \rho} J_m^2 Z'_m & \frac{-i x}{2} J'_m J_m Z'_m & \frac{m - \omega/\Omega}{2 k_{\parallel} \rho} J_m^2 Z'_m \end{pmatrix}$$

and $Z_m \equiv Z\left(\frac{\omega - m\Omega}{k_{\parallel} v_t}\right)$ is the plasma dispersion function.

A comparison with eq. (1) shows that

$$\sum_m \iint f\, \Pi_{\alpha,\beta}^{m}\, \delta(\omega - m\Omega - k_{\parallel} v_{\parallel})\, d\underline{v} =$$

$$= - \frac{4\pi e^2 v_t}{m_e} \left(\sigma_{\alpha\beta} + \sigma^*_{\beta\alpha} \right). \tag{2}$$

The power radiated by the test electrons can then be written

$$- \frac{dE}{dt} = - \frac{m_e v_t^2}{4\pi^2 \omega} \, \mathrm{Im} \int \sum_{\alpha\beta} \frac{D_{\alpha\beta}}{D} \left(\sigma_{\alpha\beta} + \sigma^*_{\beta\alpha} \right) k_\perp \, dk_\perp .$$

Since the dielectric tensor is defined by $\underline{\underline{\varepsilon}} = 1 - \frac{4\pi i}{\omega} \underline{\underline{\sigma}}$,

it follows from eq. (1) that

$$- \frac{dE}{dt} = \frac{m_e v_t^2}{16\pi^3} \, \mathrm{Re} \int \sum_{\alpha\beta} \frac{D_{\alpha\beta}}{D} \left(d_{\alpha\beta} - d^*_{\beta\alpha} \right) k_\perp \, dk_\perp .$$

Let us now introduce the integration variable $\mu = k_\perp^2 \rho^2 / 2$:

$$- \frac{dE}{dt} = \frac{u c^2}{8\pi^2 \rho^2 \omega^2} \, \mathrm{Re} \int \sum_{\alpha\beta} \frac{D_{\alpha\beta}}{D} \left(d_{\alpha\beta} - d^*_{\beta\alpha} \right) d\mu$$

where $u = KT\omega^2/\pi c^3$ is the energy density per unit angular frequency in a black body.

In the Appendix we show that to first order in the quantities $(d_{\alpha\beta} - d^*_{\beta\alpha})$ one has

$$\sum_{\alpha\beta} D_{\alpha\beta} \left(d_{\alpha\beta} - d^*_{\beta\alpha} \right) \approx 2i \, \mathrm{Im} \, D .$$

Integration over μ can easily be performed in the limit

$|\mathrm{Im}\, D| \ll |\mathrm{Re}\, D|$:

$$-\frac{dE}{dt} = \frac{\mu c^3}{4\pi e^2 \omega^2} \sum_n \left[\frac{\mathrm{Im}\, D}{\frac{\partial}{\partial \mu} \mathrm{Re}\, D} \right]_{\mu=\mu_n} \qquad (3)$$

where μ_n is a zero of the dispersion relation $\mathrm{Re}(D)=0$, and the sum is over all the zeros. In the limit of small absorption the emitted energy is therefore the sum of the contributions of the single modes.

Let us assume that the plasma is a slab with constant density and width L; the energy which arrives at the boundary is then

$$E_p = \sum_n E_{pn} = \frac{\mu c^2}{8\pi \omega} \sum_n N_{\perp n} \left(1 - e^{-2k_{In} L}\right) \qquad (4)$$

where $N_{\perp n}$ is the refractive index and

$$k_{In} = \frac{1}{e\sqrt{2\mu_n}} \left[\frac{\mathrm{Im}\, D}{\frac{\partial}{\partial \mu} \mathrm{Re}\, D} \right]_{\mu=\mu_n}$$

When $|k_I L| \gg 1$, E_{pn} is the expression one would obtain for a black body.

Section 3

Let us now consider an electron distribution function with more supra-thermal electrons than a Maxwellian which can be described as the sum of two Maxwellians:

$$f = f^{(o)} + f^{(1)} = \frac{n_e}{\pi^{3/2} v_o^3} e^{-|v|^2/v_o^2} + \frac{n_1}{\pi^{3/2} v_1^3} e^{-|v|^2/v_1^2}$$

with $v_1 >> v_o$, $n_1 v_1^2 << n_o v_o^2$.

The conductivity tensor $\sigma_{\alpha\beta}$ is the sum of the tensors corresponding to $f^{(o)}$ and $f^{(1)}$. The determinant D of the Maxwell equations is of course not the sum of the determinants $D^{(o)}$ and $D^{(1)}$ corresponding to $f^{(o)}$ and $f^{(1)}$. When, however, in the evaluation of the imaginary part of D in the thermodynamic limit of small absorption we only retain the terms linear in the anti-Hermitian parts of the $d_{\alpha\beta}$'s, the function Im(D) is the sum of two terms, proportional to $(d^{(o)}_{\alpha\beta} - d^{(o)*}_{\beta\alpha}$ and $(d_{\alpha\beta} - d_{\beta\alpha}{}^*)$ respectively; we write Im(D) $\sim D^o_I + D^1_I$ (D^1_I also depends on $d^{(o)}_{\alpha\beta}$). For the explicit expression see Appendix.
The energy emitted by the gyrating electrons can be evaluated in the same way as in Sect. 2. It is easily seen that the functions $T\,\sigma_{\alpha\beta}$ which appear in eq. (2) have now to be replaced by $T_o\,\sigma^{(o)}_{\alpha\beta} + T_1 \sigma^{(1)}_{\alpha\beta}$.
It can then be shown that the expression corresponding to $\Sigma_{\alpha\beta}\, D_{\alpha\beta}\, (d_{\alpha\beta} - d^*_{\beta\alpha})$ (in the Appendix it is denoted by $\Sigma_{\alpha\beta}\, D_{\alpha\beta}\, (\bar{d}_{\alpha\beta} - \bar{d}^*_{\beta\alpha})$) is now equal to

$$2i\,(T_o D^o_I + T_1 D^1_I).$$

In conclusion, the energy emitted per unit time, unit angular frequency and unit volume is

$$-\frac{dE}{dt} = \frac{\mu c^3}{4\pi^2 \rho^2 \omega^2} \, \mathrm{Im} \int \frac{D_I^0 + T_1 D_I^1 / T_0}{D} \, d\mu .$$

In the limit of small absorption integration over μ yields

$$E_p = \frac{\mu c^3}{4\pi \rho^2 \omega^2} \sum_n \left[\frac{D_I^0 + T_1 D_I^1 / T_0}{D_\mu} \right]_{\mu = \mu_n}$$

where μ_n is defined by $D_R(\mu_n) = 0$, and $D_\mu \equiv \frac{\partial}{\partial \mu} D_R$.

The energy which arrives at the boundary of a homogeneous slab can be derived in the same way as in the previous section. The counterpart of eq. (4) is

$$E_p = \frac{\mu c^3}{8\pi \omega^2} \sum_n k_{In} \frac{D_I^0 + T_1 D_I^1 / T_0}{D_I^0 + D_I^1} \left(1 - e^{-2 k_{In} L}\right) \quad (5)$$

with $$k_I = \frac{1}{\rho \sqrt{2 \mu_n}} \left[\frac{D_I^0 + D_I^1}{D_\mu} \right]_{\mu = \mu_n} \quad (6)$$

The dispersion properties of the waves corresponding to the $\mu_n^{'}$s are those of the "cold" electrons because we have assumed that $n_1 << n_e$ and therefore $Re(D\ \ ) \sim Re(D^{(o)}\ \ )$. The form of the radiation spectrum, however, is mainly determined by the population which preferentially absorbs a given mode μ_n because the corresponding imaginary part of the dispersion relation is the largest one. From eq. (5) it follows that the "cold" electrons determine the absorption when $T_o D_I^o >> T_1 D_I^1$ (7); in the limit $k_I L >> 1$ E_p is then proportional to the black body radiation with temperature T_o. Eq. (7) is satisfied when the energy of the "resonant" particles of $f^{(o)}$ is much larger than that of $f^{(1)}$ (the number of "resonant" particles is weighted with the relative phase of their perpendicular motion with respect to the phase of the electric field).

The other extreme is attained when the number of "resonant" hot electrons is much larger than the number of "resonant" cold electrons; this situation can occur, although we have assumed that $n_e T_o >> n_1 T_1$.

Then, always in the limit $k_I L >> 1$, E_p attains the radiation of a black-body with temperature T_1 and appears to be anomalously large for a plasma with temperature T_o .

In the following section we determine when this occurs.

Section 4

The interaction of the electrons with the electromagnetic field is particularly large when the perpendicular wave-length or a multiple thereof (for angular frequencies $\omega \sim m\ \Omega$) is comparable with the Larmor radius. One then has $N_\perp \sim c/v_t$ and if $c^2 >> v_t^{\ 2}$ a solution of the dispersion relation is (almost) electrostatic. The approximate dispersion relation for the electrostatic waves is

$$D \equiv 1 + \frac{\omega_p^2}{\Omega^2 k^2 \rho^2}\left[1 + \frac{\nu}{k_{\parallel}\rho}\sum_{n=-\infty}^{\infty} Z\left(\frac{\nu - n}{k_{\parallel}\rho}\right) e^{-\mu} I_n(\mu)\right] +$$

$$+ \frac{\omega_p^2}{\Omega^2 k^2 \rho^2}\frac{n_1 T_0}{n_e T_1}\left[1 + \frac{\nu}{k_{\parallel}\rho}\left(\frac{T_0}{T_1}\right)^{1/2}\sum_{n=-\infty}^{\infty} Z\left(\frac{\nu - n}{k_{\parallel}\rho}\left(\frac{T_0}{T_1}\right)^{1/2}\right) \cdot\right.$$

$$\left. \cdot\, e^{-\mu T_0/T_1} I_n(\mu T_0/T_1)\right] = 0, \qquad (8)$$

where Z is the "plasma dispersion function", $\nu \equiv \omega/\Omega$ ($m \leqq \nu \leqq m + \underline{1}$) and ω_p^2, ρ , μ refer to the distribution function $f^{(o)}$.

If one wants the effect of the harmonics $n \neq m$ upon the m - th to be negligible, one must have $k_{\parallel}\rho \ll 1$ because then

$$\left| Z\left(\frac{\nu - n}{k_{\parallel}\rho}\right) \Big/ Z\left(\frac{1}{k_{\parallel}\rho}\right)\right| \approx \frac{1}{k_{\parallel}\rho}\left| Z\left(\frac{\nu - n}{k_{\parallel}\rho}\right)\right| \gg 1 .$$

In this case one gets

$$D_R = 1 + \frac{\omega_p^2}{\Omega^2 k^2 \rho^2}\left(1 + \frac{\nu}{k_{\parallel}\rho} Z_R\, I_n(\mu)\right) +$$

$$\frac{\omega_p^2}{\Omega^2 k^2 \rho^2}\frac{n_1 T_0}{n_0 T_1}\left(1 + \frac{\nu}{k_{\parallel}\rho}\left(\frac{T_0}{T_1}\right)^{1/2} Z_R\, I_n(\mu T_1/T_0)\right)$$

$$\approx D_R^{\circ} \qquad (9)$$

$$D_I = \frac{\omega_p^2}{\Omega^2 k^2 \rho^2} \frac{\nu}{k_{\parallel}\rho} e^{-[(\nu-m)/k_{\parallel}\rho]^2} \Gamma_m(\mu)$$

$$+ \frac{\omega_p^2}{\Omega^2 k^2 \rho^2} \frac{\nu}{k_{\parallel}\rho} \frac{n_1}{n_e} \left(\frac{T_0}{T_1}\right)^{3/2} \cdot \tag{10}$$

$$\cdot \exp\left[-\left(\frac{\nu-m}{k_{\parallel}\rho}\right)^2 \frac{T_0}{T_1}\right] \Gamma_m(\mu T_1/T_0)$$

Some known properties of the solutions of eq. (9) will be useful in the following discussion.

The first is that when $\omega_p^2 \leqq \nu^2\Omega^2$ the real solution $\mu(\nu)$ is contained in the region $o < \nu - m \leqq \overline{\Delta\nu} < 1$, where $\overline{\Delta\nu}$ is defined by the equations

$$\begin{cases} D_R^o = 0 \\ D_\mu^o = 0 \end{cases} \tag{11}$$

When $\overline{\Delta\nu} << 1$ one can use the approximation (9) for D; one then gets

$$\left(\mu + \frac{\omega_p^2}{\Omega^2}\right)\Gamma_m' = \Gamma_m \tag{12}$$

The known approximations to the function Γ_m (see /16/) yield

$$\frac{\Gamma_m'}{\Gamma_m} \approx \frac{m^2}{2\mu^2} - \frac{1}{2\mu} \qquad (\mu >> m).$$

Hence eq. (12) has the solution $\mu \sim \frac{m^2}{3} + \frac{2}{3}\frac{\omega_p^2}{\Omega^2}$ which is independent of $k_{\parallel}\rho$.

With this value for μ and the asymptotic formula for Γ_m:

$$\Gamma_m \approx e^{-m^2/2\mu} (2\pi\mu)^{-1/2} \qquad (13)$$

eq. (9) yields

$$Z \approx - m \frac{k_{\parallel}\rho}{\sqrt{2\pi\mu}} e^{-m^2/2\mu} \frac{\omega_p^2/\Omega^2}{\mu + \omega_p^2/\Omega^2} \qquad (14)$$

and for $\overline{\Delta\nu} \gg k_{\parallel}\rho$:

$$\overline{\Delta\nu} \approx e^{-m^2/2\mu} \frac{m}{\sqrt{2\pi\mu}} \frac{\omega_p^2/\Omega^2}{\mu + \omega_p^2/\Omega^2}$$

$$\approx 0{,}45 \frac{\omega_p^2/\Omega^2}{m^2 + 5\omega_p^2/\Omega^2} .$$

The other property we shall need, is that whereas for $k_{\parallel}\rho = 0$ the real curves $\mu(\nu)$ exist for every n, are open and go to zero and to infinity for $\nu \to m$, for $k_{\parallel}\rho \neq 0$ they are closed. When $k_{\parallel}\rho$ increases, the maximum and the minimum are squeezed together, and when $k_{\parallel}\rho$ is larger than some sufficiently large value $\overline{k_{\parallel}\rho}$ there is no real solution, for a given m. Since at the extrema one has

$$D_\nu \equiv \frac{\partial}{\partial\nu} D_R = 0 \qquad (15)$$

the value $\overline{k_{\parallel}\rho}$ can be obtained as the solution of eq. (15),

together with the equations D = 0 and D_μ = 0. When $\Delta\nu << 1$ the equation $D_\nu = 0$ is equivalent to $\frac{\partial}{\partial\nu}(\nu Z) = 0$ or $\frac{\Delta\nu}{k_{\parallel}\rho} \sim 1.5$. (16)
With this value we get from eq. (14)

$$\overline{k_{\parallel}\rho} = 0.39\,\frac{\omega_p^2/\Omega^2}{m^2 + 5\,\omega_p^2/\Omega^2}$$

This equation can be written as

$$m^3 = m_o^3 \equiv \frac{0.4}{N_{\parallel}}\,\frac{c}{v_t}\,\frac{\omega_p^2}{\Omega^2} \qquad (17)$$

m_o is then the largest harmonic number with real electrostatic solutions for a given $N_{\parallel}$.

We are now in a position to discuss the conditions necessary for an anomalously high emission; as we have seen in Sect. 3 they are

$$\begin{cases} D_I^1 \gg D_I^o \\ k_I L \gg 1 \end{cases} \qquad (18)$$

where k_I is given by eq. (6).

Let us begin with the interval of (normalized) angular frequencies such that $\mu(\nu) >> m$ (remember that at $\nu = m + \overline{\Delta\nu}$ one has $\mu \sim m^2/3$). On the outer branch of the curve $\mu(\nu)$, away from the extrema, the dispersion relation does not depend on $k_{\parallel}\rho$; an important consequence is that for a given $\Delta\nu$ and the corresponding value of μ the functions D_I^o and D_I^1 strongly depend on the parameter $k_{\parallel}\rho$. In fact eqs. (18) can be written

$$\begin{cases} \exp\left[-\left(\frac{\Delta\nu}{k_{\parallel}\rho}\right)^2 (1-T_0/T_1)\right] < e^{-1.5}\frac{n_1}{n_e}\left(\frac{T_0}{T_1}\right)^{3/2} \\ \exp\left[-\left(\frac{\Delta\nu}{k_{\parallel}\rho}\right)^2 T_0/T_1\right] > \frac{\rho}{L}\frac{m^3}{m_0^3}\frac{n_e}{n_1}\left(\frac{T_1}{T_0}\right)^2 \end{cases}$$

or

$$\begin{cases} \left(\frac{\Delta\nu}{k_{\parallel}\rho}\right)^2 > \left[-1.5+\ln\left(\frac{n_e}{n_1}\left(\frac{T_1}{T_0}\right)^2\right)\right](1-T_0/T_1)^{-1} \\ \left(\frac{\Delta\nu}{k_{\parallel}\rho}\right)^2 < \frac{T_1}{T_0}\ln\left(\frac{L}{\rho}\frac{m_0^3}{m^3}\frac{n_1}{n_e}\left(\frac{T_0}{T_1}\right)^2\right) \end{cases} \qquad (19)$$

In the same way as we have deduced $m < m_o$ from eq. (16) we now get

$$m_0\,\beta_2^{-1/3} < m < m_0\,\beta_1^{-1/3} \qquad (20)$$

where

$$\begin{cases} \beta_1^2 = \dfrac{-1.5+\ln\left(\frac{n_e}{n_1}\left(\frac{T_1}{T_0}\right)^2\right)}{2.25\,(1-T_0/T_1)} \\ \beta_2^2 = \dfrac{T_1}{2.25\,T_0}\ln\left(\frac{L}{\rho}\frac{m_0^3}{m^3}\frac{n_1}{n_e}\left(\frac{T_0}{T_1}\right)^2\right) \end{cases} \qquad (21)$$

A necessary condition for the existence of radiation at the level T_1 follows from eq. (20), namely $\beta_1 < \beta_2$. From the definition of $\beta_{1,2}$ it then follows that it must be approximately

$$\frac{L}{\rho}\frac{m_0^3}{m^3}\frac{n_1}{n_e}\left(\frac{T_0}{T_1}\right)^2 > 1 \qquad (22)$$

This equation shows that there is a minimum value for the density of the "hot" electrons, which is inversely proportional to $N_{\parallel}$; hence the angle within which the radiation is anomalously high becomes smaller when the density of the hot electrons decreases. For T_1/T_o = 10 and $m = m_o$ eq. (22) yields

$$\frac{n_1}{n_e} > 10^2 \rho / L\, m_o .$$

If we now consider frequencies such that $\mu(\nu) \lesssim m$, the condition $D_I^o \ll D_I^1$ is easier to satisfy; on the other hand, the value of $k_I L$ is increasingly determined by the value of the function $\Gamma_m(\mu T_1/T_o)$, which rapidly becomes small when $\mu T_1/T_o \lesssim m$.

Numerical calculations are necessary in this frequency range. However, if we do not look for the form of the emitted line, the domain $\mu(\nu) \gg m$ we have previously investigated is enough for a comparison with the experiments. In fact, if anomalous emission is not possible at $\Delta\nu = 0(\overline{\Delta\nu})$, it will not be possible at $\Delta\nu \ll \overline{\Delta\nu}$ either. On the other hand, if it is possible there, it will extend on the side of lower frequencies, giving rise to a line whose amplitude goes to zero shortly before $\Delta\nu = 0$. The results of our discussion can be summarized as follows:

1) when $m > m_o$ no electrostatic waves propagate;

2) the width of the lines decreases as m^{-2} ;

3) the radiation is "black-body" with temperature T_1 when

$$m_o \beta_2^{-1/3} < m < m_o \beta_1^{-1/3}$$

4) when $m < m_o \ \beta_2^{-1/3}$ the radiation is proportional to $m^2 k_I$, i.e. it does not depend on m.

Section 5

If one compares eqs. (18) with the results of the model proposed some time ago for the "anomalous" emission, an interesting and important difference appears. In the previous model the parallel velocity distribution had no bearing; the emitted "anomalous" radiation was simply proportional to the density of the suprathermal electrons, without upper limit (except that the real part of the dispersion relation was essentially determined by the "thermal" electrons); self-absorption was neglected. On the contrary, eqs. (18) show that the parallel velocity distribution is essential (through the factor $e^{-(\Delta\nu/k_{\parallel}e)^2}$), whereas the perpendicular velocity distribution only appears in an integrated form (through the functions $\Gamma_m \propto \int_0^\infty J_m^2 \ (v_\perp k_\perp/\Omega) e^{-v_\perp^2/T} dv_\perp^2$ which take into account the phase of the perpendicular motion with respect to the phase of the electric field). A two-temperature electron distribution function $f^{(1)}$ would show a much larger sensitivity of the emission to a change of the "parallel" temperature than of the "perpendicular" temperature.

Only a small fraction of the emitted energy leaves the plasma, as we shall see in Sect. (6); in a slab model where only the density depends (weakly) on x, the rest is first reflected from the surface where the group velocity is zero and is then absorbed by the same population which emitted it. When also an x-dependence of the magnetic field is allowed for, the radiation at temperature

T_1 emitted in the direction of decreasing B_z is absorbed mainly by the "hot" electrons, whereas the radiation emitted in the opposite direction is absorbed by the "cold" electrons (this effect is discussed for the first two harmonics in /10/, but with opposite directions, since the authors consider extraordinary transverse waves). In fact, owing to the magnetic field gradient, a displacement δx of the wave front leads to a change $\frac{\delta\nu}{\nu} = \frac{\delta x}{R}$ in the normalized angular frequency, if we assume $B_Z = B_o\,(1-x/R)$. As we have seen in Sec. (4), the condition $D_I^o \ll D_I^1$ (which ensures that the electrons which preferentially emit and absorb energy are the "hot" ones) is only satisfied for $\Delta\nu > \beta_1 k_{\shortparallel} \rho$; hence, after a displacement δx in the direction of increasing $|B|$, with $\frac{\delta}{R} = \frac{(m+\overline{\Delta\nu})-(1\ n+\beta_1 k_{\shortparallel}\rho)}{m+\Delta\nu}$ the wave is in a region where $D_I^o \gg D_I^1$, and the energy is absorbed by the "cold" electrons. With definition (15) for $\overline{\Delta\nu}$, one gets approximately

$$\frac{\delta x}{R} \approx \frac{0.4}{m^3}\,\frac{\omega_p^2}{\Omega^2} \qquad (23)$$

The density gradient does not lead to such an effect because for $\mu \sim m^2$ one has

$$\frac{D_I^o}{D_I^1} \approx \left(\frac{T_1}{T_o}\right)^2 \frac{n_e}{n_1}\, \exp\left[-\left(\frac{\Delta\nu}{k_{\shortparallel}\rho}\right)^2 (1-T_o/T_1)\right],$$

and a small change of n_e, n_1 cannot make $D_I^o \gg D_I^1$.

It is interesting to compare the energy transfer from the "hot" to the "cold" electrons, due to the radiation, with the transfer due to the collisions. From eqs. (5) and (20) one can estimate the total energy per particle, per unit time emitted at the level T_1

$$\overline{\mathcal{E}} \approx \frac{1}{12\pi} \int \frac{c u N_\perp}{k T_1 e n_e} \frac{m_0^3}{m^3} \left(\frac{T_0}{T_1}\right)^2 d\omega \, dN_\parallel$$

$$\approx \frac{1}{12\pi} \int \sum_n \frac{c u \Omega \overline{\Delta\nu}}{k T_1} \frac{N_\perp}{n_e e} \frac{m_0^3}{m^3} \left(\frac{T_0}{T_1}\right)^2 dN_\parallel$$

(we assume a line-width $\overline{\Delta\nu}$). The largest possible value of $N_\parallel$ is the one which makes $m_o \beta_2^{-1/3}$ equal to one ; one gets $N_\parallel \leqq \beta_2^{-1} \, 0.4 \, \frac{\omega_p^2}{\Omega^2} \frac{c}{v_1}$. It thus follows that

$$\overline{\mathcal{E}} \approx 1.2 \cdot 10^5 \, n_e \, T_0^{-5/6} \, T_1^{-2/3} \qquad [keV, 10^{13} cm^{-3}]$$

In the same units the equipartition time is

$$t_{eq} \approx \frac{0.54}{\ln \Lambda} 10^{-3} \, T_1^{3/2} \, n_e^{-1}$$

It thus follows that

$$t_{eq} \, \overline{\mathcal{E}} \approx \frac{7}{\ln \Lambda} (T_1 / T_0)^{5/6} \qquad (25)$$

The estimated effect of the radiation is therefore of the same order of magnitude as the effect of the collisions.

Section 6

In the preceding sections we have evaluated the energy lost by gyrating electrons in a homogeneous plasma. The coupling of the electrostatic waves with waves which can propagate into vacuum has often been discussed. An accepted view is that the coupling is possible in the neighbourhood of the upper-hybrid resonance; i.e. in a density region where $\omega^2 \sim \omega_p^2 + \Omega^2$, or $\nu^2 \sim 1 + \omega_p^2/\Omega^2$. In experiments where the magnetic field is varied, this condition can also be satisfied for relatively large values of ν (~20).

In the usual "fusion" experiments, however, ω_p^2/Ω^2 is smaller than, or of the order of one ($\omega_p^2/\Omega^2 = 1 \cdot n_e B^{-2}$ [10^{13} cm^{-3}, Tesla]) and coupling at the upper-hybrid is no longer possible.

With this exclusion, the only region where the coupling can take place is where $|d\lambda/dx| \gg 1$, i.e. in the neighbourhood of a region where the group velocity is equal to zero ("turning point"). The "turning points" correspond to the frequencies $\nu = m + \overline{\Delta\nu}$ and to wave numbers k_x such that $\mu \sim m^2/3$, as has been shown in Sec. 4. They only exist where $\omega_p^2/\Omega^2 < \nu^2$, i.e. where coupling at the "upper hybrid" is not possible.

Here we do not give a theory of this coupling; we only deduce from physical arguments an approximation of the coupling coefficient (several recent papers [12 - 15] have been published on related arguments; unfortunately, they do not apply to the case we have just described, for $1 < m \leqq m_o$.). The slab where $|d\lambda/dx| \gg 1$ can be thought of as a surface separating two media with different dispersion properties, since its extent is smaller than or only comparable with the incident and transmitted λ's.

In the first medium three reflected waves besides a quasi-longitudinal incident wave can propagate, namely two quasi-longitudinal waves corresponding to the two real solutions of the dispersion relation (8) for a given value of ν ; and a quasi-transversal wave, with a refractive index of the order of one. The transmitted mode is the same as the reflected quasi-transversal one; but, of course, with a k_x of opposite sign. The components of

$\underline{E}$ and $\underline{B}$ parallel to the boundary plane must be continuous through the boundary, i.e.

$$\begin{cases} [E_z] = 0 \\ [E_y] = 0 \end{cases} \qquad \begin{cases} [N_x E_y] = 0 \\ [N_{\parallel} E_x - N_x E_z] = 0 \end{cases} \tag{26}$$

must be fulfilled.

These four conditions uniquely determine the reflected and transmitted waves.

The different modes will be designated by an upper index: zero for the incident wave; one for the reflected wave of the same kind as the incident one; two for the second quasi-longitudinal reflected wave; three for the quasi-transversal reflected wave; four for the transmitted wave.

Let us introduce moreover the polarizations:

$$P^i \equiv E_y^i / E_z^i \quad ; \quad Q^i \equiv E_x^i / E_z^i \qquad (i = 0, 1, 2, 3, 4)$$

From Maxwell equations one gets

$$\begin{aligned} P^i &\approx \varepsilon_{12} \, (N_x N_{\parallel} - \varepsilon_{13})^{-1} \\ Q^i &\approx N_x^2 \, (N_x N_{\parallel} - \varepsilon_{13})^{-1} \end{aligned} \tag{27}$$

Equations (26) can now be written

$$\begin{cases} [\Sigma_i E_z^i] = 0 \\ [\Sigma_i \varepsilon_{12}^i E_z^i] = 0 \end{cases} \qquad \begin{cases} [\Sigma_i P^i E_z^i] = 0 \\ [\Sigma_i N_x^i P^i E_z^i] = 0 \end{cases} \tag{28}$$

Note that mode 2 has a negative group velocity; the wave vector of the reflected wave is therefore directed toward the boundary plane.

As already stated, we consider the slab where $|d\lambda/dz| \gg 1$ as a plane separating two media with different dielectric properties. If this slab is reduced to the plane where the group velocity is zero, the mode 0 is totally reflected. With our assumption, which takes into account that wave coupling is possible in the whole slab where $|d\lambda/dx| \gg 1$, the modes 0 and 2 are characterized by different wave numbers, the difference being proportional to the thickness of the boundary layer. More precisely, $\Delta\mu \equiv \mu^{(2)} - \mu^{(o)}$ is defined by the condition that $|d\lambda/dx| \gg 1$ be fulfilled. It is not difficult to link this condition with the dispersion relation; it holds in fact that

$$\frac{d\lambda}{dx} = \frac{\pi}{2}\frac{\rho}{\mu^{3/2} D_\mu}\left(\frac{\partial D}{\partial n_e}\frac{dn_e}{dx} + \frac{\partial D}{\partial \nu}\frac{d\nu}{dx}\right) \qquad (29)$$

where $\frac{d\nu}{dx} = \frac{\nu}{\Omega}\frac{1}{R}$, since we assume $B = B_o(1 - \frac{X}{R})$.

In the neighbourhood of $D_\mu = 0$ one can write $D_\mu \sim \Delta\mu\, D_{\mu\mu}$;
one has $|d\lambda/dx| > 1$ when $\Delta\mu < \frac{\rho}{\mu^{3/2}}\frac{1}{D_{\mu\mu}}\frac{dD}{dx}$.

By taking the dispersion relation into account one finally gets

$$\frac{\Delta\mu}{\mu} < \frac{\rho}{8m}\left(\frac{n_e'}{n_e} - \frac{m}{R\Delta\nu}\right) \qquad (30)$$

Since $\Delta\nu \sim \overline{\Delta\nu}$ is proportional to m^{-2}, at the higher harmonics the second term dominates; it has the following upper limit:

$$\frac{\rho}{R\Delta\nu} = \frac{k_\parallel \rho}{k_\parallel R\Delta\nu} \leq \frac{1}{\beta_1 k_\parallel R} < \frac{1}{\beta_1}.$$

The transmission coefficient is proportional to $\Delta\mu$. In fact, the transmitted energy can be written as

$$A|E_z^{o}|^2 - A|E_z^{1}|^2 - (A+\delta A)|E_z^{(2)}|^2 - B|E_z^{(3)}|^2 \qquad (31)$$

A and B being functions of the dielectric tensor directly, and indirectly through the polarization constants P^i and Q^i;

$$\delta A \equiv A(\mu^{(2)}) - A(\mu^{(0)}).$$

It is easily seen from eqs. (24) that $E^{1}/^{3}$ are proportional to $\Delta\mu$; it follows that the transmission coefficient is approximately (to first order in $\Delta\mu$)

$$T = 1 - \frac{\delta A}{A} - \frac{|E_z^{(2)}|^2}{|E_z^{(0)}|^2} \qquad (32)$$

Since

$$A \approx \frac{|E_x|^2}{|E_z|^2} \, \mathcal{I}m \, \varepsilon_{11} \approx \frac{|E_x|^2}{|E_z|^2} D_I$$

one gets (see eqs. (10) and (12)):

$$\frac{\delta A}{A} \approx -\frac{1}{2}\frac{\Delta\mu}{\mu}$$

The value of $1 - |E_z^{(2)}/E_z^{o}|^2$ follows from eqs. (24):

$$\frac{2\Delta\mu}{\mu} N_{\parallel} \frac{-1 + \varepsilon_{12}^{(4)} / \varepsilon_{12}^{(0)}}{0.5 \cdot (1 + \varepsilon_{12}^{(4)} / \varepsilon_{12}^{(0)}) + m^2/2\mu^2}$$

which is much smaller than $\Delta\mu/\mu$. Hence

$$T \approx \frac{1}{2} \frac{\Delta\mu}{\mu} \qquad (33)$$

and finally, from eq. (25)

$$T \sim \frac{1}{16} \frac{\rho}{m} \left(\frac{n_e'}{n_e} - \frac{m}{R \Delta \nu} \right) \qquad (34)$$

The energy flow from a plasma slab, in the case when the radiation in the plasma is black-body at the temperature T_1 and is due to electrostatic waves, follows from eq. (5) and (26). One gets

$$E \approx \frac{K T_1 \omega^2}{\pi^2 c^2} \frac{N_{\perp}}{16} \frac{\rho}{m} \left(\frac{n_e'}{n_e} - \frac{m}{R \Delta \nu} \right) \qquad (35)$$

Conclusions

We have evaluated the radiation emitted by the electrons of a plasma with more suprathermal electrons than a Maxwellian one. The electron distribution function has been described as the sum of two Maxwellians; one describing the bulk of the plasma, the other the suprathermal electrons. Although we have assumed that the total energy of the suprathermal electrons is much smaller than the total energy of the bulk of the plasma, the emitted lines attain in the plasma N_1 times the black-body radiation at the temperature T_1, for some harmonic numbers up to a value $m_o \sim \beta_1^{-1/3}\left(0{,}4\,\frac{c}{N_{\parallel}v}\,\frac{\omega_p^2}{\Omega^2}\right)^{1/3}$.
The energy, which in this case is emitted by the suprathermal electrons, is partly absorbed by the bulk of the electrons, owing to the magnetic field gradient in a toroidal configuration.
This "equipartition" effect has been compared with the effect due to the collisions. The two are of the same order of magnitude, and for large (T_1/T_o) the radiation dominates. This result can be of interest for the electron cyclotron heating of a plasma.
Finally, we have evaluated the coupling of the emitted waves which are almost longitudinal, with the waves which go adiabatically into vacuum. The coupling at the upper-hybrid is not possible at higher harmonic numbers in "fusion" plasmas. The mechanism we have proposed takes into account the non-adiabaticity of the wave propagation in the neighbourhood of the region where the group velocity is zero.

Appendix

Let us write $(d_{\alpha\beta})$ as

$$\begin{pmatrix} L_1+l_1 \\ L_2+l_2 \\ L_3+l_3 \end{pmatrix}$$

where L_i , l_i (i=1,2,3) are three-component vectors; L_i are the Hermitian parts, l_i the anti-Hermitian, with $|l_i| << |L_i|$ for every component. Up to first order in $|l_i|$ one has

$$\|d_{\alpha\beta}\| = \begin{vmatrix} L_1+l_1 \\ L_2+l_2 \\ L_3+l_3 \end{vmatrix} \approx \begin{vmatrix} L_1 \\ L_2 \\ L_3 \end{vmatrix} + \begin{vmatrix} l_1 \\ L_2 \\ L_3 \end{vmatrix} +$$

$$+ \begin{vmatrix} L_1 \\ l_2 \\ L_3 \end{vmatrix} + \begin{vmatrix} L_1 \\ L_2 \\ l_3 \end{vmatrix} \qquad (1)$$

The first determinant is real, the others are imaginary.

The sum

$$\Sigma_\alpha D_{\alpha\beta} \left(d_{\alpha\beta} - d^*_{\beta\alpha} \right)$$

is the determinant of the matrix which is obtained by replacing the β-th row of $(d_{\alpha\beta})$ with its anti-Hermitian part; the sum over α and β is therefore

$$2\begin{vmatrix} l_1 \\ L_2+l_2 \\ L_3+l_3 \end{vmatrix} + 2\begin{vmatrix} L_1+l_1 \\ l_2 \\ L_3+l_3 \end{vmatrix} + 2\begin{vmatrix} L_1+l_1 \\ L_2+l_2 \\ l_3 \end{vmatrix} \qquad (2)$$

From this equation it is easily seen that to the first order in $|1_i|$ one has

$$\sum_{\alpha\beta} \sim 2i \, \mathrm{Im}\, D$$

When the distribution function is the sum of two Maxwellians one has $\sigma_{\alpha\beta} \equiv \sigma_{\alpha\beta}^{(o)} + \sigma_{\alpha\beta}^{(1)}$. The function Im D is also given in this case by eq. (1), with $L_i \equiv L_i^{(o)} + L_i^{(1)}$, $1_i \equiv 1_i^{(o)} + 1_i^{(1)}$;

Each of the three first order matrices of eq. (1) can be decomposed; e.g.

$$\begin{vmatrix} l_1 \\ L_2 \\ L_3 \end{vmatrix} = \begin{vmatrix} l_1^{(o)} \\ L_2 \\ L_3 \end{vmatrix} + \begin{vmatrix} l_1^{(1)} \\ L_2 \\ L_3 \end{vmatrix} \qquad (3)$$

Clearly one can write Im D $\sim D_I^o + D_I^1$ where D_I^o (D_I^1) is the sum of the determinants containing $1_i^{(o)}$ ($1_i^{(1)}$).

The sum

$$\sum_{\alpha\beta} D_{\alpha\beta}\left(\bar{d}_{\alpha\beta} - \bar{d}_{\beta\alpha}^{*}\right)$$

can be written as in eq. (2), where now $L_i \equiv L_i^{(o)} + L_i^{(1)}$ and $l_i \equiv T_o l_i^{(o)} + T_1 l_i^{(1)}$; e.g.

$$\begin{vmatrix} T_0 l_1^{(0)} + T_1 l_1^{(1)} \\ L_2 \\ L_3 \end{vmatrix}$$

Comparison with eq. (3) shows that

$$\sum_{\alpha\beta} \approx 2i \left(T_0 D_I^0 + T_1 D_I^1 \right) .$$

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