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On the Existence of Eigenmodes of Linear Quasi-Periodic Differential Equations and their Relation to the MHD Continuum

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Abstract

The existence of quasi-periodic eigensolutions of a linear second order ordinary differential equation with quasi-periodic coefficient $f(\omega_1 t, \omega_2 t)$ is investigated numerically and graphically. For sufficiently incommensurate frequencies ω_1, ω_2 a doubly indexed infinite sequence of eigenvalues and eigenmodes is obtained.

The equation considered is a model for the magneto-hydrodynamic "continuum" in general toroidal geometry. The result suggests that continuum modes exist at least on sufficiently irrational magnetic surfaces.

1. Introduction

For the linear second order ordinary differential equation with periodic coefficient $f(t) = f(t+\pi)$, (Hill's equation)

$$\ddot{y}(t) + [\lambda + f(t)]\, y = 0 \tag{1.1}$$

it is well known /1/ that with mild assumptions for f(t), there exists an infinite sequence of characteristic values or eigenvalues $\lambda = \lambda_n$, $n = 0,1, \ldots,$ such that the solutions $y = y_n(t)$ have the same periodicity as the coefficient f(t).

As a generalization of Hill's equation we consider the differential equation with quasi-periodic coefficient f:

$$\ddot{y}(t) + [\lambda + f(\omega_1 t, \omega_2 t)]\, y = 0 \tag{1.2}$$

where f is periodic with respect to both arguments

$$f(\theta,\phi) = f(\theta + 2\pi,\phi) = f(\theta, \phi + 2\pi) \tag{1.3}$$

but is not periodic in t in general if the ratio ω_1/ω_2 is irrational. (It is convenient here to use 2π for the period instead of π.) The purpose of the following investigation is to find out numerically - insofar as this is possible - whether eigenvalues λ_n again exist for this generalized equation. Eigenvalues are defined here by the analogous requirement that for $\lambda = \lambda_n$ the solutions $y = y_n(t)$ be quasi-periodic with the same quasi-periodicity as the coefficient f(t):

$$y_n = u(\omega_1 t, \omega_2 t) \tag{1.4}$$

with

$$u(\theta,\phi) = u(\theta + 2\pi,\phi) = u(\theta,\phi + 2\pi). \tag{1.5}$$

As we shall briefly explain below this eigenvalue problem arises naturally in the theory of the so-called MHD continuum in general toroidal geometry.

Equation (1.2) is equivalent to the partial differential equation of the parabolic type

$$(\omega_1 \frac{\partial}{\partial\theta} + \omega_2 \frac{\partial}{\partial\phi})^2 \, y\,(\theta,\phi) + \left[\lambda + f\,(\theta,\phi)\right] y = 0 \qquad (1.6)$$

with the real characteristics

$$\theta = \omega_1 t + c_1 \;, \quad \phi = \omega_2 t + c_2 \qquad (1.7)$$

with c_1, c_2 = const. A quasi-periodic eigenmode corresponds to a solution with periodic boundary conditions on the square $0 \leqq \theta \leqq 2\pi$, $0 \leqq \phi \leqq 2\pi$. While for elliptic operators the existence of such solutions is guaranteed by the theory of Sturm - Liouville, no equivalent theorems are known for parabolic equations. The existence of eigenmodes is therefore uncertain a priori.

Difficulties in analytic investigations of eq. (1.2) or (1.6) usually come from the problem of "small denominators". If $y\,(\theta,\phi)$ is Fourier analyzed in θ and ϕ the operator d^2/dt^2 corresponds to $-\,(n_1\omega_1 + n_2\omega_2)^2$ whose inverse becomes arbitrarily small if the integers n_1 and n_2 are appropriately chosen in the limit $|n_1|,|n_2| \to \infty$. This problem makes the generalization of the Floquet theory /2/ of systems of periodic differential equations to systems of quasi-periodic differential equations so difficult and incomplete /3 - 6/ . We shall discuss in Section 3 below some results relevant to us.

The paper is organized as follows: Section 2 presents a short discussion of the physical problem which prompted the investigation. For a quasi-periodic δ-function-type $f\,(\omega_1 t,\, \omega_2 t)$ equation (1.2) is transformed into a recurrence relation in Section 3, and the general features of the numerical solution /7/ together with pertinent analytic results /5/,/6/ are recalled. In Section 4 numerically obtained eigenvalues and eigensolutions are presented and discussed. Section 5 contains a critical discussion.

2. The MHD continuum in general toroidal geometry

For a plasma confined in a toroidal equilibrium configuration and described by magneto-hydrodynamic equations the linearized equations of motion may be put in the form /8/:

$$\frac{d}{d\psi} \underline{X} = \underline{\underline{A}} \cdot \underline{X} + \underline{\underline{B}} \cdot \underline{Y} \tag{4.1a}$$

$$\underline{\underline{L}} \cdot \underline{Y} = \underline{\underline{K}} \cdot \underline{X} \tag{4.1b}$$

where $\underline{X}$ and $\underline{Y}$ are vectors with two and four components, respectively and together describe the perturbed fluid motion and the perturbed magnetic field. $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{K}}$ and $\underline{\underline{L}}$ are matrix operators containing derivatives in the magnetic surfaces ψ = const. Derivatives out of the surfaces are explicitly indicated in eq . (4.1a). The operator $\underline{\underline{L}}$ is particular in that it only contains derivatives along the magnetic field lines on ψ = const. The existence of equilibria with a continuous set of nested toroidal magnetic surfaces is non-trivial in the general case but is assumed here.

If eq. (4.1b) can be solved for $\underline{Y}$ an equation for the radial variation of $\underline{X}$ is obtained. With suitable boundary conditions a set of discrete eigenvalues ω^2 may be determined, where an ansatz $\sim \exp(i\omega\tau)$ is made for the time dependence of the perturbations. If however

$$\underline{\underline{L}}(t,\omega^2,\psi)\ \underline{Y}(t) = 0 \tag{4.2}$$

has a non-trivial solution - here t is a coordinate along the field line considered - another set of eigenvalues ω^2 results directly from eq. (4.2). If $\underline{\underline{L}}$ is a continuous function of the radial coordinate ψ the same is true of the eigenvalues ω^2. Hence the name "continuum" for the eigenmodes and eigenvalues of eq. (4.2).

Equation (4.2) consists of a system of four ordinary first order differential equations along a field line. It may be written in the form

$$\dot{\underline{Y}}(t) = \underline{\underline{\Omega}}(t,\omega^2,\psi)\ \underline{Y} \tag{4.3}$$

The matrix $\underline{\underline{\Omega}}$ contains quasi-periodic functions of t since the equilibrium depends periodically on poloidal and toroidal angles θ and ϕ, respectively, which may be chosen such that the linear relations

$$\theta = B^\theta\ t + \text{const}\ , \qquad \phi = B^\phi\ t + \text{const} \tag{4.4}$$

hold along the magnetic field $\underline{B}$. Here $B^\theta = \underline{B}\cdot\nabla\theta$, $B^\phi = \underline{B}\cdot\nabla\phi$. As boundary conditions for eq. (4.3) it is required that $\underline{Y}(t)$ also be periodic in the angles θ and ϕ. The existence of such eigenmodes is, however, uncertain and the present investigation is intended to contribute to this problem.

In equilibria with cylindrical or axial symmetry or with closed field lines the matrix $\underline{\underline{\Omega}}$ becomes either constant or a periodic function of t. In these cases continuum eigenmodes exist (see e.g. /9-14/) and have been proposed for efficient local plasma heating ("Alfvén wave heating"). Their radial dependence, which in cylindrical geometry includes a logarithmic divergence, has been preliminarily discussed in /15/,/16/ for general toroidal geometry.

In the limit of low plasma pressure eq. (4.3) to lowest order reduces to

$$\frac{d}{dt}\left[a\frac{d}{dt}(by)\right] + c\omega^2 y = 0 \tag{4.5}$$

where the quasi-periodic coefficients a,b,c are defined in /17/. Clearly, equation (1.2) which we shall investigate for particular $f(\omega_1 t,\omega_2 t)$ is a simplified model equation of eq. (4.5), where λ, B^θ and B^ϕ correspond to ω^2, ω_1 and ω_2, respectively. The case $\omega^2>0$ and $q = B^\phi/B^\theta$ irrational will be considered here exclusively.

3. Numerical solution for δ-function-type coefficient

We consider eq. (1.2) in the form

$$\ddot{y}(t) + \left[\omega^2 + f(\omega_1 t,\ \omega_2 t)\right] y = 0 \tag{3.1}$$

with particular 2π-quasi-periodic functions

$$\begin{aligned} f(t) &\equiv f(\theta(t),\phi(t)) \\ &= \sum_{n=-\infty}^{+\infty} (-1)^n \left[F_1\ \delta(t-nT_1+c) + F_2\ \delta(t-nT_2)\right] \qquad (3.2) \\ &= \sum_{n=-\infty}^{+\infty} C_n\ \delta(t-t_n) \end{aligned}$$

with

$$\theta = \omega_1\cdot(t + c), \quad \phi = \omega_2 t, \quad \omega_i = \pi/T_i, \quad i = 1,2 \tag{3.3}$$

$f(t)$ is an imitation of $F_1 \cos(\omega_1 t + c) + F_2 \cos \omega_2 t$ with δ-functions. F_1 and F_2 are arbitrary amplitudes. Together with the factor $\pm$ 1 they are collectively called C_n.

Between the δ-functions the solution of eq. (3.1) is

$$y(t) = a \cos \omega t + b \sin \omega t \tag{3.4}$$

Integration across the δ-functions at $t = t_n$ yields a jump condition for the derivative $\dot{y}$, while y is continuous. From the jump conditions the recursion

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 + c_n \sin 2\omega t_n & 2c_n \sin^2 \omega t_n \\ -2c_n \cos^2 \omega t_n & 1 - c_n \sin 2\omega t_n \end{pmatrix} \cdot \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} \qquad (3.5)$$

for the amplitudes a_n, b_n for $t_n < t < t_{n+1}$ may be derived, with $c_n = C_n/(2\omega)$. The recursion allows fast and accurate numerical determination of y(t) and $\dot{y}(t)$ at multiples of T_1 and T_2. It was performed partly on an AMDAHL 470 V/6 and partly on a CRAY-1. Different word lengths on the two computers allow evaluation of round-off errors.

For each "run" not only the amplitudes F_1, F_2 and the frequencies ω, ω_1, ω_2 have to be specified but also the phase (-difference) c and the initial direction of the vector (y(o), $\dot{y}(o)/\omega$). The latter is done by specifying the coefficient r, with $o \leqq r \leqq 1$, in the ansatz

$$y(o) = \cos r\pi, \quad \dot{y}(o)/\omega = \sin r\pi \qquad (3.6)$$

The result of many such runs have been reported in /7/. Both, cases with stochastic and cases with ordered behaviour in phase space were presented and the integrability of eq. (3.1), written as a Hamiltonian system, was discussed.

Cases with unbounded solution for $t \to \infty$, i.e. $n \gg 1$, were found. Detailed investigation reveals that the solutions y(t) become unbounded if the eigenvalue parameter ω is located in a doubly indexed infinite sequence of gaps situated in the vicinity of $\omega = \omega_o$ which are defined by the resonance condition

$$n_o \, \omega_o + n_1 \, \omega_1 + n_2 \, \omega_2 = 0 \qquad (3.7)$$

with $n_o = 2$ and n_1, n_2 arbitrary integers. As functions of ω_2

the gaps become bands, see Figs. 1a, 1b. In Fig. 1a the band structure in the region $o < \omega < 2$, $o < \omega_2 < 2$ is shown for $F_1 = F_2 = 0.2$. Owing to the course grid many bands are only intermittently visible. In Fig. 1b a magnified view at increased amplitudes $F_1 = F_2 = 0.5$ shows more details. Some (n_o, n_1, n_2) triples are indicated together with the corresponding lines (3.7), for arbitrarily fixed $\omega_1 = 1$. The bands are slightly shifted from the position (3.7) owing to the finite amplitudes F_1, F_2 and they are deflected somewhat whereever they cross each other. A high growth rate corresponds to a small number in Figs. 1a, 1b. The values of ω and ω_2 are indicated at the top of the figures and in their right margin. Bands with n_1 and n_2 large are not easily detected numerically because they are very narrow and the growth rates become exceedingly small.

Equation (3.1) was investigated analytically by Dinaburg and Sinai /5/ and Ruessmann /6/ for the complementary case of smooth functions $f(\omega_1 t, \omega_2 t)$. They find that if ω_1/ω_2 is "sufficiently" irrational and ω^2 is sufficiently large the solution is of the generalized Floquet type:

$$y(t) = e^{i\nu t}\, u(\omega_1 t, \omega_2 t) + c.c. \tag{3.8}$$

where ν is real and u is 2π - quasiperiodic, provided ω is outside a doubly infinite sequence of gaps which are again situated close to $\omega = \omega_o$ as given by eq. (3.7). Since ν is real the solutions therefore are bounded outside the same set of gaps as in our case with δ-function pulses. Inside the gaps, however, the properties of the solutions were not specified in /5/,/6/ and their width cannot easily be compared with ours.

It will become clear in the next section that regarding the search for eigenmodes it would be helpful to know whether the solutions y(t) may possibly be represented in the generalized Floquet form (3.8) for all ω and ω_1, ω_2, but of course with ν

being complex in general (and the factor exp ($i\nu t$) being replaced by the more general expression exp ($i\underline{\underline{K}}t$) with constant matrix $\underline{\underline{K}}$.) In general, however, this representation is not possible, see /3/, not even for the first order equation $\dot{y} = f(\omega_1 t, \omega_2 t)\, y$ as discussed in /4/,/18/. There always tend to be restrictions on the frequencies involved caused by the previously mentioned problem of "small denominators".

4. Eigenmodes

As stated in the Introduction a solution $y(t)$ of eq. (1.2) is called here an eigenmode if it has the same quasi-periodic behaviour as the coefficient $f(t)$, i.e. if it satisfies equations (1.4), (1.5).

It is straight forward to test any numerically obtained solution with graphical methods whether it is an eigenmode or not, - within the natural limits of numerical and graphical methods in general, of course: At multiples of the basic quasi-period $\tau_1 = 2T_1$ for an eigenfunction the argument $\omega_1 t$ of $u\ (\omega_1 t, \omega_2 t)$ by definition of ω_1 is a multiple of 2π so that according to eq. (1.5) the function u is constant with respect to its first argument. Only the periodic dependence on the complementary argument $\omega_2 t$ remains, and analogously at multiples of $\tau_2 = 2T_2$. Hence $y_1 = y(t = n\tau_1)$ plotted versus $\omega_2 t$, modulo 2π, and $y_2 = y(t = n\tau_2)$ plotted versus $\omega_1 t$, modulo 2π for an eigenfunction each yield a well defined curve in the limit $n \to \infty$, displaying the functional dependence of $y(t)$ on the two sub-arguments $\omega_1 t$ and $\omega_2 t$. "Trivial" eigenfunctions of this type (see below) are shown in Figs. 2a, 3a.

Another useful representation is the pair of phase space diagrams $\dot{y}/\omega$ versus y, plotted at $t = n\tau_1$ and $t = n\tau_2, n = 1,2,\ldots$. For eigenmodes a closed curve results for $n \to \infty$, or rather a curve with discontinuities in $\dot{y}$ since in our case only y is continuous but $\dot{y}$ in general is not (except if $y = o$). Figures 2b, 2c, 3b, 3c, for example, correspond to the cases 2a and 3a.

For all solutions which are not eigenmodes the graphs show a two-dimensional distribution of scattered points /7/ instead of curves except for subharmonics of eigensolutions, i.e. solutions with quasi-periods $m \cdot 2\pi$, $m \geqq 2$. Such subharmonics, however, may be identified by the fact that for them y has more than one branch.

In order to search for eigenmodes the following procedure has been applied. A ω_1-ω_2-quasi-periodic eigensolution was constructed for a

ω_1-periodic f(t). In small steps a ω_2-periodic contribution was then added to f(t) and the eigenvalue parameter ω^2 was adjusted each time so that the solutions looked as like as eigenmodes as possible. In the Appendix the construction of the initial "trivial" eigenmodes is explained. Such an eigenmode with eigenvalue $\omega = \Omega = 0.28675534$ is shown in Figs. 2a-2c and 3a-3c for two values of the initial value parameter, r = 0.111 and r = 0.500, respectively, for later reference. The amplitude is $F_1 = 0.1$ and the frequencies are $\omega_1 = 1$, $\omega_2 = 1/\sqrt{2}$. The same frequencies are used throughout in the following for all cases considered ; see Section 5 for discussion.

Unfortunately, it turns out that the intended procedure does not work. Once the originally vanishing amplitude F_2 reaches a few percent of F_1 even the best fit of ω does not yield well defined curves but some structure of finite width. Figure 4 shows how poorly one such "optimum" fit, $\omega = 0.27504376$, works when F_2 is pushed up to equality with F_1, for N = 16 000 iterations. Additional parameters are r = c = 0. If the number N of iterations is increased it turns out that the "optima" obtained are in fact unstable, i.e. $|y|$ grows slowly but without bounds.

There is a further "correlation" of eigenmodes with unboundedness of solutions. As mentioned in Section 3, under certain conditions the solutions y(t) are of the form (3.8). Clearly, the solutions are eigenmodes if

$$\nu = m_1\omega_1 + m_2\omega_2 \tag{4.1}$$

with m_1, m_2 being arbitrary integers, because exp $(im_i\omega_i t)$ is 2π-periodic in $\omega_i t$, i = 1,2. Inspection, however, shows /6/ that for ν given by eq. (4.1) the eigenvalue parameter ω is exactly in one of the "forbidden" gaps, eq.(3.7), which for our δ-function-type f(t) were shown in the last section to be connected with unboundedness of the solutions.

Although the foregoing results seem to give evidence against the existence of eigenmodes, such modes may nevertheless be found. Consider again eq. (3.1) with ω_1-periodic coefficient, equivalent to Hill's equation. It is well known /1/ that ω_1-periodic eigenmodes occur exactly if the eigenvalue parameter is at the boundary between bounded and unbounded behaviour of the solutions, i.e. at the boundary of gaps which for small amplitude are situated approximately at $\omega = \omega_o$

$$\omega_o = n_1 \omega_1 \tag{4.2}$$

with integer n_1. (At the boundary of the gaps close to $\omega = (2n_1+1)\omega_1/2$ the solutions have the period $2\tau_1$. Such subharmonic solutions do not interest us here.) At this position of ω, however, there is another linearly independent solution of the form $y(t) = t\, u\, (\omega_1 t)$, i.e. a solution with unbounded secular behaviour; see /19/, for example for Mathieu's equation. If the initial values $y(o)$, $\dot{y}(o)$ are chosen at random, the solution will always pick up a secular contribution and mask the existence of the eigenmode. For periodic δ-function coefficient this situation can be studied analytically (see Appendix).

These considerations suggest that in looking for quasi-periodic eigenmodes it is not enough to adjust the eigenvalue parameter ω properly; the initial condition specified by the coefficient r in eqs. (3.6) has to be as well. Otherwise at best a secular solution of the type $y(t) = t\, u\, (\omega_1 t, \omega_2 t)$ might be seen. Indeed, the above mentioned "optimum" solution shown in Fig. 4 grows essentially linearly in t, viz. in the number N of iterations.

In consequence, the search for eigenmodes was modified in the following way: The amplitudes and phase were fixed at $F_1 = F_2 = 0.1$ and $c = 0$. Then, with arbitrarily fixed initial condition (direction) $r = 0$ a crude search with the eigenvalue parameter ω was made for the position of the gap close to $\omega = \omega_o$, with

$$\omega_o = \omega_1 - \omega_2 \tag{4.3}$$

Even values $n_1 = -2$, $n_2 = 2$ were selected in order to avoid the possible construction of subharmonic solutions. With the approximate position of the gap known a detailed investigation of the behaviour of y(t) in the region of the left and right boundaries of the gap was made. In particular, max $|y(t)|$ was determined on a two-dimensional ω - r (= initial value) grid. In the most promising regions of this grid the solutions were visually checked to see how far they corresponded to an eigenmode. And indeed an eigenmode was found on each boundary of the gap. The two eigenvalues are $\omega = \Omega' = 0.27495798 \pm 3 \cdot 10^{-9}$ (see Fig. 5a - 5c) and $\Omega'' = 0.27504376 \pm 7 \cdot 10^{-9}$ (see Fig. 6a - 6c). The corresponding initial value parameters, see eq. (3.6), are $r' = 0.500 \pm 5 \cdot 10^{-4}$ and $r'' = 0.111 \pm 1 \cdot 10^{-3}$, respectively. The parameters ω and r, in particular ω, have to be determined more and more precisely if one wants to go to higher and higher numbers N of iterations. $N = 1.28 \times 10^5$ was used to determine the eigenvalues above. In Figs. 5a - 6c N = 16000 and only every 2nd iteration is plotted.

The modes in Figs. 5 and 6 are the generalization of the modes from the periodic case $F_2 = 0$, Figs. 2 and 3, to the quasi-periodic case $F_2 = F_1$. Both eigenvalues Ω', Ω'' are roughly 4% smaller than Ω. If F_2 (or F_1) is decreased both Ω' and Ω'' increase and the width of the gap $\Omega'' - \Omega'$ shrinks until at $F_2 = 0$ both eigenvalues coalesce into $\omega = \Omega$, the gap disappears and the eigenmode is degenerate with respect to the initial value r. In general the modes have a discontinuity in $\dot{y}$ at $t = n\,T_1$ and $t = n\,T_2$. The modes should be indexed with (-2,2) corresponding to the gap index (n_1, n_2).

In order to check the effect of the phase difference c (see eq. (3.2)) on the eigenvalues, the search for eigenmodes was repeated with c changed from zero to 0.5π i.e. $\Delta t = 0.5T_1$, on the right hand boundary of the same gap as before. Figures 7a - 7c show the resulting eigenmode. Its eigenvalue $\omega = \Omega''_c = 0.27504378 \pm 1.6 \cdot 10^{-8}$ agrees with the previous value Ω'' for $c = 0$ within the limits of accuracy aspired. Hence, as expected for incommensurate ω_1 and ω_2, the effect of the initial phase difference disappears for sufficiently large N. The "proper" initial value is $r''_c = 0.6085 \pm 1 \cdot 10^{-4}$.

From the discussion above it is obvious that the existence of eigenmodes is not restricted to the particular values of the amplitudes, gap indices and phase F_i, n_i, c, $i = 1,2$ used. Eigenmodes with other parameters have indeed been constructed. It might happen, however, that for large amplitudes some of the eigenvalues disappear when different gaps overlap.

5. Discussion and conclusions

It has been shown within the limits of numerical and graphical methods, that the quasi-periodic differential equation (3.1), (3.2) possesses a doubly indexed infinite sequence of eigenvalues $\omega = \Omega'_{m,n}$ and $\omega = \Omega''_{m,n}$ and eigensolutions $y'_{m,n}$, $y''_{m,n}$ with the same quasi-periodicity as the equation itself. The eigenvalues are situated at the edge of "gaps" in the vicinity of $\omega = \omega_o$, where $\omega_o = m\omega_1 + n\omega_2$, m, n integer. Inside the gaps the solutions are unbounded. This situation is the complete analogue of the properties of Hill's equation with periodic coefficient. It seems plausible that the result is true of more general, linear, quasi-periodic differential equations. The existence of subharmonic solutions with quasi-periods $2\tau_1$ and/or $2\tau_2$ was investigated more in passing. They exist at the edges of gaps with half-integer m and/or n.

A particular point which deserves discussion is the choice of $q = \omega_1/\omega_2$. In the paper $q = \sqrt{2}$ is used throughout, i.e. an irrational number, as intended. On the other hand, in the computer all numbers are truncated so that q becomes rational. During the computations, however, it was monitored whether two δ-function pulses ever coincided again if two pulses did so initially; in other words, whether $\omega_1/\omega_2 = N_1/N_2$ for N_1, $N_2 \leqq N$, the number of iterations. (Coincidence here is defined as $|N_1T_1 - N_2T_2| \leqq 10^{-12}$, a small number but larger than the round-off error in double-precision operation.) This did not occur up to $N = 1.28 \times 10^5$, the highest number used. The ratio $q = \sqrt{2}$ was therefore still "effectively irrational".

There is a second necessary criterion for "effective irrationality" of q: The number of iterations has to be large enough, so that the effect of the initial phase difference c, eq. (3.2), gets lost. Consider, for example, $\omega_1 = 1$, $\omega_2 = \sqrt{4+\varepsilon}$, $|\varepsilon| << 1$.

This implies that $T_1 \approx 2T_2$. The order in which the pulses $n\,T_1$ and $m\,T_2$, n, m = 1,2,..., follow each other gets mixed up only for $N > N_o$, where $2T_2\,N_o = T_1(N_o+1)$, i.e. for $N \gtrsim 8/|\varepsilon| \gg 1$. Thus, the criterion of "phase scrambling" leads to the requirement of an exceedingly large number of iterations if q is very close to a rational number m/n with small m,n. This was indeed observed numerically. For the case $F_1 = F_2 = 0.5$, $\omega_1 = 1$, $\omega_2 = (4 + 1\text{x}10^{-8})^{1/2}$ and c = 0 an eigenvalue was found at $\omega = 1.126872$ while for $c = 0.5\pi$ it changed to $\omega = 1.137703$, even at $N = 2\cdot 10^5$ iterations. In contrast, the eigenvalues for $\omega_2 = 1/\sqrt{2}$ were independent of c up to at least 7 decimal places. Such problems with rational versus irrational numbers have their counterpart in the requirement of "strong incommensurability" $|n_1\omega_1 + n_2\omega_2| \geqq \Omega(n)$ where Ω is a sufficiently fast decreasing function of $n = \max\,(|n_1|,|n_2|)$ in the analytic treatment of eq. (3.1) (see /5/,/6/).

In applying the foregoing considerations to the problem of the MHD continuum in general toroidal geometry (see Sections 1 and 2) it seems plausible that eigenmodes exist on "sufficiently irrational" surfaces while their existence for other values of q is less certain.

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Appendix

In the periodic case, say $F_2 = 0$, the solution y, $\dot{y}$ can readily be obtained from eqs. (3.4), (3.5) after one full period $\tau_1 = 2T_1$ as a function of the initial values $y(0)$, $\dot{y}(0)$. The result is

$$\underline{Y}(\tau_1) = \underline{\underline{A}} \cdot \underline{Y}(0) \tag{A1}$$

with

$$\begin{aligned} A_{11} &= \cos 2z - f \sin z \cos z \\ A_{12} &= \sin 2z - f \sin^2 z \\ A_{21} &= -\sin 2z - f \sin^2 z - f^2 \sin z \cos z \\ A_{22} &= \cos 2z + f \sin z \cos z - f^2 \sin^2 z \end{aligned} \tag{A2}$$

$$\text{where } \underline{Y} = (y, \dot{y}/\omega)^T, \quad f = F_1/\omega, \quad z = \pi \frac{\omega}{\omega_1} \tag{A3}$$

According to the Floquet theory /2/ the solution of eq. (3.1) in the present case is of the form

$$y(t) = e^{i\nu t} u(\omega_1 t) + c.c \tag{A4}$$

with $u(\theta+2\pi) = u(\theta)$, provided $\nu \neq n_1\omega_1/2$ where in addition to (A4) there is a solution of a different type (see below). The exponent ν is related to the eigenvalue λ of the matrix $\underline{\underline{A}}$ by

$$\lambda = 0.5\,(1 \pm \sqrt{S^2-4}) \;;\; S(\omega) = A_{11} + A_{22} \tag{A5}$$

$$\exp(i\nu\tau_1) = \lambda \tag{A6}$$

It is therefore possible to construct quasi-periodic solutions of the periodic differential equation by setting $\nu = m_1\omega_1 + m_2\omega_2$ with integer m_1, m_2 and numerical solution of eqs. (A5, A6) for the eigenvalues $\omega = w_n$, $n = 1,2, \ldots$. For $F_1 = 0.1$, $\omega_1 = 1$, $\omega_2 = 1/\sqrt{2}$, $m_1 = 1$ and $m_2 = -1$ the eigenmode with the lowest eigenvalue $w_o = \Omega$ is shown in Figs. 2a - 3c for two different initial conditions (see Section 4).

The solutions (A4) are unbounded for complex ν. The transition between real and complex ν occurs at $\lambda = \pm 1$. The case $\lambda = 1$ corresponds to $\omega = n\omega_1$ with integer n. $\underline{\underline{A}}$ in this case goes over into the identity matrix, which implies that the solution is ω_1 - periodic for all F_1. It is an eigenmode corresponding to $m_2 = 0$ above. There is a degeneracy here: the usually existing finite ω region of unbounded solutions has collapsed and disappeared. This degeneracy, which does not occur in Mathieu's equation, is due to the infinite number of harmonics of equal amplitude which build up the δ-functions.

The case $\lambda = -1$ implies

$$F_1 = \pm 2\omega \text{ ctg } z \tag{A7}$$

which is satisfied by two infinite sets of $\omega = \alpha_n, \beta_n$ which are situated pairwise to the left and right of $(2n + 1)\omega_1/2$, $n = 0,1, \ldots$. Between each conjugate pair the solutions are unbounded. It is easily checked with eqs. (A1), (A2) that for ω satisfying eq. (A7) the solutions are subharmonic with period $2\tau_1$, $\underline{Y}(\tau_1) = -\underline{Y}(0)$, provided the initial values satisfy

$$\frac{\omega y(0)}{\dot{y}(0)} = 0 \quad , \quad \frac{\omega y(0)}{\dot{y}(0)} = - \text{ tg } z \tag{A8}$$

at $\omega = \alpha_n$ and $\omega = \beta_n$ respectively. For all other initial conditions but with the same ω the solutions at $t = m\tau_1$ have the secular form $y = a + m\, b$, $\dot{y} = c + m\, d$ with two alternating sets of constants a,b,c,d for m even and odd. (In addition, $b = 0$ for $\omega = \alpha_n$.)

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Figure captions:

Fig. 1a: Unstable bands in the (ω, ω_2) grid for $\omega_1 = 1$ and $N = 10^4$ iterations. Higher numbers correspond to weaker instability. Amplitudes $F_1 = F_2 = 0.2$.

Fig. 1b: Enlarged section of Fig. 1a, $F_1 = F_2 = 0.5$. Lines $n_o\omega + n_1\omega_1 + n_2\omega_2 = 0$ are indexed as (n_o, n_1, n_2).

Fig. 2: "Trivial" eigenmode y for periodic case $F_1 = 0.1$, $F_2 = 0$, $\omega_1 = 1$, $\omega_2 = 1/\sqrt{2}$. Eigenvalue $\Omega = 0.28675534$, initial direction $r = 0.500$, initial phase $c = 0$. y as function of $\omega_2 t$ and $\omega_1 t$ (Fig. a) and phase space diagrams $(y, \dot{y}/\omega)$ at multiples of periods τ_1 (Fig. b) and τ_2 (Fig. c). $N = 8000$.

Fig. 3: Same as Fig. 2 with $r = 0.111$.

Fig. 4: "Optimum" solution for $F_1 = F_2 = 0.1$, $\omega_2 = 1/\sqrt{2}$, $r = 0$, $c = 0$ for $\omega = 0.27504376$. $N = 16\ 000$

Fig. 5: Eigenmode for $F_1 = F_2 = 0.1$, $\omega_2 = 1/\sqrt{2}$, $r = 0.500$, $c = 0$ with eigenvalue $\Omega' = 0.27495798$.

Fig. 6: Eigenmode for $F_1 = F_2 = 0.1$, $\omega_2 = 1/\sqrt{2}$, $r = 0.111$, $c = 0$ with eigenvalue $\Omega'' = 0.27504376$.

Fig. 7: Eigenmode for $F_1 = F_2 = 0.1$, $\omega_2 = 1/\sqrt{2}$, $r = 0.6085$, $c = 0.5\pi$ with eigenvalue $\Omega''_c = 0.27504378$.

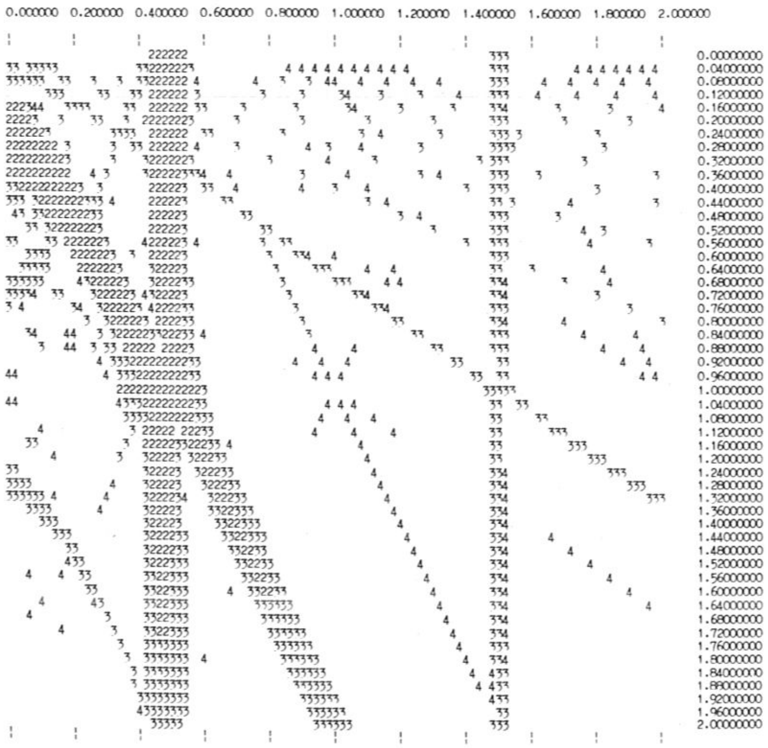
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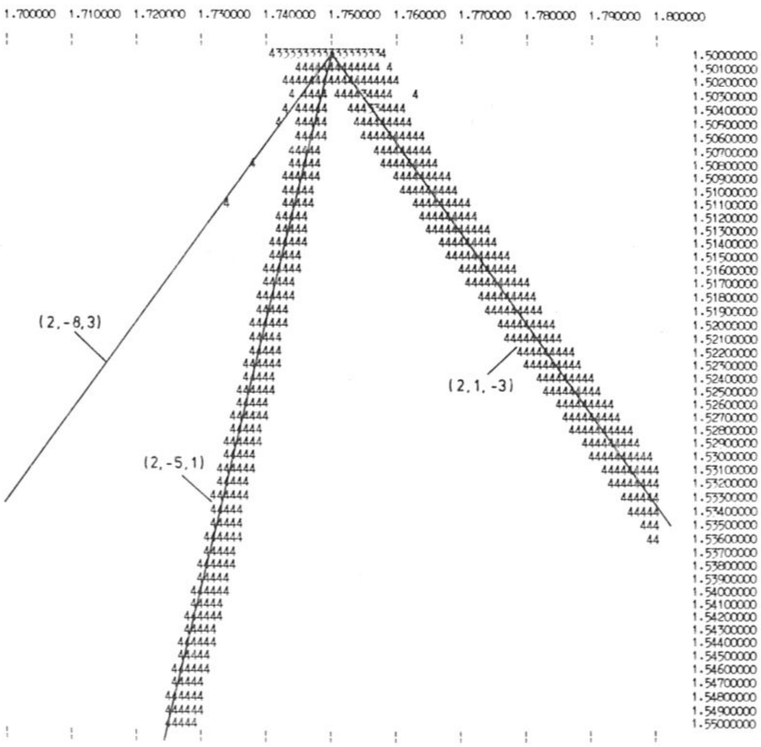
Fig. 1a)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(2,1,-3)

Fig. 1b)

a) b) c)

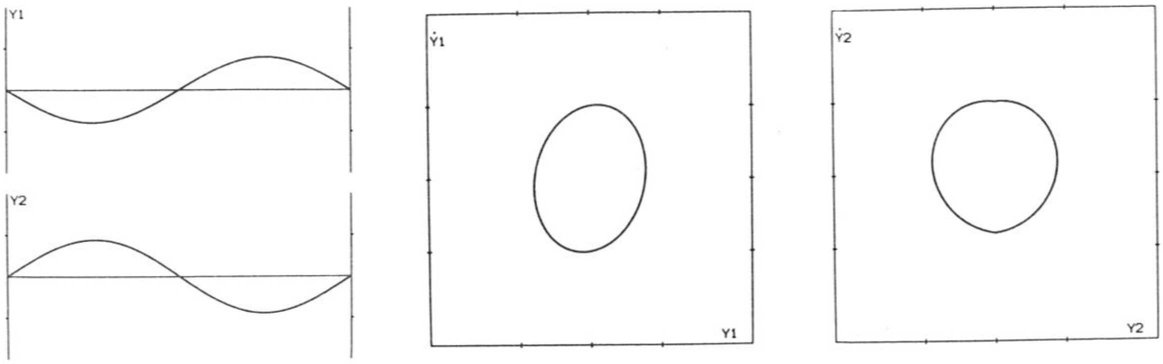

Fig. 2

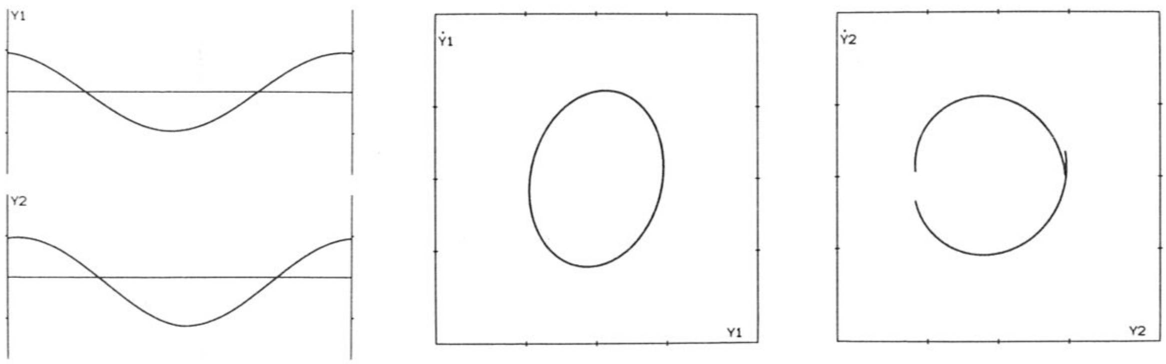

Fig. 3

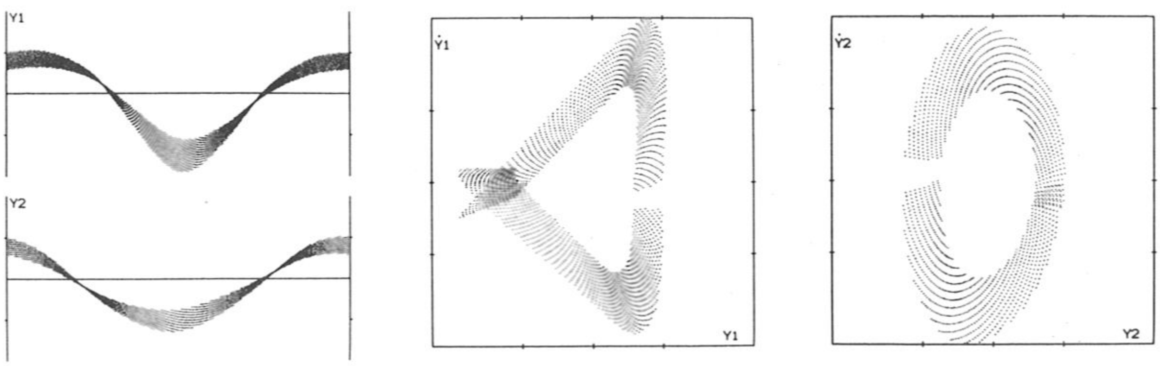

Fig. 4

a) b) c)

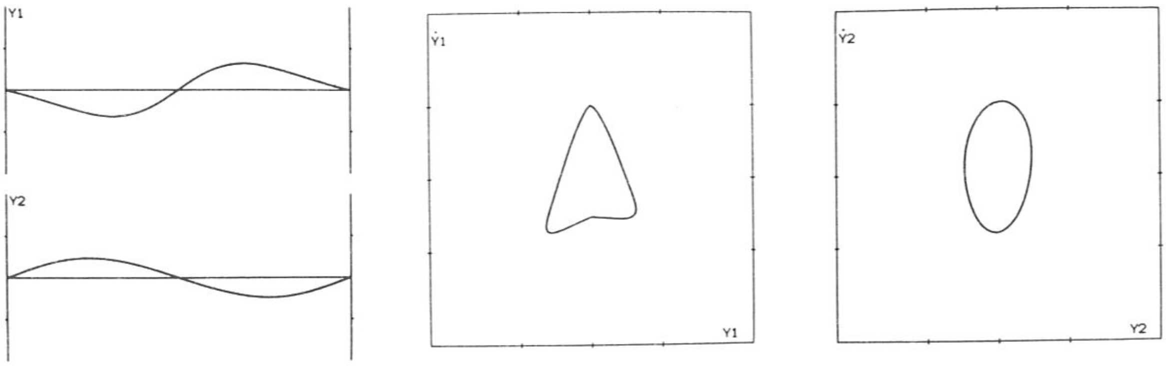

Fig. 5

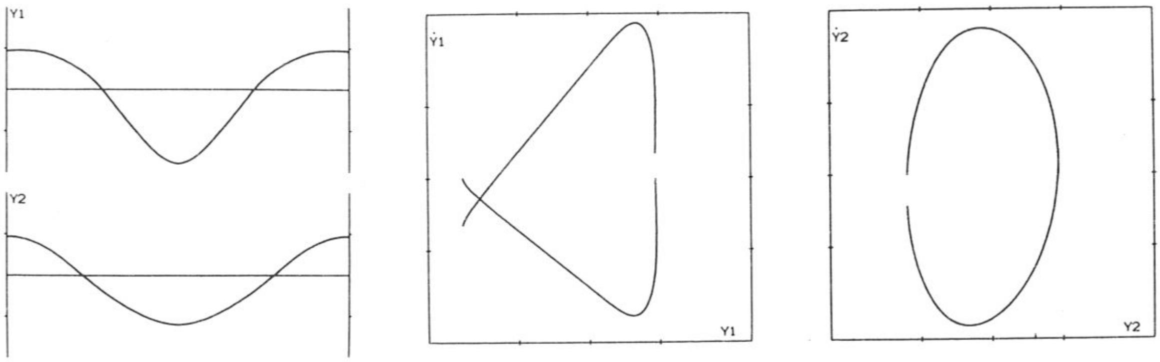

Fig. 6

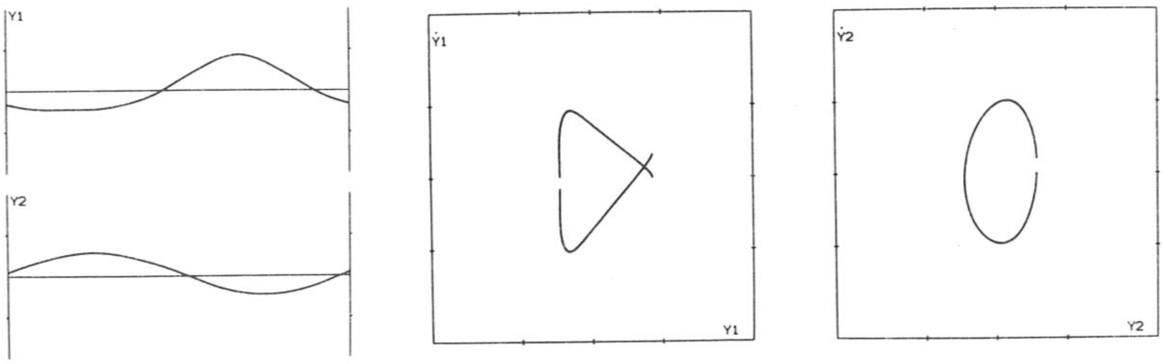

Fig. 7