NONLINEAR BEHAVIOR OF TEARING MODES *

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IPP 6/192 January 1980

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Abstract

This lecture deals with the nonlinear properties of individual tearing modes. First the purely resistive case is considered. The nonlinear island growth is a slow diffusion process on the resistive skin time scale. The different effects determining the saturation are outlined. Tearing modes in a high temperature plasma are thendiscussed, where, in addition to resistivity, diamagnetic and viscous effects are important. Nonlinearly, these so-called drift tearing modes may behave quite similarly to purely resistive modes, since diamagnetic effects are quenched at a certain island size depending on the rate of cross-field plasma diffusion.

* Lecture given at the International School of Plasma Physics Varenna, Italy, August 27 - September 8, 1979

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

PURELY RESISTIVE TEARING MODES

Tearing modes lead to a new more complex magnetic topology called magnetic islands. The nonlinear theory of the tearing instability therefore deals with the behavior of magnetic islands of finite size. Let me introduce a system of resistive MHD equations that take into account the most important properties of tearing modes in low β plasmas, and which shall be the model equations of the first part of this lecture. I consider a plasma imbedded in a strong magnetic field $\vec{B}_o$. For sufficiently simple geometry, to which I restrict myself, such as plane or cylindrical configurations, $\vec{B}_o$ is homogeneous. In addition there is a current $\vec{j}$ flowing within the plasma essentially along $\vec{B}_o$. It generates a poloidal field component $\vec{B}_\perp$

$$\begin{aligned} &\vec{B} = (\vec{B}_\perp\ ,\ B_o\hat{z}) \quad , \quad B_o >> B_\perp \\ &\vec{B}_\perp = \hat{z} \times \nabla\psi \quad , \quad \nabla^2\psi = j \end{aligned} \tag{1}$$

Tearing modes grow essentially perpendicularly to $\vec{j}$, the basic geometry being two-dimensional. Hence ψ is the vector potential in the direction of the ignorable coordinate and therefore a flux function, which is conserved in the limit of vanishing resistivity. The plasma motion, too, is in the plane perpendicular to $\vec{B}_o$ and hence incompressible

$$\vec{v}_\perp \cong c\,\frac{\vec{E} \times \vec{B}_o}{B_o^2} = \hat{z} \times \nabla\phi \tag{2}$$

$$v_{\shortparallel} \cong 0$$

From the parallel component of Ohm's law

$$E_{\shortparallel} = \eta j \tag{3}$$

and the definition of $E_{\shortparallel}$ one obtains the equation

$$\frac{\partial\psi}{\partial t} + \vec{v}\cdot\nabla\psi = \eta j - E_o \tag{4}$$

where E_o is the electric field externally applied along $\vec{B}_o$ to maintain a resistive equilibrium, corresponding to the loop voltage in a tokamak. An equation for the stream function ϕ , introduced in (2), is obtained by taking the curl of the equation of motion thus eliminating the pressure. Because of the two-dimensionality of the system only the z-component of $\nabla \times \vec{v}$ is non-vanishing, $\hat{z}\cdot(\nabla \times \vec{v}) = \nabla^2\phi$, satisfying the equation

$$(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla)\nabla^2\phi = \hat{z}\cdot(\nabla\psi \times \nabla j) \ , \tag{5}$$

where $\rho = \rho_o = 1$ is assumed for simplicity. Equations (4), (5) are our model equations written in the following units: a typical poloidal magnetic field $B_{\perp o}$, the corresponding Alfven speed $v_A = B_{\perp o}/(4\pi\rho_o)^{1/2}$, and a typical width a of the current carrying region, e.g. the plasma radius. In these units η is the inverse magnetic Reynold's number S

$$\eta = S^{-1} = \frac{\eta c^2}{4\pi v_A\, a} \propto \frac{\rho^{1/2}}{T_e^{3/2} a} \qquad (6)$$

Characteristic values of S are

$S \sim 10^6 - 10^7$ for present day tokamaks

$\sim 10^8$ for reactor plasmas

Quasilinear Theory

The linear growth rate of the tearing mode is proportional to a fractional power of η , $\gamma \sim \eta^{3/5}$, intermediate between an MHD process and resistive diffusion. This property of the instability is, however, already changed at a very low amplitude, where the influence of plasma inertia becomes insignificant, as has been shown by Rutherford[1]. He considered the quasilinear change δj_o of the current density $j_o(x)$ within the resistive layer $\delta_s \sim \eta^{2/5}$ around the singular surface x_s, where $\vec{k}\cdot\vec{B} = 0$. Equation (4) yields

$$\frac{\partial \delta\psi_o}{\partial t} + \langle \tilde{\vec{v}}\cdot\nabla\tilde{\psi} \rangle = \eta\delta j_o \qquad (7)$$

Since the skin time of the resistive layer $\delta_s^2/\eta \sim \eta^{-1/5}$ is small compared with the growth time $\gamma^{-1} \sim \eta^{-3/5}$, the current distribution relaxes instantaneously within δ_s. Hence the first term on the l.h.s. of eq. (7) is negligible and δj_o becomes

$$\delta j_o = \frac{1}{\eta} \langle \vec{v} \cdot \nabla\tilde{\psi} \rangle \tag{8}$$

For $\tilde{\psi} = \psi_1(x)\cos ky$, $\tilde{\phi} = \phi_1(x)\sin ky$ one has $\delta j_o = k\psi_1\phi_1'/2\eta$. Insertion into eq. (5) gives

$$\frac{\partial\phi_1''}{\partial t} = k(\psi_o'\psi_1'' - \psi_1 j_o') - k\psi_1\delta j_o'$$

which may be rearranged as

$$\left(\frac{\partial}{\partial t} + \frac{k^2\psi_1^2}{2\eta}\right)\phi_1'' = \text{linear terms} \tag{9}$$

If $k^2\psi_1^2/2 > \gamma$ the first term on the l.h.s. of eq. (9), which represents the effect of inertia, is negligible. Using

$$\gamma \sim \eta^{3/5}\Delta'^{4/5}(\vec{k}\cdot\vec{B})'^{2/5} \tag{10}$$

$$\delta_s \sim \frac{\eta\Delta'}{\gamma} \tag{11}$$

and the definition of the island width W

$$W = 4\ (\psi_1/|\psi_o''|)^{1/2} \tag{12}$$

it is readily found that inertia becomes unimportant as soon as the island width exceeds the width of the resistive layer. The tearing instability thus changes its character at a small amplitude $W \sim \delta_s$. Since further growth proceeds on the resistive time scale as a sequence of equilibrium states $j = j(\psi,t)$, it has been argued that there is no such thing as a tearing instability in a real (non-idealized) plasma configuration, but only diffusive growth or decay of magnetic islands.

Nonlinear Island Growth

The diffusion process just mentioned is elegantly described by a set of generalized differential equations[2]. Averaging eq. (4) over the magnetic surface ψ enclosing the volume V and writing $\psi(\vec{x},t) = \psi(V(\vec{x},t),t)$,one obtains the diffusion equation

$$\frac{\partial\psi(V,t)}{\partial t} = \eta\frac{\partial}{\partial V} K \frac{\partial\psi}{\partial V} \tag{13}$$

where

$$K = < |\nabla V|^2 >_V \tag{14}$$

is an inductance coefficient V being normalized such that $\oint ds/|\nabla V| = 1$. K characterizes the geometry (island size etc.) which at each instance is determined by Ampere's law

$$\nabla^2\psi = j(V,t) \tag{15}$$

Equations (13) to (15) were solved numerically for the case of island growth in a symmetric plane sheet pinch. For more general asymmetric configurations, however, the numerical treatment would be quite involved. For small islands eq. (13) reduces to a simple equation for the island width W

$$\frac{dW}{dt} = \frac{\pi}{2}\Delta'\eta \tag{16}$$

i.e. W increases linearily with time, a result first obtained in ref. 1. Equation (16) can also be derived using a rather crude argument. For small W eq. (4) becomes

$$\frac{\partial\psi_1}{\partial t} = \eta\psi_1'' \tag{17}$$

In the regime of exponential tearing mode growth where $W < \delta_s$ one has (because of the constant ψ_1 property)

$$\psi_1'' \cong \frac{\psi_{1+}' - \psi_{1-}'}{\delta_s} = \frac{\Delta'\psi_1}{\delta_s} \tag{18}$$

For $W > \delta_s$, however, the current-carrying layer is given by the island width. Replacing δ_s by W in eq. (18) and substituting the result in eq. (17), one obtains eq. (16) up to a numerical factor of the order unity.

The same argument also leads to the correct time dependence of small but finite islands in the case where the dominant dissipation in Ohm's law is the perpendicular electron viscosity rather than the resistivity. Instead of eq. (17) one has

$$\frac{\partial \psi_1}{\partial t} = -\mu_e \nabla^2 j_1 \tag{19}$$

By analogy with eq. (18) $\nabla^2 j_1$ is approximated by

$$\psi_1'''' \cong \frac{\Delta'}{\delta_s^3} \psi_1$$

Replacing δ_s by W for $W > \delta_s$ in this relation and inserting the result in eq. (19), one obtains

$$\frac{dW^3}{dt} \cong \mu_e \Delta'$$

and hence

$$W \cong (\mu_e \Delta' t)^{1/3} \tag{20}$$

in agreement with a recent more exact calculation[3)].

Saturation Island Width

For not too large islands a theory of island saturation has been given in ref.4. The basic equation is

$$\frac{dW}{dt} = \eta(\Delta'(W) - \alpha W) \tag{21}$$

where $\Delta'(W)$ is the difference of the derivatives of ψ_1 taken at the radii corresponding to the inner and outer edges of the island. The main stabilizing effect comes from the decrease of Δ' with growing island width; it represents the decrease of free magnetic energy, which is the main energy source for tearing modes. The relation $\Delta'(W) = 0$, in general, gives a good estimate of the saturation island width. The α-term in eq.(21) contains additional nonlinear effects, in particular the influence of a self-consistent change of the resistivity profile, i.e. the electron temperature[6]. Let me give a simple estimate of this effect. Averaging eq.(4) over a flux surface ψ makes the covective term vanish

$$\left\langle\frac{\partial\psi}{\partial t}\right\rangle_{\psi} = \eta(\psi)j(\psi) \tag{22}$$

Let us consider eq.(22) at the O-point and the X-point, where the average term on the l.h.s. degenerates to a local expression

$$\frac{\partial\psi^{o,x}}{\partial t} = (\eta j)^{o,x} \tag{23}$$

(the average value over the separatrix ψ_s is identical with the value at the X-point because of the dominating weight of the latter in the average). For not too large islands

$$\psi \cong \psi_o + \psi_1 \cos ky \tag{24}$$

is a good approximation, higher harmonics being small. Inserting eq.(24) into eq.(23), one obtains

$$2\frac{\partial \psi_1}{\partial t} = (\eta j)^o - (\eta j)^x$$

$$\cong 2\eta_o j_1 + 2\eta_1 j_o$$

$$j_1 \cong \frac{\Delta'(W)}{W}\psi_1 \quad , \quad \eta_1 = \eta'\psi_1 \quad , \quad \eta' \equiv \frac{\delta\eta}{\delta\psi}$$

hence

$$\frac{dW}{dt} = \eta\Delta'(W) + j_o\eta' W \tag{25}$$

This result shows that for $\eta' > 0$, i.e. T_e lower in the islands than on the separatrix, the islands become larger, while $\eta' < 0$ leads to smaller islands. $\eta(\psi)$ is determined by the electron energy balance. It turns out that for reasonable assumptions on heat sources and heat conduction the temperature profile is rather flat within the islands. The saturation island width is only slightly larger than in the case $\eta = \eta_o(r)$, which is often used in the 2D-simulations, and which corresponds to $\eta' < 0$.

Application to Tokamak Theory

Tearing modes obviously play an important role in tokamaks. Because of the geometrical complexity the nonlinear behavior of magnetic islands has thus far only been investigated in the cylindrical tokamak approximation, to which attention will therefore be restricted. In this geometry individual nonlinear tearing modes generate a helically symmetric configuration. All quantities depend only on r and $\theta - nz/mR$, and the flux function to be used in eqs.(4) and (5) is the so-called helical flux function ψ_* defined by

$$\frac{\partial \psi_*}{\partial r} = B_\theta - \frac{n}{m}\frac{r}{R} B_o$$

$$j = \nabla^2 \psi_* + 2 \frac{n}{mR} B_o \qquad (26)$$

The eigenfunction ψ_1 in a cylindrical plasma is of the form shown in Fig.1 [6], being large inside the resonant radius r_s. It resembles an internal kink mode, which is MHD stable according to Newcomb's criterium. This is in general not true of the (m,n) = (1,1) mode, which plays a special role and which will be discussed in a subsequent lecture.

The saturation island width is essnetially determined by the current distribution. A general tendency is that peaked profiles lead to small islands, while flat-topped, square-shaped profiles may give rise to large islands[4),7)], primarily owing to the strong current density gradients at the resonant radius. It is rather tempting to interpret this behavior as due to the island width being proportional to the gradient of the original current profile, but this would be an oversimplification. The island size is a functional of the global current behavior, though the vicinity of the resonant surface has a particularly large weight. Nor is there any simple relation between the original Δ' or the linear growth rate, and the final island size.

The theory of tearing mode saturation as just outlined has been applied quite successfully to explain the amplitudes of Mirnov oscillations observed in tokamaks[5)] and similar devices.

TEARING MODES IN HIGH TEMPERATURE PLASMAS

At high plasma temperatures eqs.(4) and (5) give a rather poor approximation since, in addition to resistivity, various other non-ideal effects are important. In recent years the linear theory has been substantially refined by including diamagnetic effects, viscosity, parallel plasma flow, drift-kinetic description of the electrons etc., and various different types of unstable modes have been found. As will be shown, the nonlinear behavior of these modes may, however, be quite similar to the behavior of purely resistive tearing modes treated in the first paragraph.

Two-fluid Model Equations for Drift-Tearing Modes

A rather simple generalisation of eqs. (4) and (5) for a high temperature plasma is obtained within the framework of two-fluid theory. Again let us restrict ourselves to cylindrical geometry, denoting the helical flux function defined in eq. (26) by ψ for simplicity. Ohm's law (3) is generalized by including the electron pressure term

$$E_{\shortparallel} = \eta j - \frac{T_e}{en} \nabla_{\shortparallel} n \qquad (27)$$

using $\nabla_{\shortparallel} T_e = 0$. The incompressible part $\vec{u}$ of the ion velocity contains the ion diamagnetic drift

$$\vec{u} = \vec{v}_{i*} + c\,\frac{\vec{E}\times\vec{B}}{B^2} = \hat{z}\times\nabla\phi \qquad (28)$$

while the compressible part due to the polarization drift can be expressed by $\nabla_{\shortparallel} j$ because of quasi-neutrality. An additional smallness parameter α arises through the polarisation and diamagnetic drifts

$$\alpha = \frac{c}{\omega_{pi} a}\,\frac{B_{\theta o}}{B_o} \qquad (29)$$

The basic equations replacing eqs.(4) and (5) are

$$\frac{\partial\psi}{\partial t} + \vec{u}\cdot\nabla\psi = \eta j - \alpha\,\frac{T_e + \gamma_i T_i}{n}\,(\hat{z}\times\nabla\psi)\cdot\nabla n \qquad (30)$$

$$\frac{\partial n}{\partial t} + \vec{u}\cdot\nabla n = \alpha\,(\hat{z}\times\nabla\psi)\cdot\nabla j - \nabla_{\shortparallel} n v_{\shortparallel} \qquad (31)$$

$$\frac{\partial n v_{\shortparallel}}{\partial t} = -\,(T_e + \gamma_i T_i)\nabla_{\shortparallel} n + \mu_{\shortparallel} n \nabla_{\shortparallel}^2 v_{\shortparallel} \qquad (32)$$

$$\begin{aligned}\frac{\partial\nabla^2\phi}{\partial t} &+ (\vec{u} - \vec{v}_{i*})\cdot\nabla\nabla^2\phi - (\hat{z}\times\nabla n)\cdot\nabla\frac{u^2}{2} \\ &= (\hat{z}\times\nabla\psi)\cdot\nabla\nabla^2\psi + \nabla\cdot\mu_{\perp}\nabla\nabla^2\phi\end{aligned} \qquad (33)$$

Here temperatures are measured in $B_{\theta o}^2/4\pi$ and hence are of the order of the poloidal beta,

$$\beta_p = 2(T_e + T_i)$$

A polytropic gas law is assumed for the ions

$$\nabla p_i = \gamma_i T_i \nabla n$$

The properties of the unstable modes described by eqs.(30)-(33) are primarily determined by the ratio ω_*/γ_T , $\omega_* = \omega_*(r_s)$ is the diamagnetic drift frequency and γ_T the tearing mode growth rate for $\omega_* \to 0$. The linear dispersion relation is approximately given by (for details see for example ref. 9)

$$\omega(\omega - \omega_{i*})(\omega - \omega_{e*})^3 = i\gamma_T^5 \qquad (34)$$

which immediately yields the limiting cases

$$\omega = \begin{cases} i\gamma_T \, , & \omega_* \ll \gamma_T \\ \omega_{e*} + i\gamma_T(\gamma_T/\omega_*)^{2/3} \, , & \omega_* \gg \gamma_T \end{cases} \qquad (35)$$

It is seen that for large ω_* the growth rate is strongly reduced.

Nonlinear Behavior of Drift Tearing Modes

Generalizing the treatment outlined at the beginning, it can readily be shown[8)] that, just as in the case of the purely resistive tearing instability, the exponential growth is terminated as soon as the island size W exceeds the current-carrying layer width δ_s; for $W > \delta_s$ inertia is negligible, $j = j(\psi)$. Here δ_s depends on the ratio ω_*/γ_T through the relation $\delta_s \cong \eta\Delta'/\gamma$.

Simultaneously, a quasi-linear flattening of the density profile at the resonant surface takes place, reducing the drift-frequency $\omega_* \propto n_o'(r_s)$. Let us estimate the mode amplitude or the island size at which the density gradient $n_o'(r_s)$ vanishes. From eq.(31), neglecting the r.h.s., one obtains the following approximate diffusion equation

$$\frac{\partial n_o}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} rD \frac{\partial n_o}{\partial r} \tag{36}$$

$$D_n = \frac{2\gamma}{\omega^2}\left(\frac{m}{r_s}\right)^2 |\phi|^2 \tag{37}$$

$$\cong \gamma\delta_s^2 \quad \text{for} \quad W \cong \delta_s \ , \quad |r - r_s| \lesssim \delta_s$$

It can readily be seen that for $W \gtrsim \delta_s$ a density plateau is formed and hence the diamagnetic frequencies vanish. The development of γ and ω with growing amplitude is shown qualitatively in Fig.2. For island size $W > \delta_s$ the density becomes a flux function $n = n(\psi)$, and eqs.(30) and (33) essentially reduce to the corresponding equations (4) and (5) for the purely resistive case. Hence in the present model the saturation island size does not depend on the parameter α, i.e. on diamagnetic effects, but only on the current profile and the behavior of the resistivity $\eta(\psi,t)$; island rotation can only be due to plasma rotation. Numerical solution of eqs. (30) to (33) confirms these qualitative arguments.

Influence of Cross-field Plasma Diffusion

The model (30) to (33) can, however, not be applied directly to interpret observations in tokamaks. It turns out that plasma cross-field diffusion, which is not accounted for in eq.(31), strongly competes with the quasi-linear flattening of the density profile just described. Comparing the diffusion coefficient (37) for the latter process

$$D_n \cong \gamma\delta_s^{\,2} \sim \nu_e(\frac{c}{\omega_{pe}})^2(\beta_p\Delta' a\alpha)^{2/3} \qquad (38)$$

with neoclassical plasma diffusion

$$D_\perp \cong \nu_e\rho_e^{\,2}q^2(\frac{R}{r})^{3/2} \sim 10^2\nu_e\rho_e^{\,2} \qquad (39)$$

$$= 10^2\ \beta\nu_e(\frac{c}{\omega_{pe}})^2$$

it is seen that neoclassical diffusion may already dominate over the quasi-linear flattening. By including particle diffusion (with appropriate particle sources to maintain a certain overall density profile) the density gradient thus remains finite for $W \gtrsim \delta_s$ and the $\nabla_{\parallel} n$ term in eq.(30) continues to couple n and ψ. However, as the islands increase in size, perpendicular convection $\vec{u}$ as well as parallel flow $v_{\parallel}$ in eq.(31) will increase the tendency to make n a flux function and thus flatten the density profile, finally overcoming the effect of cross-field diffusion. To estimate the necessary island size, one may tentatively replace $\gamma\delta_s^2$ by γW^2 ($\cong \Delta'\eta W$ because of eq.(16)). Equating γW^2 with the observed plasma diffusion coefficient gives the island size W_o , above which n_o' and hence ω_* should be substantially reduced and the arguments given in the previous section should apply.

Experimental observation of plasma conditions during Mirnov oscillation activity do in fact reveal a temperature plateau in the island region, but no visible density profile flattening. This is consistent with the observed mode frequency being roughly equal to the diamagnetic frequency given by the average density gradient instead of the pressure gradient. Only if the $m = 2$ mode grows to large amplitudes prior to a major disruption, does the frequency decrease to zero. It is tempting to identify this slowing down with a flattening of the density profile (still unobserved experimentally) for $W > W_o$, as discussed above.

SUMMARY

The nonlinear evolution of individual tearing modes is discussed. For magnetic island size W exceeding the thin resistive layer δ_s carrying the perturbed current, the mode growth proceeds on the resistive diffusion time scale. The saturation width is primarily determined by the exhaustion of the magnetic energy reservoir and is approximately given by $\Delta'(W) = 0$. In a high temperature plasma further non-ideal effects such as diamagnetic drifts and parallel flow have to be taken into account. Though the linear properties strongly depend on these effects, in particular on the ratio ω_*/γ_T, the nonlinear development and saturation may be the same as in the purely resistive case because the diamagnetic effects are quenched at a certain island size depending on the rate of plasma cross-field diffusion. The nonlinear theory of tearing modes can explain a number of effects observed in connection with Mirnov oscillations.

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Figure Captions

Fig. 1 Eigenfunctions ψ, ϕ of the resistive tearing mode in a cylindrical plasma, $m = 2$, $\eta = 10^{-6}$

Fig. 2 Time development of growth rate and frequency of drift tearing modes. Island size $W(t_o) = \delta_s$

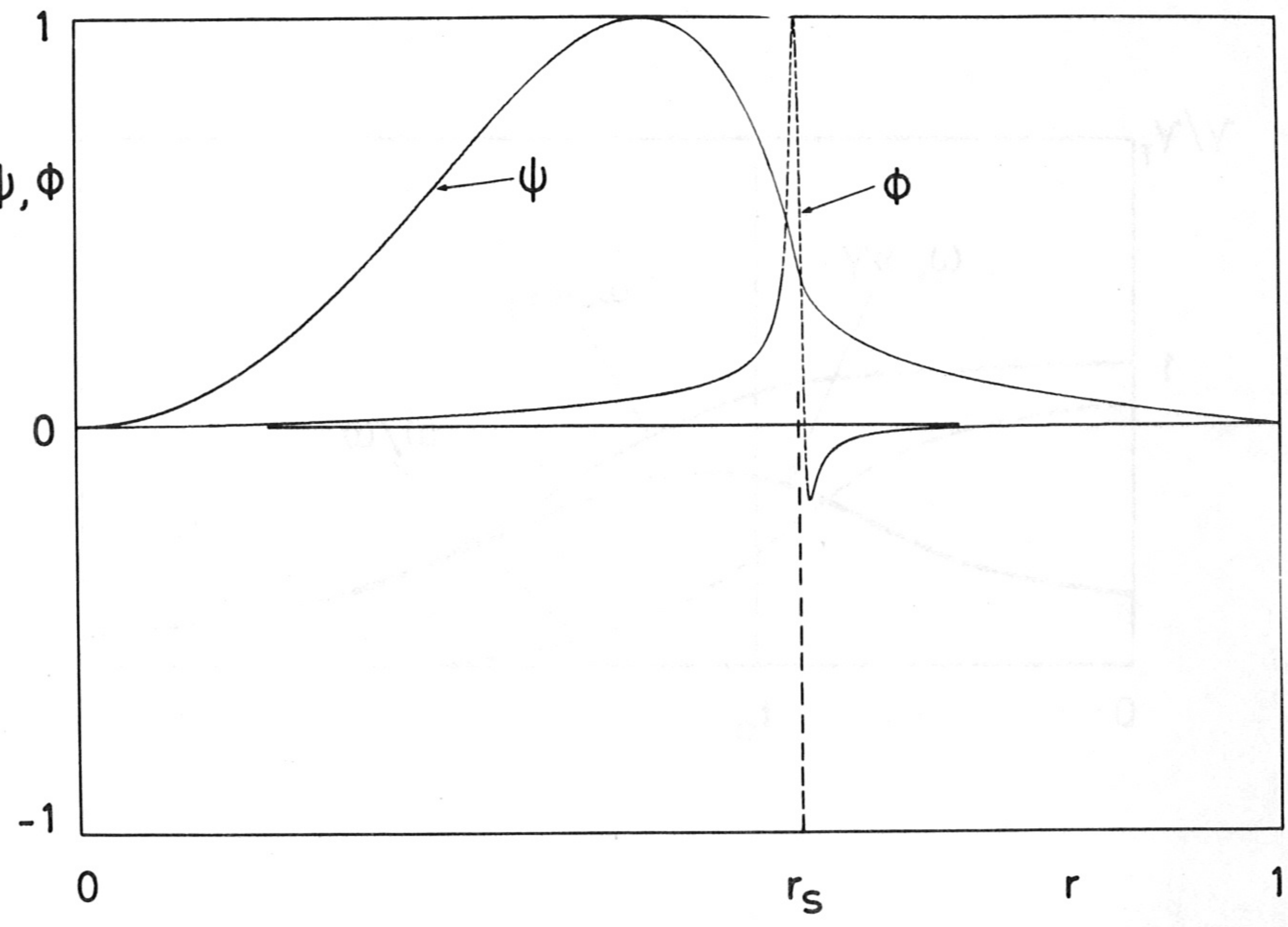

Fig. 1

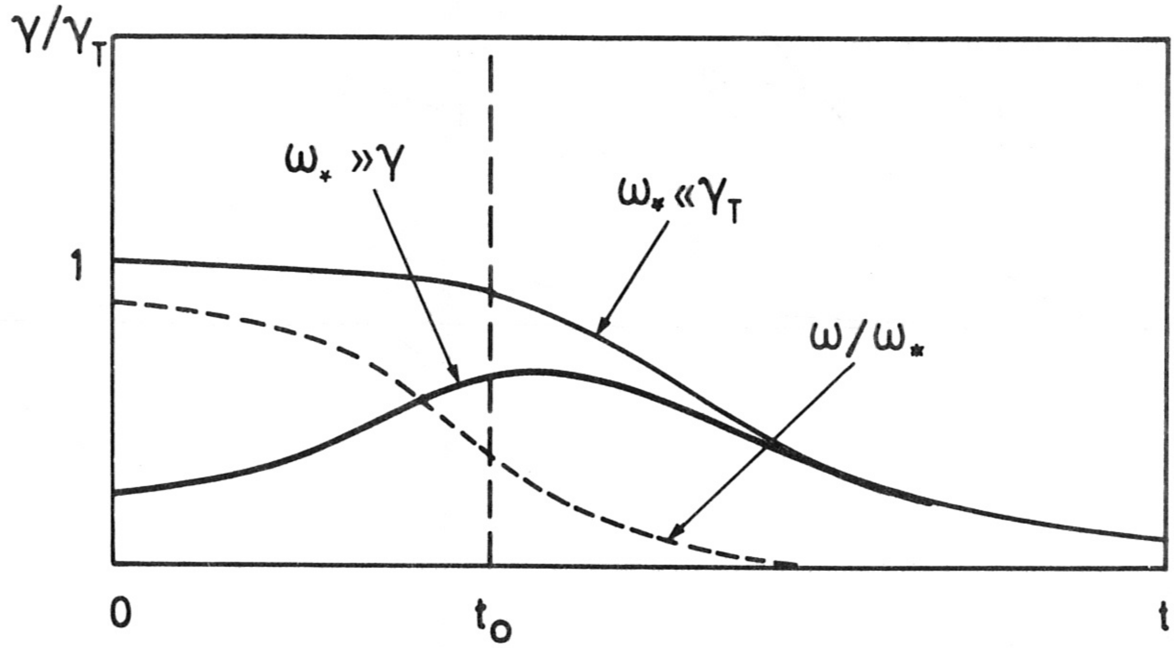

Fig. 2