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Improved Formulas for Trapped-Ion Anomalous Transport in Tokamaks without and with Shear

- Extended Version -

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Abstract

More refined numerical calculations of trapped-ion anomalous transport in a 2-D slab, trapped-fluid model suggest an anomalous diffusion coefficient $D \approx 3.5 \times 10^{-2} \delta_0 a^2 \nu_i^{eff}$ for a tokamak plasma without shear. This supersedes earlier results. The new formula is independently confirmed by two different analytical calculations. One of them uses a similarity analysis of unabridged Kadomtsev-Pogutse-type trapped-fluid equations and the "multiperiodic" spatial structure of the saturated trapped-ion wave found in both the earlier and the recent numerical calculations. The other calculation yields a class of exact nonlinear solutions of the trapped-fluid equations. The new shearless result is used to derive the anomalous diffusion with shear effect by a method described in an earlier paper. The new transport formulas have been numerically evaluated for several tokamaks in an IPP report, where the results are shown in graph form.

1. INTRODUCTION

Earlier numerical calculations by SAISON et al. (1976, 1978) of the anomalous diffusion coefficient D caused by the dissipative trapped-ion instability in a (toroidal) plasma without shear, approximated by a slab model, gave a BOHM-like diffusion formula. Recently, this earlier shearless result was used to derive an anomalous diffusion formula that takes shear effects into account (SARDEI and WIMMEL, 1979).

Now, more refined numerical solutions (SARDEI and WIMMEL, to be published elsewhere) of the unabridged KADOMTSEV-POGUTSE trapped-fluid equations (see SAISON et al., 1978, and further literature cited therein) in the form of an initial-value problem with appropriate boundary conditions have provided a more accurate result for anomalous transport without shear: see Table I. As in the earlier calculations, the time-asymptotic plasma state shows a coherent nonlinear wave-type solution (rather than turbulence). The existence of a saturated state is guaranteed by the use of unabridged trapped-fluid equations (see SAISON et al., 1978). From Table I the following average scaling can be deduced for the range $0.5 \leq m_{marg} \leq 6.0$:

$$D \approx 3.5 \times 10^{-2} \, \delta_0 \, \nu_i \, a^2 \, , \qquad (1.1)$$

where cgs units are used, a is the minor plasma radius, ν_i is the effective collision frequency of the trapped ions, i.e.

$$\nu_i \approx \nu_{ic} / \delta_0^2 \quad , \tag{1.2}$$

ν_{ic} being the 90° Coulomb collision frequency, and δ_0 is the fraction of trapped particles in equilibrium. The value $\nu_i / \nu_e = 1.17 \times 10^{-2}$ was used in the calculations; this is valid for a deuterium plasma with $T_i = T_e$ according to BRAGINSKII (1965). The quantities m_{marg}, defined in eq. (1.4), and ν_i / ν_e are the only relevant dimensionless equilibrium parameters, as is shown in Sec. 2. Moreover, m_{marg} is an approximate expression (valid for $\nu_i \ll \nu_e$) of the marginally stable mode number m_y. The new diffusion coefficient of eq. (1.1) is independent of the magnetic field strength B. From this new shearless result the corresponding new formula for anomalous diffusion in a plasma with shear can again be obtained by the general method of SARDEI and WIMMEL (1979). This derivation will be done in Sec. 4.

The other main subject of this paper is to confirm the new shearless formula, eq. (1.1), by two independent methods. This is done in Secs. 2 and 3. The first method (Sec. 2) combines a similarity analysis of the unabridged trapped-fluid equations (see SAISON et al., 1978) with a simple property common to the earlier and the recent numerical solutions. The scaling of eq. (1.1) follows. The "simple property" means that the saturated trapped-ion wave pattern is nearly "multiperiodic" in the ignorable y-direction of

the slab model used (see Fig. 1), with a dominant wave number m_y that, for broad-spectrum initial conditions, is given by

$$m_{dom} \approx 4\, m_{marg}\,, \tag{1.3}$$

with

$$m_{marg} = b\,(\nu_e \nu_i)^{1/2} / (2\pi v_0)\,. \tag{1.4}$$

Equation (1.3) holds for $m_{marg} \gtrsim 1/4$. "Multiperiodicity" means that the wave pattern has several (more than one) periods in the y-direction, the fundamental period b′ being given by $b' = b/m_{dom} < b$, where b is the extent of the slab in the azimuthal y-direction. In addition, ν_e is the effective collision frequency of the trapped electrons, and v_0 is the trapped diamagnetic drift velocity, viz.

$$v_0 = (\delta_0 c T) / [2 e B r_n (1 - \delta_0)] > 0\,, \tag{1.5}$$

where gaussian cgs units are used, T is the temperature in erg, $r_n = n^o/n^o_x$ is the scale length of the radial density profile of the trapped particles (subscript x denoting the radial derivative), and the remaining notation is standard. The equilibrium parameters in the formulas for ν_i, ν_e, and v_o must be understood as averages over x, so that ν_i, ν_e, and v_o are constants. The quantities B and δ_0 are taken as constants too.

The second method of confirmation consists in constructing exact nonlinear solutions of the unabridged trapped-fluid equations

(see SAISON et al., 1978) using the ansatz

$$n^i = n^0(x) + n_1^i(x) + n_2^i(x) \cdot \cos\left[Ky - \omega t - \varphi^i(x)\right]$$

for the trapped-particle densities, with $K = 2\pi m/b$. These solutions represent stationary nonlinear waves similar to those developing at late times in numerically solving the initial-value problem with initial perturbations monochromatic in m_y. From these solutions an analytic expression for the accompanying anomalous diffusion coefficient can be obtained (Sec. 3).

We should also explain why the earlier calculations (SAISON et al., 1976, 1978) gave a different result. Common to the earlier and recent calculations is the use of a 2-D slab model, with $0 \leq x \leq a$, $0 \leq y \leq b$, x being the radial and y the azimuthal coordinate; and in both instances the initial-value problem for the unabridged trapped-fluid equations [see SAISON et al., 1978] was numerically solved with the appropriate boundary conditions (see Sec. 2) and using LELEVIER's first-order, "one-sided", explicit difference scheme, which is a specialization of the CARLSON scheme (see RICHTMYER and MORTON, 1967). The recent calculations differ in that they employ an additional (GAUSSian) spectral cut-off in both K_x and K_y, set $b = \pi a$ instead of $b = a$, use a finer y - grid for large values of m_{marg} and m_{dom}, and evaluate a diffusion coefficient D averaged over x, y, and t rather than over y only and taken at the special values $x = a/2$, $t = t_{final}$. Even though the unabridged trapped-fluid equations used by us exactly provide for

saturation of the density amplitudes (see SAISON et al., 1978), an additional cut-off (or damping) at short wavelengths is advisable because the drift approximation breaks down there, the linear growth rate γ increases indefinitely with K_y, and without cut-off discontinuous solutions would not be excluded. We chose a mathematically simple cut-off procedure, rather than one that would quantitatively represent an actual physical process. This seems appropriate because the anomalous transport proves to be mainly produced by long-wavelength parturbations (see also Appendix C). In addition, this approach provides the opportunity to test various cut-off and damping procedures as to their consequences for numerical stability and convergence. From our calculations and tests we conclude that the deviation of the earlier numerical results (SAISON et al., 1978) from the present more refined ones stems mainly from the lack of a cut-off in the earlier ones and from the fact that the numerical grid was too coarse for large values of m_{dom} and m_{marg}. For a more detailed discussion see Appendix C. It should also be noted that the unabridged trapped-fluid equations that we use avoid three severe approximations employed in the original work of KADOMTSEV and POGUTSE (1970). These approximations (not used by us) destroy the saturation of the instability and remove an important symmetry of the equations (see Appendix C).

It could be argued that anomalous transport should be determined from

nonlinear kinetic equations rather than from a fluid approximation. In Appendix C we point out, however, that this would be beyond present means of research. Neither any relevant numerical nor reliable analytical calculations of this kind can now be made. Another point concerns the existence of non-fluid-type varieties of the trapped-ion instability for which magnetic drifts are destabilizing and hence essential (see TAGGER et al., 1977, and TANG et al., 1977). Again, there is at present no adequate way of dealing with these in the nonlinear regime, and it appears that the nonlinear behaviour of the simpler, fluid-type instability ought to be investigated before one attacks these non-fluid-type instabilities (see Appendix C). Whether it would be a sound approach to modify the trapped-fluid equations by adding terms that imitate microscopic effects seems to be a complex mathematical question and remains to be seen.

Granted that several microscopic processes are missing from the trapped-fluid equations used by us, the mathematics by which these equations are solved is correct. This is confirmed by the agreement of the results of three different methods, namely numerical solution of the initial-value problem, similarity analysis, and construction of particular analytical solutions. Hence we argue that our results are the only ones in trapped-particle anomalous transport theory that are based on a systematic and consistent mathematical analysis. Clearly, the basic equations, i.e. the physics involved, should be improved in the future,

if possible. However, in view of the complex mathematical problems involved it would have been unreasonable not to begin research on the basis of the simplest set of pertinent equations that is available.

2. THEORETICAL DERIVATION OF THE TRAPPED-ION ANOMALOUS TRANSPORT WITHOUT SHEAR (SIMILARITY ANALYSIS)

The unabridged KADOMTSEV-POGUTSE trapped-fluid equations (see SAISON et al., 1978) will be used in the form (j = i,e):

$$\tilde{n}^j_t + \nu_j \tilde{n}^j + v_0 \varrho_y + A\left(\tilde{n}^i_x \tilde{n}^e_y - \tilde{n}^i_y \tilde{n}^e_x\right) = 0, \qquad (2.1)$$

with the anomalous diffusion coefficients $D_{i,e}$ for trapped ions and electrons respectively, both being given by the ambipolar formula

$$D \approx \frac{\delta_0 A}{\nu_e n^0_x} \langle \varrho_t \cdot \varrho_y \rangle \qquad (2.2)$$

(see WIMMEL, 1976a, and SAISON et al., 1978). Here $\tilde{n}^j(x,y,t)$ are the trapped particle density perturbations, the subscripts x,y,t designate derivatives, $\varrho = \tilde{n}^i - \tilde{n}^e$, $n^0(x)$ is the trapped-particle density in equilibrium, $\delta_0 = n^0/N^p$ = const is the relative number of trapped particles in equilibrium, the pointed brackets denote the average over x, y, and a suitable interval of t, and the additional constant A is defined as

$$A = \frac{cT}{2eBN^p(1-\delta_0)}, \quad T = \frac{2T_e T_i}{T_e + T_i}, \qquad (2.3/4)$$

N^p being the x-averaged total plasma density of trapped plus untrapped particles (of one species). The other quantities were defined in Sec. 1 or are standard. Contrary to some of our earlier work, D is defined relative to ∇N^p rather than ∇n^0, and the sign of v_0 has been defined

differently, viz. $\nu_o > 0$. Equation (2.2) is an approximation that holds if $\nu_i \ll \nu_e$ (see Sec. 3).

By passing to dimensionless quantities we carry out a similarity analysis and obtain another formula for D. The transformation used is

$$\left.\begin{aligned} t &= \nu_e^{-1}\tau\,, \quad x = a\xi\,, \quad y = b\eta\,, \\ \tilde{n}^j &= (a/r_n)\cdot n^o\left(\tfrac{a}{2}\right)\cdot \mu^j(\xi,\eta,\tau)\,, \\ \varrho &= (a/r_n)\cdot n^o\left(\tfrac{a}{2}\right)\cdot \sigma(\xi,\eta,\tau)\,, \end{aligned}\right\} \tag{2.5}$$

where $r_n = n^o\left(\frac{a}{2}\right)\Big/ n_x^o\left(\frac{a}{2}\right) = \text{const}$ is the scale of the radial density variation in equilibrium. Equations (2.1) and (2.2) are then transformed into

$$\mu_\tau^j + \frac{\nu_j}{\nu_e}\mu^j + \frac{v_o}{\nu_e b}\sigma_\eta + \frac{v_o}{\nu_e b}\left(\mu_\xi^i \mu_\eta^e - \mu_\eta^i \mu_\xi^e\right) = 0 \tag{2.6}$$

and

$$D \approx -\delta_o \frac{a^2 v_o}{b} \langle \sigma_\tau \cdot \sigma_\eta \rangle\,. \tag{2.7}$$

The boundary conditions (see SAISON et al., 1978) require periodicity in y, with the period b, and

$$\tilde{n}^j(x=0) = \tilde{n}^j(x=a) = 0 \tag{2.8}$$

for all y and t. This transforms into periodicity in η, with the period 1, and

$$\mu^j(\xi=0) = \mu^j(\xi=1) = 0. \tag{2.9}$$

Consequently, the solutions μ^j depend on ξ, η, τ, the initial conditions, and the dimensionless equilibrium parameters

$$c_1 = \nu_i / \nu_e \quad , \quad c_2 = v_0 / (\nu_e b) . \tag{2.10}$$

If the dependence on the initial conditions is irrelevant for late times, it follows that

$$D \approx \delta_0 \frac{a^2 v_0}{b} g(c_1, c_2) \tag{2.11}$$

or, equivalently,

$$D \approx \delta_0 \nu_i a^2 g_1 \left(\frac{\nu_i}{\nu_e} , m_{marg} \right), \tag{2.12}$$

where m_{marg} is defined in eq. (1.4). The question of the irrelevance of the initial conditions was discussed ealier (SAISON et al., 1978; SARDEI and WIMMEL, 1979). The numerical calculations support the assumption that D is approximately independent of initial conditions. In comparing eq. (2.12) with numerical results one should also consider the spectral cut-off used in the more recent calculations. This cut-off was employed in order to regularize the trapped-fluid equations at short wavelengths. The cut-off was applied to the time-increments of the trapped-particle densities. In order to derive eq. (2.12) it has been tacitly assumed that D does not noticeably depend on the cut-off half-widths over K_x and K_y, if these half-widths are reasonably chosen. We show in Sec. 3 that at least the K_y cut-off does not influence D in the case of solutions that are monochromatic in K_y. A more general investigation of this point is not practical with present-day computers, but

there are indications that the dependence of D on the cut-off may be weak.

It is possible to eliminate the dimensionless parameter m_{marg} from eq. (2.12) and arrive at a more explicit formula for D. In fact, the solutions to the fluid equations and the diffusion coefficient must remain virtually unaltered when the original y-period b is replaced by the approximate fundamental period $b' = b/m_{dom}$ of Sec. 1, with m_{dom} given by eqs. (1.3) and (1.4). In eq. (2.12) m_{marg} is then replaced by

$$m'_{marg} = m_{marg} \cdot \frac{b'}{b} = \frac{1}{4} , \qquad (2.13)$$

and $g_1 (c_1 , m'_{marg})$ is a function of ν_i/ν_e alone. The diffusion coefficient now assumes the final form (for $m_{marg} \gtrsim 1/4$)

$$D \approx \delta_0 \nu_i a^2 \hat{g}\left(\frac{\nu_i}{\nu_e}\right). \qquad (2.14)$$

This confirms the scaling of eq. (1.1). We note that D is independent of the magnetic field B, the azimuthal period b, and the length scale r_n of the radial density variation. Of course, the existence conditions in parameter space of the fluid-type trapped-ion instability (see Sec. 4 and Appendix C) must also be considered. For instance, when $r_n \to \infty$, with the other parameters kept constant, then m_{marg}, the mode frequency and wave number formally go to infinity. The condition $K_y R_i < \pi$ (R_i being the ion gyro-radius) is then violated; the fluid-type instability disappears, and the pertinent anomalous

diffusion vanishes. But within the existence range of the fluid-type trapped-ion instability (see Sec. 4 and Appendix C) the shearless D is indeed independent of B, b, and r_n, even though the independence of r_n may seem somewhat paradoxical. It should be noted that the final result of eq. (2.14) only uses the "multiperiodicity" of the solutions (see Sec. 1); a sinusoidal dependence on y is not required. It should also be noted that the scaling of eq. (2.14) is compatible with the low-ß, collisional, quasineutral scaling derived by CONNOR and TAYLOR (1977), if δ_0 and ν_i/ν_e are introduced as additional parameters.

3. EXACT ANALYTICAL SOLUTIONS OF THE NONLINEAR FLUID EQUATIONS

This method employs more assumptions than the method of Sec. 2, but the results also contain more detailed information than eq. (2.14). In particular, it is possible to take the spectral cut-off in the y-direction explicitly into account and show that it does not enter the formula for the anomalous diffusion coefficient D. As in Sec. 2, shear effects are omitted here.

We start with the unabridged trapped-fluid equations (see SAISON et al., 1978) supplemented by the K_y cut-off, viz. (j = i,e):

$$\tilde{n}^j_t + \Gamma_y \left\{ \nu_j \tilde{n}^j + A\left(n^i_x n^e_y - n^i_y n^e_x \right) \right\} = 0, \qquad (3.1)$$

where Γ_y is the (GAUSSian) spectral cut-off in the y-direction, and $n^j = n^0 + \tilde{n}^j$ are the trapped-particle densities. Equations (3.1) can be expressed by $\varrho = n^i - n^e$ alone, viz.

$$\varrho_{tt} + (\nu_e + \nu_i)\Gamma_y \varrho_t + \nu_i \nu_e \Gamma_y^2 \varrho - v_0 (\nu_e - \nu_i)\Gamma_y^2 \varrho_y + A\Gamma_y^2 \left\{ \varrho_x \Gamma_y^{-1} \varrho_{yt} - \varrho_y \Gamma_y^{-1} \varrho_{xt} \right\} = 0. \qquad (3.2)$$

In order to solve eq. (3.2) for late times, we use the stationary, monochromatic wave ansatz

$$\varrho = \varrho_1(x) + \varrho_2(x) \cdot \cos\left[K(y - ut) - \varphi(x) \right], \qquad (3.3)$$

where $\omega = Ku = \text{const}$, $K = 2\pi m/b$, and m is an integer. The cut-off factor of the mode m is denoted by g_m; with $g_o = 1$ for $m = 0$. Assuming n_x^o, v_o, ν_i, ν_e to be constants and substituting eq. (3.3) in eq. (3.2) yields three equations for the case $\rho_2 \neq 0$, viz.

$$u = - g_m v_o \frac{\nu_e - \nu_i}{\nu_e + \nu_i} , \tag{3.4}$$

$$\frac{d\rho_1}{dx} = - \frac{\nu_e - \nu_i}{\nu_e + \nu_i} n_x^o \left\{ 1 - \frac{\nu_i \nu_e}{\omega_o^2} \left(\frac{\nu_e + \nu_i}{\nu_e - \nu_i} \right)^2 \right\} , \tag{3.5}$$

$$\frac{d(\rho_2^2)}{dx} = \frac{2\nu_i \nu_e}{\omega_o^2} \cdot \frac{\nu_e + \nu_i}{\nu_e - \nu_i} \cdot n_x^o \cdot \rho_1(x) , \tag{3.6}$$

with $d\rho_1/dx = \text{const}$, $\omega_o = K v_o$. Here one has $v_o = A n_x^o < 0$, the sign differing from that of eq. (1.5). These equations are exact in the sense that the nonlinear terms of eq. (3.2) do not produce second harmonics from the ansatz used. Equation (3.5) can be written in another form by using the exact expression for the marginally stable mode number (see SAISON et al., 1978), viz.

$$\hat{m}_{marg} = \frac{b\sqrt{\nu_e \nu_i}}{2\pi |v_o|} \cdot \left| \frac{\nu_e + \nu_i}{\nu_e - \nu_i} \right| . \tag{3.7}$$

Equation (3.5) then becomes

$$\frac{d\rho_1}{dx} = - \frac{\nu_e - \nu_i}{\nu_e + \nu_i} n_x^o \left\{ 1 - \left(\frac{\hat{m}_{marg}}{m} \right)^2 \right\} . \tag{3.8}$$

In the case $\varrho_2 = 0$, eqs. (3.4) and (3.5) do not apply, and eq. (3.6) assumes the alternative form

$$\varrho_2 \cdot \frac{d\varrho_2}{dx} = - \frac{g_m \nu_i \nu_e}{A K \omega} \varrho_1 \,. \tag{3.6a}$$

On choosing $\varrho_1(x)$ antisymmetric and $\varrho_2(x)$ symmetric with respect to $x = a/2$, one gets for $0 < x < a$

$$\varrho_1(x) = \frac{d\varrho_1}{dx} \cdot \left(x - \frac{a}{2} \right) \tag{3.9}$$

and

$$\varrho_2(x) = \left[C\, x(a-x) \right]^{\frac{1}{2}} , \tag{3.10}$$

with

$$C = \left(\frac{m_{marg}}{m} \right)^2 (n_x^0)^2 \left\{ 1 - \left(\frac{\hat{m}_{marg}}{m} \right)^2 \right\} . \tag{3.11}$$

The boundary conditions require $\varrho = 0$, $n^j v^x = 0$ at $x = 0$ and $x = a$. These boundaries are then characteristic surfaces, the fluid equations leaving n_x^j undetermined. This permits one to put $\varrho_1 = \varrho_2 = 0$ at the boundaries. Then eq. (3.6a) is satisfied at the boundaries. The function $\varrho_1(x)$ is discontinuous at $x = 0$ and $x = a$. Our numerical results make it probable that an additional K_x cut-off would smoothen out these discontinuities of ϱ_1. The solution found does not determine the values of K, and hence of m, ω_0, ω. Also $\varphi(x)$ is left undetermined. In fact, the dominant wave vector K (and, hence, also m, ω_0, ω) appears to remain variable in the numerical calculations, too; in the sense

that K can be influenced by the initial conditions. For broad-spectrum initial conditions the dominant m satisfies the relation $m \approx 4m_{marg}$, but monochromatic initial conditions allow the dominant m to deviate from this standard value. It should be noted, by the way, that the above solutions are independent of the cut-off factor g_m, except eq. (3.4) for the phase velocity u.

From the above solutions one can then derive the respective ambipolar anomalous diffusion coefficients D. One has the definition

$$D = - \langle n^i v^x \rangle / N_x^p \, . \tag{3.12}$$

Here $\underline{v} = c \, [\underline{E} \times \underline{B}] / B^2$, with $v^x = -A \varrho_y$ because of quasineutrality (see SAISON et al., 1978). The densities n^i are given by

$$n^i = n^0 + \frac{1}{\nu_e - \nu_i} \left(g_m^{-1} \varrho_t + \nu_e \varrho \right), \tag{3.13}$$

$$n^e = n^0 + \frac{1}{\nu_e - \nu_i} \left(g_m^{-1} \varrho_t + \nu_i \varrho \right). \tag{3.14}$$

The pointed brackets in eq. (3.12) average over y and t. On using the solution for ϱ one obtains

$$D(x) = \frac{\delta_0}{2} \cdot \frac{\omega_0^2}{\nu_e + \nu_i} \cdot \frac{\varrho_2^2 (x)}{(n_x^0)^2} \, , \tag{3.15}$$

with $\omega_0 = K v_0$. Substituting ϱ_2 and doing an additional average over x finally yields

$$D = \frac{\delta_0 a^2}{12} \frac{\nu_i \nu_e}{\nu_i + \nu_e} \left\{ 1 - \left(\frac{\hat{m}_{marg}}{m} \right)^2 \right\} . \qquad (3.16)$$

Contrary to the analysis of Sec. 2, here m and m_{marg} (or $\hat{m}_{marg}$) are completely arbitrary, i.e. $m_{marg} > \frac{1}{4}$ or $m \equiv m_{dom} = 4 m_{marg}$ are not required. As in an earlier paper (WIMMEL, 1976a), unstable modes, with $m > \hat{m}_{marg}$, yield outward diffusion ($D > 0$), while damped modes, with $m < \hat{m}_{marg}$, formally yield inward diffusion ($D < 0$). For $m \gg m_{marg}$, $\nu_i \ll \nu_e$, eq. (3.16) gives $D \approx \frac{1}{12} \delta_0 \nu_i a^2$, which agrees with the numerical scaling of eq. (1.1) and the theoretical scaling derived in Sec. 2. The absolute value of D here derived is about twice the value determined from numerical calculations (with broad-spectrum initial perturbations). This relatively small discrepancy may stem from the assumption of monochromaticity, the lack of a K_x cut-off in the analytic derivation, and numerical grid effects. It should be noted that the diffusion coefficient of eq. (3.16) is independent of the K_y cut-off factor g_m. This supports the omission of cut-off effects in the similarity analysis of Sec. 2. To check for consistency we consider the case $\nu_i \to \nu_e$. The instability is absent and D must be zero for $\nu_i = \nu_e$. In eq. (3.16) $\hat{m}_{marg}$ and the unstable $m > \hat{m}_{marg}$ go to infinity for $\nu_i \to \nu_e$ [see eq. (3.7)]. The existence condition $K_y R_i < \pi$ of the fluid-type trapped-ion instability (see Sec. 4 and Appendix C) is then violated, whence D = 0. This argument resembles that on the r_n-independence of D (Sec. 2). Equation (3.16) is further discussed in Appendix A.

4. ANOMALOUS TRAPPED-ION TRANSPORT WITH SHEAR EFFECT

The anomalous trapped-ion transport without shear can be modified, by a general method presented earlier (SARDEI and WIMMEL, 1979), so as to include shear effects. This will be done for the new result of eq. (1.1) in this section. Comparison with experiments should then be done using the modified formulas eqs. (4.1) and (4.2) below . In applying the following results to experiments one should also remember that the anomalous trapped-ion diffusion is always ambipolar (see SAISON et al., 1978) so that an anomalous trapped electron diffusion equal to that of the ions is always present.

The method mentioned above consists in replacing in the shearless formula, eq. (1.1), the minor radius a by the distance Δa of two properly chosen mode-rational surfaces (where $K_{\parallel} = 0$). This yields

$$D_s \approx 3.5 \times 10^{-2} \, \delta_0 \, \nu_i \, (\Delta a)^2 , \tag{4.1}$$

where D_s is the anomalous diffusion coefficient with shear, and Δa is given (see SARDEI and WIMMEL, 1979) by

$$\Delta a = \min \left\{ a , \, r_q , \, (r_q / m_{dom}) \right\} , \tag{4.2}$$

with m_{dom} defined by eq. (1.3), and $r_q = q/q'$ = length scale of the radial variation of the safety factor $q = 2\pi/\iota$. The replacement of a by Δa takes into account the effective radial localization by strong Landau damping of the dominant trapped-ion modes in a sheared toroidal

field (see GLADD and ROSS, 1973, and SARDEI and WIMMEL, 1979). The validity of this method was discussed extensively by SARDEI and WIMMEL (1979), and this discussion will not be repeated here. However, the reader is reminded that in evaluating eqs. (4.1) and (4.2) one must also consider the existence conditions for the fluid-type trapped-ion instability (see Appendix C). These are: $2\pi\nu_j < \omega_{bj}$, $\omega < \omega_{bj}$, $K_y R_i < \pi$, and $K_x R_{Bi} < \pi$, i.e. at least $R_{Bi} < \Delta a$. Here ω_{bj} are the bounce frequencies of the trapped ions and electrons, $\omega \approx \omega_0$ is the dominant mode frequency, R_i is the ion gyro-radius, and R_{Bi} is the trapped-ion banana width. If not all existence conditions are satisfied for $m = m_{dom}$, then, as an approximation, one can put $D_s = 0$. By this device one neglects the possibility that other unstable modes, with $m_{marg} < m < 4m_{marg}$, might take over and yield non-vanishing anomalous transport. This does not provide a general problem because the transition to $D_s = 0$ often takes place in a parameter range where the anomalous energy transport caused by D_s is smaller than the neoclassical energy transport by the ions and/or the anomalous energy transport by electrons. In using the fluid equations and the above conditions one also neglects possible contributions of modes that may exist outside the parameter range determined by the above conditions (see Appendix C). As an additional point, KADOMTSEV and POGUTSE (1970) indicate that a transition to the collisionless trapped-ion instability should occur whenever $\nu_e \lesssim K_y v_0$ or, equivalently, $m \gtrsim m_{marg} \cdot (\nu_e / \nu_i)^{1/2}$. A BOHM-type diffusion is suggested for this case, but trapped-ion depletion may well enforce a lower diffusion level (see WIMMEL, 1976b, and Appendix B).

Let us consider the three cases following from eq. (4.2). When $\Delta a \equiv \min\{\dots\} = a$, then eq. (1.1) is recovered, with D replaced by D_{s1}. When $\Delta a \equiv \min\{\dots\} = r_q$, then

$$D_{s2} \approx 3.5 \times 10^{-2}\ \delta_0\, \nu_i\, r_q^2 , \tag{4.3}$$

which is usually similar to eq. (1.1). However when $\Delta a \equiv \min\{\dots\} = r_q / m_{dom}$, then substituting m_{dom} from eqs. (1.3) and (1.4) and putting $b = 2\pi r$ leads to

$$D_{s3} \approx 2.2 \times 10^{-3}\ \delta_0 \left(\frac{r_q}{r}\right)^2 \frac{v_0^2}{\nu_e} . \tag{4.4}$$

This formula applies in a wide range of equilibrium parameters, e.g. for $r_q \leq a$ and $m_{marg} > 0.25$, whence $m_{dom} > 1$. It contains the factor v_0^2/ν_e, which recalls the formula by KADOMTSEV and POGUTSE (1970), viz. $D_{KP} \approx (\delta_0/2)\cdot(v_0^2/\nu_e)$. We want to emphasize, though, that eq. (4.4) differs from D_{KP} in these five respects:

a) Equation (4.4) contains the geometrical factor $(r_q/r)^2$. Such an additional factor is necessary in order that the formula be compatible with the trapped-fluid equations and the necessary boundary conditions.

b) The numerical factor in eq. (4.4) is much smaller than the one occurring in D_{KP}.

c) The physics involved in deriving eq. (4.4) (boundary conditions, shear effect) is different from the one used in deriving D_{KP}.

d) Equation (4.4) is only a special branch of a more general formula [eqs. (4.1), (4.2)].

e) Via eq. (1.1), eq. (4.4) is derived from extensive numerical calculations; and it is confirmed by the two independent theoretical analyses of Secs. 2 and 3.

It should be noted that the fact that the numerical factor in eq. (4.4) is small does not, by itself, make trapped-ion transport, with shear effects included, irrelevant. Recent studies by COPPI and MAZZUCATO (1979), and PFIRSCH et al. (1979) have shown that the anomalous energy transport by electrons can also be expected to be smaller than previously anticipated (in the case of large tokamak machines). An additional discussion of equivalent energy confinement times as they follow from eqs. (4.1), (4.2) is given in Appendix B.

5. CONCLUSION

Improved results for the anomalous transport in tokamaks caused by the dissipative trapped-ion instability have been derived. Three independent methods have been used in order to arrive at reliable and consistent results, viz. numerical solution of the initial value problem, similarity analysis, and construction of special nonlinear solutions. As a basic set of equations, unabridged trapped-fluid equations (see SAISON et al., 1978) were used. The present results are the final ones of a series of more preliminary results. To date, essentially four different nonlinear fluid theories of the anomalous trapped-ion transport exist that are all based upon the trapped-fluid equations, originally proposed (in an approximate form) by KADOMTSEV and POGUTSE (1970). These theories and their authors are listed in Table II, where the resulting anomalous diffusion coefficients have been classified according to their B-dependence and the absence or presence of shear effects. Theory N^{o} 4 is the one presented in this paper. We have listed only 2-D fluid theories (without and with shear effect) that lead to simple scaling laws for D. The 2-D work by COHEN and TANG (1978) is not listed because the only scaling law they give holds only in the marginally unstable case and yields an infinitely large diffusion coefficient in the limit of vanishing weak Landau damping (i.e. for $\eta_i \equiv d \ln T_i / d \ln N^P \to \frac{2}{3}$). See SARDEI and WIMMEL (1979) for a more extended discussion. See also Appendix D.

In an IPP report (WIMMEL, 1980) the trapped-ion anomalous energy transport with shear has been evaluated numerically and compared with anomalous energy transport by electrons (see COPPI and MAZZUCATO, 1979, PFIRSCH et al., 1979) and neoclassical ion heat transport for several tokamaks. Preliminary calculations showed the following:

a) In the PLT heating experiment (EUBANK et al., 1979) the trapped-ion anomalous energy transport is smaller than twice the neoclassical energy transport and equals about 5% of the anomalous energy transport by electrons.

b) In JET the trapped-ion energy transport is generally larger than the anomalous energy transport by the electrons.

c) In INTOR the trapped-ion energy transport allows relatively large values of τ_E and $N\tau_E$ and does not, by itself, prevent ignition of the plasma.

The correctness and validity of our results and methods are carefully discussed in Appendix C. Apart from the quantitative aspect of these results there are two points of more general import. Firstly, our results rigorously refute the widely used and believed approximation formula $D \sim \gamma / K_\perp^2$ (see Appendix C). Secondly, they also refute the belief that anomalous transport from the trapped-ion instability

(or in general) are necessarily connected with "turbulence". In our case, the plasma state and anomalous transport are determined instead by saturated nonlinear coherent waves.

APPENDIX A. A SIMPLE INTERPRETATION OF THE ANOMALOUS DIFFUSION COEFFICIENT

The general expression for the anomalous trapped-ion diffusion coefficient

$$D = \frac{\delta_0 A}{(\nu_e - \nu_i)\, n_x^\circ} \langle \rho_t \cdot \rho_y \rangle \,, \tag{A.1}$$

can be put in a form that permits a simple physical interpretation.

On using the quasineutrality relation

$$\rho = \frac{2 e N^p (1-\delta_0)}{T} \phi \,, \tag{A.2}$$

ϕ being the electric potential, and introducing the fact that the time-asymptotic solutions are travelling waves (in the y-direction), with $\omega / K_y = u$, one obtains

$$D = - \frac{\delta_0 u}{(\nu_e - \nu_i) v_0} \langle (v^x)^2 \rangle \; g_m^{-1} \,, \tag{A.3}$$

where $v^x = c E^y / B$ is the x - component of the $\underline{E} \times \underline{B}$ drift velocity, and g_m is the spectral cut-off factor for $K_y = 2\pi m / b$. Equation (3.4) gave $u = - g_m v_0 \left(\frac{\nu_e - \nu_i}{\nu_e + \nu_i} \right)$ for a monochromatic wave, which yields a cut-off independent result:

$$D = \frac{\delta_0}{\nu_e + \nu_i} \langle (v^x)^2 \rangle \,. \tag{A.4}$$

Even though this formula looks similar to a random-walk result, it has been derived and is valid for a coherent wave pattern. Comparison with eq. (3.16) yields

$$\langle (v^x)^2 \rangle = \frac{a^2}{12} \nu_i \nu_e \left\{ 1 - \left(\frac{\hat{m}_{marg}}{m} \right)^2 \right\} , \tag{A.5}$$

which for $m \gg \hat{m}_{marg}$ simplifies to an m-independent expression:

$$\langle (v^x)^2 \rangle \approx \frac{a^2}{12} \nu_i \nu_e . \tag{A.6}$$

Hence, the fact that D is independent of m (for $m \gg \hat{m}_{marg}$), r_n, and B can be reinterpreted to mean that the square-averaged v^x saturates independently of these quantities. The same conclusion holds for the square-averaged y-derivative of ρ because

$$D = \delta_0 \frac{A^2}{\nu_e + \nu_i} \langle (\rho_y)^2 \rangle , \tag{A.7}$$

whence

$$\langle (\rho_y)^2 \rangle \approx \frac{1}{12} \nu_i \nu_e \frac{a^2}{A^2} . \tag{A.8}$$

APPENDIX B. SCALING LAWS FOR THE EQUIVALENT ENERGY CONFINEMENT TIME

From the anomalous diffusion coefficient with shear effect, D_s, of eq. (4.1) an equivalent energy confinement time τ_E can be defined by

$$\tau_E = (r_m\, a)/(2\, D_s). \tag{B.1}$$

In order to obtain scaling laws for τ_E, we shall put $r \sim r_q \sim a$ in D_s, but keep r_n and δ_0 as independent parameters.

For low temperatures the existence conditions of the fluid-type instability are violated (see Sec. 4 and Appendix C), and from $D_s = 0$ one obtains $\tau_E = \infty$. For higher temperatures, but with $m_{marg} > \frac{1}{4}$, one has $D_s = D_{s3}$. It thus follows that $\tau_E = \tau_{E3}$, with

$$\tau_{E3} \propto \left(\frac{r_m}{a}\right)^3 a^4\, \frac{N\, B^2}{T^{7/2}}\, \frac{1-\delta_0}{\delta_0^5}\,. \tag{B.2}$$

The strong dependence on r_n/a requires that τ_{E3} be evaluated by a numerical transport calculation by which the density profiles are determined self-consistently. For high temperatures, with $(\nu_i/\nu_e)^{\frac{1}{2}} < m_{marg} < 1/4$, and assuming $r_q \lesssim a$, one obtains $\tau_E = \tau_{E2}$, with

$$\tau_{E2} \propto \frac{r_m}{a}\, \frac{T^{3/2}}{N}\, \delta_0\,. \tag{B.3}$$

This scaling is independent of B and a. Consequently, in the validity range of τ_{E2} (or τ_{E1}) a larger tokamak with a stronger B-field does

not perform better than a smaller one with a weaker B-field. However, by sufficiently increasing B and/or a one may also increase m_{marg} to fall in the range $m_{marg} > 1/4$ so that the alternative scaling of eq. (B.2) again applies. The scaling of m_{marg} is given by

$$m_{marg} \propto \frac{r_n}{a} a^2 \frac{N B}{T^{5/2}} \frac{1-\delta_0}{\delta_0^3} . \qquad (B.4)$$

For still higher temperatures, with $m_{marg} < \sqrt{\nu_i/\nu_e}$, KADOMTSEV and POGUTSE (1970) assume a collisionless trapped-ion instability and BOHM-like diffusion. A plausible ansatz for this regime could be $D_{CL} = \max (D_{s2}, D_{Bohm})$. However, this ansatz neglects the fact that the collisional D_s of eq. (4.1) may lead to a considerable depletion of the trapped ions, and that the trapped ions are supplemented only slowly by collisions. This fact will probably modify D_{CL} to read

$$D_{CL} \approx \min\left\{D_{s2}, \max\left(D_{s2}, D_{Bohm}\right)\right\} \equiv D_{s2} . \qquad (B.5)$$

It then follows that eq. (B.3) approximately holds in the whole high-temperature regime with $m_{marg} < 1/4$. This can only be correct if the trapped-ion depletion does not lead to anomalous collisions caused by some micro-instability (inverse loss-cone instability, see WIMMEL, 1976b). The argument also rests on the usual assumption, inherent in the fluid theory, that the untrapped particles do not contribute to the anomalous trapped-particle diffusion. The validity of D_{s1} or D_{s2} for $m_{marg} < \frac{1}{4}$ rests on the results of Sec. 3.

APPENDIX C. QUESTIONS AND COMMENTS

The referees of this paper raised a number of questions that require comment which we think will be of interest to the reader.

One question refers to the relation between the original, approximated fluid equation by KADOMTSEV and POGUTSE (1970) and the unabridged trapped-fluid equations used by us [see eq. (2.1)], in connection with the question of saturation (see SAISON et al., 1978, and further literature cited therein). The unabridged trapped-fluid equations used by us constitute a system of two first-order partial differential equations, while the approximate equation used by KADOMTSEV and POGUTSE (1970) is just a single first-order equation. In fact, KADOMTSEV and POGUTSE (1970) employed three severe approximations, viz. $\nu_i = 0$, $\partial n_e / \partial t = 0$, and, in the convection term of the trapped-electron equation, $n_e \equiv n_o$. These approximations destroy the saturation of the instability, distort the linear dispersion equation for the unstable branch at large values of K_y (the damped branch is removed altogether), and destroy an important symmetry of the equations. It should be noted that, for instance, LAQUEY et al. (1975) and COHEN and TANG (1978) use these approximations in their work, with the consequence that they do not find saturation without adding further dissipation terms. Never-

theless, these approximations and their consequences are not discussed in the literature (see, for example, the review paper by TANG, 1978) except in our earlier work (SAISON et al., 1978, and further papers cited therein). In the unabridged trapped-fluid equations used by us the (nonlinear) convection terms in the two trapped-fluid equations are identical. Hence, a linear equation is obtained by subtraction of the two equations. This symmetry is destroyed by the three approximations mentioned. When the unabridged trapped-fluid equations are used, as in our case, the saturation of density amplitudes can be rigorously proved (SAISON et al., 1978). If, in addition, short wavelengths are cut off or sufficiently damped as in our case, saturation of the density gradient amplitudes also follows. The unabridged trapped-fluid equations that we use, together with a discussion of their main properties, are already found in our earlier papers (WIMMEL, 1976a, 1976b; SAISON et al., 1978; SARDEI and WIMMEL, 1979).

Secondly, the referees ask whether the trapped-fluid equations are not too simple to describe the dissipative trapped-ion instability. It is argued that anomalous trapped-particle transport ought to be derived from nonlinear kinetic equations. These are in fact two different questions, and the question of non-fluid-type branches of the trapped-ion instability is also involved (see further below). Granted that the linearized theory of trapped-ion modes is in a comparatively advanced state, even though even there considerable problems concerning the

collision terms, boundary conditions, numerical evaluation, and comprehensive parameter studies remain (see, for example, MARCHAND et al., 1979, 1980), there are, nevertheless, at the present time insuperable obstacles to solving this nonlinear kinetic problem. This is easily demonstrated. Let us consider, firstly, attempts at solving the problem numerically. Our study of the trapped-fluid equations has shown that about 64 grid points (or more) are needed for each spatial dimension, and that the number of necessary time steps is about 10^4 or larger. Let us also assume that the numerical grid in two-dimensional $(v_\perp, v_\parallel)$ space consists of $16 \times 32 = 2^9$ points. It then follows that at least $2 \times 64^3 \times 2^9 \times 10^4 \approx 2.7 \times 10^{12}$ values of f_i and f_e must be dealt with. Replacing, conservatively, each drift-kinetic equation, including collision terms, by only 10 multiplications yields 2.7×10^{13} multiplications for one run of the initial-value problem. We assume a vectorized multiplication time on the CRAY-1 of 0.5×10^{-7} sec. One single run then takes a computing time of 1.3×10^6 sec $\approx$ 15 days. To derive scaling laws for the anomalous transport coefficients a comprehensive parameter study is required. This necessitates many runs, and the total computing time is increased accordingly. It should also be noted that the computer would have to run about 400 times faster in order to get the computing time for one single run down to 1 hour. This rules out numerical solutions for the time being.

It should also be noted that the numerical solution of this problem would probably be difficult even in case the computing time could be made reasonable. Such a linearly unstable, multidimensional, nonlinear initial-value problem, with largely differing multiple time scales, can be expected to exhibit a certain degree of intrinsic stochasticity, in the sense that certain features of the solutions will sensitively depend on initial conditions, equilibrium parameters, and numerical or analytical approximations. Numerical methods that guarantee numerical stability and convergence are then not available. Rather must they be sought by intelligent trial and error.

Consider, secondly, attempts at solving the (kinetic) problem analytically. An analytic solution of the nonlinear drift-kinetic equations, with collisions and self-consistent fields, in the way of an initial-value problem, can only be attempted with the aid of drastic approximations. However, as discussed above, this nonlinear kinetic problem can be expected to be sensitive with regard to such approximations. In particular, certain standard approximations, e.g. quasilinear theory or random-phase, wave-kinetic equations, cannot describe coherent wave phenomena and hence do not permit general application. Since numerical solutions are not available, it is also difficult to check the reliability of analytical results obtained with the aid of approximations. The standard estimate of transport coefficients, viz. $D \sim \gamma / K_\perp^2$, cannot be used either. Firstly,

this formula is without general foundation because many more expressions would be permitted for dimensional reasons. Secondly, our above results explicitly refute this simple ansatz for D. In the present case a different relation, of the form $D \sim \frac{1}{3} \delta_0 \gamma / K_x^2$, does in fact approximately hold; but the dominant values of K_x and K_y, the latter determining γ, must be obtained from the (numerical) solution of the initial-value problem. Consequently, this formula does not save one the task of solving the trapped-fluid equations or the self-consistent kinetic equations. Moreover, this relation may be invalid for more complex problems. In general, D will depend on the structure and the parameters of the basic equations, on geometry and boundary conditions, and on the consequent properties, particularly the coherence properties and characteristic amplitudes, of the solutions.

Linearized kinetic theory shows that other varieties of the dissipative trapped-ion instability exist for which magnetic drifts are destabilizing and hence essential (see TAGGER et al., 1977, and TANG et al., 1977). This non-fluid type of trapped-ion instability is not now amenable to adequate nonlinear analysis because a fluid model does not apply, and the nonlinear kinetic problem cannot now be solved, as explained above.

In Secs. 2,3, and 4 a number of "existence conditions" for the

fluid-type dissipative trapped-ion modes considered by us are mentioned. Let us explain this terminology. These conditions are usually introduced as necessary conditions of validity of pertinent fluid or kinetic equations. For instance (see Sec. 4), the conditions $2\pi\nu_j < \omega_{bj}$ characterize the banana regime, where trapped and untrapped particles can be distinguished. Using the drift approximation for particle trajectories requires $K_y R_i < \pi$ and $K_x R_i < \pi$ because the $\underline{E} \times \underline{B}$ drift loses its validity if the $\underline{E}$ - field considerably varies over one ion gyro-radius. If, in addition, an average over the bounce motion of the trapped particles is applied and the banana width R_{Bi} is neglected in comparison with the pertinent radial wavelength the conditions $\omega < \omega_{bj}$ and $K_x R_{Bi} < \pi$ must hold. All these conditions are implied when the trapped-fluid equations are used. In our case, these validity conditions are at the same time existence conditions for the instability considered because in our calculations this instability is defined by the trapped-fluid equations. Consequently, we are led to assume that the mode amplitudes and the anomalous transport vanish when not all of the above conditions are satisfied.

A final question is whether our spectral cut-off is physically motivated or ad hoc. This concerns an important point that will be explained in some detail. First of all, our earlier numerical calculations (SAISON et al., 1978) did not use any additional cut-off or damping terms;

however, the numerical scheme (see Sec. 1) provided for some damping at short wavelengths. Similarly to the present results, a coherent wave pattern was obtained rather than turbulence, with the spectrum peaked at large wavelengths. On the other hand, a short-wavelength cut-off (or damping) is clearly advised by, for instance, the fact that the unstable modes grow owing to $\underline{E} \times \underline{B}$ drifts, and the drift approximation breaks down at wavelengths comparable with the ion gyro-radius. In addition, the trapped-fluid equations are mathematically ill defined at short wavelengths because the linear growth rate γ increases indefinitely with K_y, and discontinuous solutions can exist (see SAISON et al., 1978). One will then prefer a cut-off or damping that is independent of the numerical scheme and the numerical grid. The next question is then whether one should use some complicated cut-off that explicitly describes certain physical damping mechanisms, but is possibly not well understood mathematically; or whether one should construct mathematically simple cut-off procedures whose effects upon the solutions, numerical stability and convergence can be checked and studied. We chose the latter approach because it was already known from the earlier calculations (SAISON et al., 1978) that in the present problem anomalous transport is determined by long-wavelength perturbations. This has been confirmed by our present results. The coherent, long-wavelength solutions found numerically for the nonlinear initial-value problem confirm this assumption. Consequently, we use the assumption that the anomalous transport is only weakly

dependent on the cut-off lengths chosen, as long as the cut-off lengths are small relative to the dominant wavelengths. A definite proof of this assumption is not now practical, considering the computing times that would be required on present-day machines, e.g. the CRAY-1 computer we used. The analytical result for D derived in Sec. 3 is in fact completely independent of the K_y cut-off.

For a critical discussion of other authors' work with various nonlinear equations related to our problem, see Appendix D.

APPENDIX D. ON RELATED WORK

A critical assessment of related nonlinear work of some other authors is presented here. Our critique is not intended to detract from such work. Despite its shortcomings we consider the work discussed to be worthwhile and original. On the other hand, some of the comments of one of our referees lead us to believe this evaluation of related work will help readers to arrive at a better assessment of our results as presented in this paper.

The first study to be discussed is that of KADOMTSEV and POGUTSE (1970). These authors were the first to formulate trapped-particle fluid equations as a macroscopic model for the dissipative trapped-ion instability, both in the linear and nonlinear regimes. Unfortunately, they introduced the three approximations mentioned in Appendix C and an additional expansion in ν_e^{-1}, thus arriving at eq. (9.5) of their paper. It is easily seen that this approximate equation does not yield amplitude saturation because it follows that

$$\frac{d}{dt}\int_0^a dx \int_0^b dy\,(n_+')^2 = \frac{2\,v_*^2\,\delta^2}{\nu_e}\int_0^a dx\int_0^b dy\left(\frac{\partial n_+'}{\partial y}\right)^2 > 0 \qquad \text{(D.1)}$$

(in the notation of those authors) if the density perturbations n_+' are periodic in y (with period b) and vanish at the boundaries $x = 0$, $x = a$. (We thank one of the referees of this paper for suggesting a

similar result for boundaries at $x = \pm\infty$.) Independently, it was shown using the unabridged trapped-fluid equations that their putting $\nu_i = 0$ would lead to vanishing anomalous diffusion (WIMMEL, 1976). The neglect of radial boundary conditions by KADOMTSEV and POGUTSE (1970) is unjustified because the explicit dependence of anomalous diffusion on the radial extent of the slab is easily derived by a simple similarity analysis (this paper, and SARDEI and WIMMEL, 1979). In fact, it was shown that the diffusion formula proposed by KADOMTSEV and POGUTSE (1970) is in contradiction to the unabridged trapped-fluid equations including radial boundary conditions (see SARDEI and WIMMEL, 1979). The assumption of isotropic turbulence in the work of KADOMTSEV and POGUTSE (1970) has been explicitly refuted by numerical analysis of the unabridged trapped-fluid equations (see SAISON et al., 1978, and the present paper).

A second nonlinear study to be mentioned is that of LAQUEY et al. (1975). Again, the authors use only one approximate equation (first-order in time) that requires adding an extra Landau damping term in order to secure amplitude saturation for $\eta_i < \frac{2}{3}$. For values $\eta_i \geq \frac{2}{3}$ there is no saturation. In addition, the analysis is only one-dimensional in space, which is tantamount to again neglecting radial boundary conditions. A further shortcoming is the restriction to $\nu_i = 0$ (see above). Finally, an equilibrium spectrum and an anomalous diffusion formula are found; but COHEN et al. (1976) later showed that both are invalid

because the equilibrium spectrum is linearly unstable. Additional questionable points were discussed by SAISON et al. (1978).

A third nonlinear study to be mentioned is that of COHEN et al. (1976). The authors use the same basic approximations as LAQUEY et al. (1975), but do obtain specialized wave equilibria that are linearly stable. In addition, they discuss the case $\nu_i \neq 0$ and dispersion effects due to finite ion banana width. But their results are nevertheless to be questioned because the authors restrict themselve to a one-dimensional analysis, neglect radial boundary conditions, and use only <u>one</u> approximate trapped-fluid equation, first-order in time, instead of the unabridged system of two trapped-fluid equations. Their analysis is again restricted to cases with $\eta_i < \frac{2}{3}$. Particularly, they obtain infinite anomalous diffusion for $\eta_i = \frac{2}{3}$ (and for $\eta_i = 1$, which is, however, outside their parameter range). These and other questionable points were more fully discussed by SAISON et al. (1978).

The fourth nonlinear study to be mentioned, that of COHEN and TANG (1978), supersedes the preceding two by furnishing a <u>two-dimensional</u> analysis, so that a comparison with our two-dimensional work would, in principle, be possible. This study again uses a two-fluid theory that contains additional terms that simulate several microscopic effects. However, this advantage is again compensated by the use of a simplified fluid description that consists of only <u>one</u> approximate fluid equation

(first-order in time, see above). Consequently, saturation is obtained only for $\eta_i < \frac{2}{3}$, as in the second and third studies mentioned above. In addition, COHEN and TANG (1978) restrict themselves to specialized forms of wave spectra and wave-wave interactions. For an assumed three-wave interaction their result remains inconclusive because the resulting equilibrium spectrum turns out to be linearly unstable. No scaling law for transport is given in this case. For an assumed four-wave, one-mode self-interaction, the result holds only for rather small saturation amplitudes close to marginal stability. Here a scaling law is given [eq. (45) of their paper], but that result cannot be compared with ours because taking the limit of vanishing Landau damping in their formula, i.e. letting $\eta_i \to \frac{2}{3}$, makes their anomalous diffusion coefficient D go to infinity. Two possible reasons for these discrepancies can readily be named. Firstly, the use of a drastically approximated version of a set of basic equations for describing a complex, multi-dimensional, nonlinear, initial-value problem with largely differing multiple time scales may completely change the character of the solutions. Secondly, a similar problem arises with respect to approximate methods of evaluation of the basic equations. In particular, our numerical work shows that rather long relaxation times to the saturated state prevail. Then, quite erroneous final states may be obtained by discarding most degrees of freedom and classes of interactions involved, keeping only a few Fourier modes. The general problem of accessibility of final states cannot be reliably and realistically investigated in such an approximate way.

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m_{marg}	$D/(\delta_o \nu_i a^2)$	$D/(\delta_o a v_o)$	D/D_{KP}	$(e\Phi/T)_{max}$	m_{dom}	m_{cy}	m_{oy}	N_y
0.5	4.10×10^{-2}	4.43×10^{-3}	8.21×10^{-2}	1.43×10^{-1}	3	4	8	64
1.0	3.37×10^{-2}	7.27×10^{-3}	2.69×10^{-1}	1.42×10^{-1}	4(+ 5)	8	16	64
2.0	3.40×10^{-2}	1.47×10^{-2}	1.09	1.54×10^{-1}	8(+ 9)	16	32	128
3.0	4.19×10^{-2}	2.71×10^{-2}	3.02	1.73×10^{-1}	14(+15)	32	64	256
4.0	3.39×10^{-2}	2.93×10^{-2}	4.34	1.58×10^{-1}	16(+18)	32	64	256
5.0	3.02×10^{-2}	3.26×10^{-2}	6.04	1.61×10^{-1}	20(+...)	40	80	256
6.0	2.77×10^{-2}	3.59×10^{-2}	7.98	1.73×10^{-1}	22(+...)	48	96	256

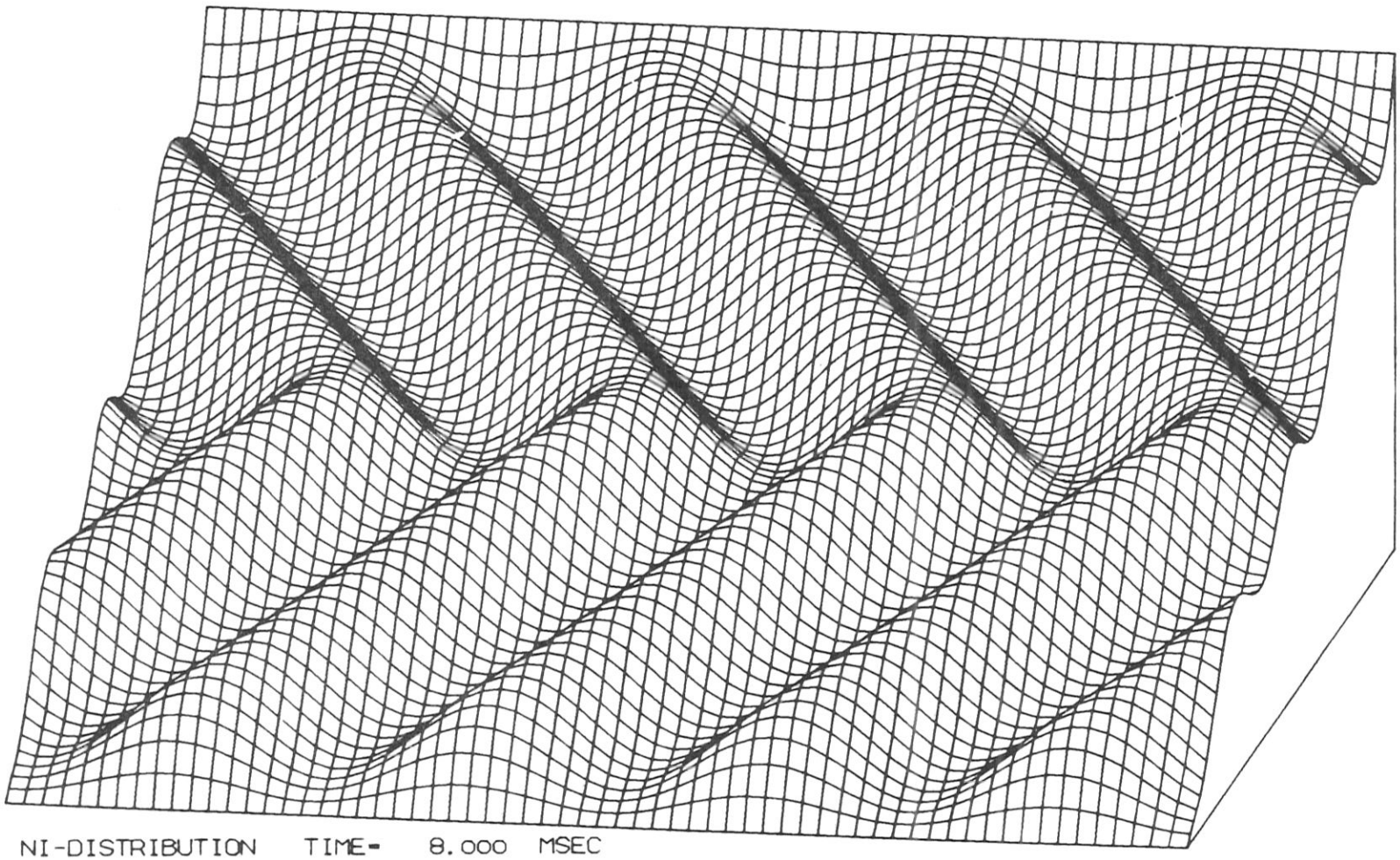
Table I. Anomalous trapped-ion diffusion coefficient D versus m_{marg}, the marginally stable m_y, for $\nu_i/\nu_e = 1.17 \times 10^{-2}$. Quantities: δ_o = fraction of trapped particles, a = minor plasma radius, v_o = trapped diamagnetic drift velocity, ν_j $(j = i, e)$ = effective collision frequencies of trapped particles, D_{KP} = Kadomtsev - Pogutse diffusion coefficient, Φ = electric potential, m_{dom} = dominant azimuthal mode number m_y, m_{cy} = half-width of numerical cut-off of m_y, m_{oy} = highest m_y present in the initial perturbation, N_y = number of grid intervals in the y-direction, $m_{cx} = 16$ = half-width of cut-off of m_x, $N_x = 64$ = number of grid intervals in the x-direction. The values of $(e\Phi/T)_{max}$ hold for $\delta_o = 1/2$.

N°	Authors	Year	Without Shear	With Shear
1	Kadomtsev Pogutse	1967/70	$D = D_{KP} \propto B^{-2}$	
2	Saison Wimmel Sardei	1976/78	$D \propto B^{-1}$	
3	Sardei Wimmel	1979		$D_{s1}, D_{s2} \propto B^{-1}$ --- $D_{s3} \propto B^{-3}$
4	Sardei Wimmel	1980	$D \propto B^{0}$	$D_{s1}, D_{s2} \propto B^{0}$ --- $D_{s3} \propto B^{-2}$ $\neq D_{KP}$

Table II. Classification of the fluid theories of trapped-ion anomalous transport according to the B-dependence of the resulting anomalous diffusion coefficients D and D_s.

Figure Caption

Fig. 1. Trapped-ion density distribution $n_i(x,y)$ in the x-y plane, for $m_{marg} = 1$; example of a saturated state of the instability at late times, with the initial perturbations monochromatic in K_y.



NI-DISTRIBUTION TIME= 8.000 MSEC