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Reduced, three-dimensional, nonlinear equations for high-β plasmas including toroidal effects

R. Schmalz

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Abstract

The resistive MHD equations for toroidal plasma configurations are reduced by expanding to the second order in ε, the inverse aspect ratio, allowing for high $\beta = \mu_0 p/B^2$ of order ε. The result is a closed system of nonlinear, three-dimensional equations where the fast magnetohydrodynamic time scale is eliminated. In particular, the equation for the toroidal velocity remains decoupled.

I. Introduction

In order to follow the dynamics of toroidal plasmas numerically, it seems highly desirable to reduce the original magnetohydrodynamic equations. A great simplification was achieved by the equations of Strauss (1977), who used an expansion procedure in the inverse aspect ratio $\varepsilon \ll 1$. He derived the lowest-order nonlinear 3-D equations for high-β (β of order ε) plasmas. Later work (e.g. Biskamp and Welter 1977, Carreras, Hicks, Lee and Waddell 1977, Edery, Pellat and Soule 1979) included toroidal effects by taking into account the next higher order in ε, but was restricted to low-β plasmas. In the present paper we derive the reduced, nonlinear equations including both high-β and toroidal effects.

II. Basic equations

The model equations underlying the following considerations are

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v}:\underline{v}) = - \nabla p + \underline{j} \times \underline{B} , \tag{1}$$

$$\frac{\partial B}{\partial t} = \nabla \times (\underline{v} \times \underline{B} - \eta \, \underline{j}), \tag{2}$$

$$\nabla \times \underline{B} = \mu_o \underline{j}, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = Q, \tag{4}$$

$$\frac{\partial p}{\partial t} + \nabla \cdot (p\underline{v}) = (\gamma-1)(-p\nabla\cdot\underline{v} + \eta j^2 + \nabla\cdot(\underline{\underline{\kappa}}\cdot\nabla T)) . \qquad (5)$$

All symbols have the usual meaning, and Q represents a mass source. As usual we shall later neglect the last two terms in eq.(5).

As a first step, we introduce dimensionless quantities by setting, for example, $p = p_o\tilde{p}$, $t = t_o\tilde{t}$ etc. We define

$$\beta := \frac{\mu_o p_o}{B_o^2}$$

and the magnetic Reynolds number

$$S := \frac{\mu_o a v_o}{\eta_o} = \frac{t_R}{t_H} ,$$

where a is a typical length scale and the characteristic velocity is $v_o = v_A = B_o/\sqrt{\mu_o\rho_o}$. Omitting all tildes we obtain

$$\left(\frac{a}{v_A t_o}\right) \frac{\partial}{\partial t}(\rho\underline{v}) + \nabla \cdot (\rho\underline{v}:\underline{v}) = -\beta\nabla p + \underline{j} \times \underline{B} , \qquad (6)$$

$$\left(\frac{a}{v_A t_o}\right) \frac{\partial}{\partial t} \underline{B} = \nabla \times (\underline{v} \times \underline{B} - \frac{\eta}{S} \underline{j}) , \qquad (7)$$

$$\nabla \times \underline{B} = \underline{j} , \qquad (8)$$

$$\left(\frac{a}{v_A t_o}\right) \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\underline{v}) = Q , \qquad (9)$$

$$\left(\frac{a}{v_A t_o}\right) \frac{\partial p}{\partial t} + \nabla \cdot (p\underline{v}) = (1-\gamma)p\, \nabla\cdot\underline{v} . \qquad (10)$$

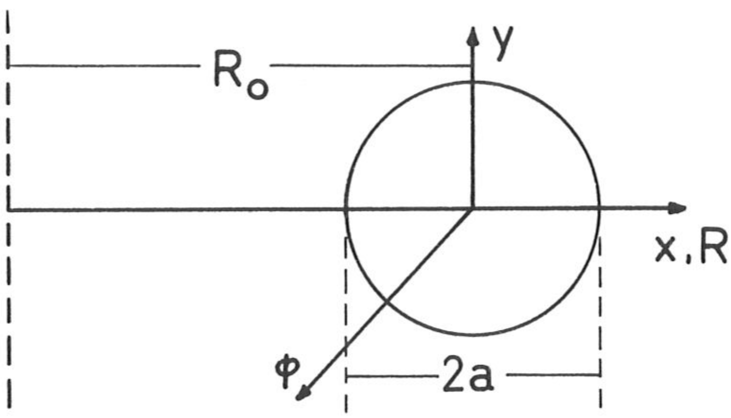
III. Reduction of the equations

We introduce dimensionless cylindrical coordinates (R,y,ϕ), where R is measured in units of R_o, and y in units of a, which represents the minor radius of the plasma. Furthermore, we replace R by x through

$$R = 1 + \varepsilon x ,$$

with

$$\varepsilon := \frac{a}{R_o} .$$



If $\varepsilon \ll 1$, eqs. (6) to (10) may be expanded using ε as an ordering parameter, provided this is possible. We assume the following ordering (see Strauss 1977):

$$B_\phi = O(1), \quad B_\perp = O(\varepsilon), \; \rho = O(1), \; v_\perp = O(\varepsilon),$$

$$v_\phi = O(\varepsilon^2), \; \beta = O(\varepsilon) \text{ and } S \gg \varepsilon^{-1}.$$

For this ordering to be consistent with eqs. (6) - (10) it follows that the natural time scale for the dynamics is $t_o = a/v_A/\varepsilon = R_o/v_A$, which means that the fastest hydrodynamic time scale $t_H = a/v_A$ is eliminated. Toroidal gradients are in general of order ε ($_\perp$ means poloidal):

$$\nabla = \nabla_\perp + \frac{\varepsilon}{R} \underline{e}_\phi \frac{\partial}{\partial\phi} \text{, e.g. } \nabla\phi = \frac{\varepsilon}{R} \underline{e}_\phi \quad .$$

We stress that all variables in the following equations are of order one and the order of magnitude of each term is explicitly expressed by powers of ε.

The magnetic field $\underline{B}$ is decomposed into a vacuum toroidal field plus a correction due to plasma currents. For the vector potential of the latter we make the ansatz

$$\underline{A} = R^2 \nabla\chi \times \nabla\phi + \psi \nabla\phi \tag{11}$$

with the gauge $\nabla \cdot (\underline{A}_\perp/R^2) = 0$.
So we have

$$\underline{B} = \frac{1}{R} \underline{e}_\phi + \nabla \times \underline{A} = \nabla\psi \times \nabla\phi + \varepsilon^2 \nabla_\perp \chi_{,\phi} + \frac{1-\varepsilon R^2 \tilde{\Delta}_\perp \chi}{R} \underline{e}_\phi \ , \tag{12}$$

where

$$_{,\phi} := \frac{\partial}{\partial\phi} \ ,$$

$$\tilde{\Delta}_\perp := \frac{1}{R}\frac{\partial}{\partial x} \left(R \frac{\partial}{\partial x}\right) + \frac{\partial^2}{\partial y^2} =: \Delta_\perp + \frac{\varepsilon}{R} \frac{\partial}{\partial x}$$

and $\nabla \cdot \underline{B} = 0$ is obviously satisfied.

The current density is obtained by taking the curl of eq. (12):

$$\underline{j} = -\nabla(R^2 \Delta\chi) \times \nabla\phi + \frac{\varepsilon^2}{R^2} \nabla_\perp \psi_{,\phi} - \Delta^* \psi \nabla\phi \ . \qquad (13)$$

Here Δ^* represents the standard toroidal operator $R^2 \nabla \cdot (\frac{1}{R^2} \nabla_\perp)$.

Let us first review two basic results obtained by Strauss (1977):

To lowest order in ε the momentum balance equation (6) yields

$$\beta p = \varepsilon \, \Delta_\perp \chi + \text{const.} + O(\varepsilon^2), \qquad (14)$$

and from the pressure equation (10) it follows that

$$\frac{Dp}{Dt} = O(\varepsilon) \ . \qquad (15)$$

Here the total time derivative is given in lowest order by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + [f,U] + O(\varepsilon),$$

where a convenient abbreviation

$$[f,U] := \nabla f \times \nabla U \cdot \underline{e}_\phi$$

is used. U represents the velocity stream function

$$\underline{v}_\perp = \nabla U \times \nabla\phi + O(\varepsilon^2) \ .$$

From eqs. (14) and (15) we deduce

$$\frac{\partial}{\partial t} \Delta_\perp \chi + [\Delta_\perp \chi, U] = O(\varepsilon) . \tag{16}$$

Let us now proceed to the next order in ε. Instead of eq. (7) we use the equation for the vector potential

$$\varepsilon \frac{\partial \underline{A}}{\partial t} = \underline{v} \times \underline{B} - \varepsilon \nabla \Phi - \frac{\eta}{S} \underline{j} , \tag{17}$$

where Φ is the electrostatic potential.

Operating with $\underline{e}_\phi \times$ on eq. (17) and making use of eq. (16), the perpendicular velocity may be written

$$\underline{v}_\perp = R^2 \nabla U \times \nabla \phi + O(\varepsilon^3) . \tag{18}$$

A connection between U and Φ is established by taking the divergence of eq. (17), divided by R^2:

$$\Delta^*(U+\Phi) = \varepsilon \nabla \cdot (\Delta_\perp \chi \nabla_\perp U) + O(\varepsilon^2) . \tag{19}$$

We notice that in the limiting cases $\varepsilon \to 0$ or low β ($\chi = O(\varepsilon)$) we simply have $U = - \Phi$. Although for high β the two potentials are different, eq. (18) tells us that the "toroidal incompressibility" $\nabla \cdot (\underline{v}_\perp / R^2) = 0$ is maintained to second order. This is surprising since for $\beta = O(\varepsilon)$ the actual toroidal field may strongly deviate from the 1/R dependence.

On the other hand, taking the ϕ-component of eq. (17) yields

$$\frac{D\psi}{Dt} = - \Phi_{,\phi} - \varepsilon \nabla\chi_{,\phi} \cdot \nabla_{\perp} \Phi + \frac{\eta}{\varepsilon S} \Delta^{*}\psi + O(\varepsilon^{2}) \ . \tag{20}$$

Here the total time derivative is given by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + R \left[f,U\right] + O(\varepsilon^{2})$$

and v_{ϕ} does not yet contribute.

To simplify the density equation, we define

$$\tilde{\rho} := R^{2}\rho$$

and obtain from eq. (9)

$$\frac{D\tilde{\rho}}{Dt} = \frac{QR^{2}}{\varepsilon} + O(\varepsilon^{2}) \quad . \tag{21}$$

If processes which do not strongly depend upon the density gradient are considered, we may set $Q = 0$ and, for example, $\tilde{\rho} = 1$ or, alternatively, $\rho = 1$ and $Q = 2\ \varepsilon^{2}\left[\chi,U\right]$. In both cases eq.(21) is satisfied and can be omitted.

We now go on to the momentum equation (6).

The ϕ-component of the vorticity

$$W: = \frac{R^2}{\varepsilon^2}\nabla\phi \cdot \nabla\times\tilde{\rho}\underline{v}_\perp = R^2\nabla \cdot (\tilde{\rho}\nabla_\perp U) + O(\varepsilon^2) \tag{22}$$

obeys an equation which follows by operating with $\nabla\phi \cdot \nabla\times$ on eq. (7), multiplied by R^2:

$$\frac{DW}{Dt} = R\,[\Delta^*\psi,\psi] + (1 - \varepsilon\Delta_\perp\chi)\,\Delta^*\psi_{,\phi} + 2\,\frac{\beta}{\varepsilon}\,R^2\,[\chi,p] -$$

$$- \varepsilon\nabla\Delta_\perp\chi \cdot \nabla_\perp\psi_{,\phi} + \varepsilon\nabla \cdot (\Delta_\perp\psi\nabla_\perp\chi_{,\phi}) - \tag{23}$$

$$- \frac{R^2}{2}\,\nabla \cdot (\frac{\tilde{\rho}}{R}\,\nabla\,(R^2|\nabla_\perp U|^2)\times\,\underline{e}_\phi) + O(\varepsilon^2)\ .$$

This equation determines W and in turn U via eq. (22). The remaining toroidal component of eq. (6) leads to

$$\varepsilon\,\frac{Dv_\phi}{Dt} = -\frac{1}{R}\,(\frac{\beta}{\varepsilon}\,p + \frac{\varepsilon}{2}\,|\nabla_\perp\psi|^2)_{,\phi} + \frac{1}{R^2}\,[I,\psi] + \varepsilon\nabla_\perp I \cdot \nabla\chi_{,\phi} + O(\varepsilon^2)\ . \tag{24}$$

Here the definition

$$I: = -\,R^2\tilde{\Delta}_\perp\chi \tag{25}$$

was introduced and v_ϕ is replaced by $\varepsilon^2 v_\phi$ according to our ordering. We notice that in eq. (24) I is required to the next higher order exceeding eq. (14). An equation for I in the required order follows from the divergence of eq. (6), multiplied by R^2 and profiting by $\nabla\cdot\underline{v} = O(\varepsilon^2)$. We find (for Q = 0)

$$\nabla \cdot ((1+\varepsilon I)\nabla_\perp I) = - \frac{\beta}{\varepsilon}\nabla \cdot (R^2 \nabla p) + 2\varepsilon [U_{,x}, U_{,y}] -$$

$$- \varepsilon\nabla \cdot (\Delta_\perp \psi \nabla_\perp \psi) + O(\varepsilon^2) \; . \tag{26}$$

The most important observation here is that the v_ϕ equation (24) remains completely decoupled from the former ones, and eq. (26) is only necessary for solving eq. (24). Therefore, the lowest-order relation (14) is sufficient if we are not interested in v_ϕ.

It remains to expand eq. (10), which gives

$$\frac{Dp}{Dt} = 2\varepsilon\gamma p \, [U,x] + O(\varepsilon^2) \; . \tag{27}$$

To summarize, the relevant equations are:
(14) for χ, (19) for Φ, (20) for ψ, (21) for $\tilde{\rho}$,
(22) for U, (23) for W and (27) for p. Equations (24) for v_ϕ and (26) for I are decoupled. Let Γ be the boundary of a cross-section ϕ=const, then the boundary conditions $U(\Gamma) = 0$, $\Phi(\Gamma) = 0$, $p(\Gamma) = 0$, $\chi(\Gamma) = 0$ and $\psi(\Gamma) = 0$ may be chosen. The latter can only be achieved by adding $\Phi_o = RE_\phi(\Gamma) = \text{const.}$ to eq. (20). For a synopsis we collect here our equations for the case $Q = 0$, $\tilde{\rho} = 1$:

$$\frac{D\psi}{Dt} = - \Phi_{,\phi} - \varepsilon \nabla \chi_{,\phi} \cdot \nabla_{\perp} \Phi + \frac{\eta}{\varepsilon S} \Delta^* \psi + \Phi_o , \tag{28}$$

$$\frac{DW}{Dt} = R \left[\Delta^*\psi, \psi\right] + (1 - \varepsilon \Delta_{\perp}\chi) \Delta^*\psi_{,\phi} + 2 \frac{\beta}{\varepsilon} R^2 \left[x, p\right] -$$

$$- \varepsilon \nabla_{\perp} \Delta_{\perp}\chi \cdot \nabla\psi_{,\phi} + \varepsilon \nabla \cdot (\Delta_{\perp}\psi \nabla_{\perp}\chi_{,\phi}) , \tag{29}$$

$$\frac{Dp}{Dt} = 2\varepsilon\gamma p \left[U, x\right], \tag{30}$$

$$\Delta^*\Phi = - \Delta^* U + \varepsilon \nabla \cdot (\Delta_{\perp}\chi \nabla_{\perp} U), \tag{31}$$

$$\tilde{\Delta}_{\perp} U = \frac{W}{R^2} , \tag{32}$$

$$\Delta_{\perp}\chi = \frac{\beta}{\varepsilon} p - c . \tag{33}$$

The terms including ε give rise to toroidal effects of first order. The equations of Strauss are recovered from (28) to (33) simply by letting $\varepsilon \to o$ (remember $R = 1+\varepsilon x$). The low-β case is obtained by assuming $\chi = 0(\varepsilon)$ and $\beta=0(\varepsilon^2)$. In both limiting cases we may set $U = -\Phi$ and forget eqs. (31) and (33) in the second case. Furthermore, by omitting the time derivatives we have an expansion of the equilibrium equations as studied by, for example, Yoshikawa (1974) by means of the Grad-Shafranov equation ($\frac{\partial}{\partial \phi} = 0$) .

IV. The energy equation

Multiplying eq. (1) by $\underline{v}$ and integrating over the whole volume with the boundary condition $\underline{v}$(surface) = 0 and using eqs. (2)-(5) leads to the general energy equation

$$\frac{\partial}{\partial t} \int (\rho \frac{v^2}{2} + \frac{B^2}{2\mu_o} + \frac{P}{\gamma-1}) \, d\tau =$$

$$= - \int Q \frac{v^2}{2} \, d\tau + \oint \underline{n} \cdot (\kappa_\perp \nabla_\perp T - \underline{S}) \, df, \qquad (34)$$

where $\underline{S} = \frac{1}{\mu_o} \underline{E} \times \underline{B}$

represents the Poynting flux.

We now multiply eq. (23) by U/R^2, eq. (20) by $\Delta^*\psi/R^2$ and integrate with the help of eq. (19) to obtain the appropriate expansion of eq. (34) in dimensionless form:

$$\frac{\partial}{\partial t} \int (\frac{1}{2} \tilde{\rho} \, |\nabla_\perp U|^2 + \frac{1}{2R^2} |\nabla_\perp \psi|^2 + \frac{\beta}{\varepsilon} \frac{p}{\gamma-1}) \, d\tau =$$

$$= - \int \frac{Q}{2\varepsilon} |\nabla_\perp U|^2 \, d\tau - \int \Phi_o \frac{\Delta^*\psi}{R^2} \, d\tau , \qquad (35)$$

if the surface integral vanishes. Here the pressure equation was used in its original form (5).

V. Conclusions

The equations derived in this paper will be primarily used to study resistive instabilities (toroidal mode coupling etc.). So we are prepared to solve at least six coupled equations for six scalar quantities ψ, W, p, Φ, U and χ. Only three of them need time integration, the others require inversions of Laplace-like operators. However, the structure of the equations is in principle not different from the structure of the equations mentioned as limiting cases. The numerical methods already developed should thus be applicable as well.

The validity of the equations is limited to large aspect ratios, the error being in general of order ε^2. Formally also a time limitation $t \lesssim t_H/\varepsilon^2$ arises from the expansion. However, in practice the effective local values of ε and S are often considerably reduced by the presence of sharp gradients in resistive layers etc. So this restriction should not be taken too seriously.

To look ahead, it seems hardly promising to go further to the next order equations because the simple expression for $\underline{v}_\perp$ and the complete decoupling of v_ϕ are lost. Moreover, the main toroidal effects should already be included in the equations derived above.

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