MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

THEORY OF THE M = 1 MODE IN HIGH TEMPERATURE PLASMAS

D. Biskamp

and

H. Welter

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Abstract

The linear and nonlinear properties of the m = 1 kink mode have been investigated numerically and analytically within two fluid theory taking into account diamagnetic drifts, ion viscosity μ and plasma diffusion κ in addition to resistivity η. In the regime of strong diamagnetic effects, $\omega_*/\gamma_T > 1$, the linear growth rate is a very sensitive function of κ, being strongly reduced and even negative around $\kappa \gtrsim \eta$. Nonlinearly the mode is found to saturate at finite amplitude in contrast to the resistive regime $\omega_*/\gamma_T < 1$, where complete reconnection occurs. The main stabilizing effect is a nonlinear azimuthal flow near the resonant radius; hence there is a strong dependence on μ. A quasi-linear theory can explain the main numerical results. Brief application to tokamak experiments is given.

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1. Introduction

In a toroidal current-driven plasma configuration the mode with (dominant) poloidal and toroidal mode numbers (m, n) = (1,1) may play an important role. This mode is destructively unstable if the safety factor q at the plasma boundary a is smaller than unity, which limits operation to $q(a) > 1$. In this case the $q = 1$ surface is located within the plasma column, and since the $m = 1$ kink mode is confined to the interior of this surface, the mode can only affect the central part of the plasma. In tokamaks and similar devices the sawtooth oscillations, in particular the internal disruptions, are attributed to this mode.

Since in ideal MHD theory the $m = 1$ mode is only weakly, if at all, unstable [1)2)3)], nonideal effects dominate its linear and particularly nonlinear properties. In the low-temperature regime resistivity η is the most important effect. The growth rate $\gamma \sim \eta^{1/3}$ [4)] is much larger than in the case of (MHD stable) $m \geqq 2$ tearing modes with $\gamma \sim \eta^{3/5}$. In the resistive regime the nonlinear behavior of the $m = 1$ mode is quite well understood. Computer simulations[6)] have confirmed the picture given by Kadomtsev[7)] of an $m = 1$ magnetic island growing without saturation until completely filling the interior of the original $q = 1$ surface. It is said that the mode leads to complete reconnection of the helical magnetic flux inside this surface. When the system reassumes cylindrical symmetry, the final state is determined by total helical flux conservation since the process occurs on a time scale fast compared with the resistive skin time a^2/η. The magnetic reconnection is accompanied by plasma motion which drives the hot central plasma into the region surrounding the $q = 1$ surface. This picture gives a very plausible interpretation of the internal disruption.

In high-temperature plasma, however, further effects play a role in addition to resistivity, for instance diamagnetic drifts and viscosity. The linear theory of the $m = 1$ mode was recently generalized to include several of these effects[8)9)]. The main result is a strong reduction of the growth rate if the diamagnetic frequency exceeds the resistive growth rate $\omega_* > \gamma_T$, which is quite similar to the case of

$m \geqq 2$ drift tearing modes. Using these linear results, in particular the strong dependence of the growth rate on the magnetic shear, a theory of the sawtooth oscillation has been given[10] which assumes that the nonlinear time scale is given by the linear growth time, and that the mode does not saturate just as in the resistive case $\omega_* < \gamma_T$. This assumption is based on the somewhat vague argument that the final state is the one of lowest magnetic energy consistent with the single constraint of global helical flux conservation independent of the particular nonideal effects, which only determine the time scale.

In recent neutral beam heated tokamak experiments, however, the behavior of the sawtooth oscillations is different and more complex[12], and some observations indicate the presence of a finite amplitude $m = 1$ mode not leading to an internal disruption. Since in these experiments plasma temperatures are particularly high, diamagnetic and viscous effects should play an important role. In the present paper we have therefore investigated the nonlinear behavior of the $m = 1$ mode in the high-temperature regime. Our main result is that, roughly speaking, for $\omega_* > \gamma_T$ the $m = 1$ instability in fact saturates at a finite amplitude. In Sec. 2 we briefly discuss the basic equations. Section 3 gives some new results on the linear theory, in particular the effect of plasma diffusion. Section 4 is devoted to the nonlinear properties. We present results of numerical simulations and discuss the dominant stabilization process. Section 5 gives a summary and a brief application to tokamak experiments.

2. Basic Equations

The model equations we are treating have been discussed in several previous publications[9)13)]. Let us briefly repeat the main approximations implied. Regarding geometry we restrict ourselves to the lowest order in the so-called tokamak expansion, where the inverse aspect $\varepsilon = a/R$ and the ratio of poloidal to toroidal magnetic field components B_θ/B_o are small relative to unity such that the safety factor $q \cong \varepsilon B_o/B_\theta$ is finite. Hence the equilibrium has cylindrical symmetry, while the nonlinearly perturbed state is helically symmetric, depending only on r and $\theta - z/R$ for $m = 1$. The magnetic field is written in terms of the helical flux function ψ and the "toroidal" field B_o, B_o = const in the following form:

$$\begin{aligned} \vec{B} &= \vec{h}\times\nabla\psi + B_o\vec{h} \\ &\cong \hat{z}\times\nabla\psi + B_o\vec{h}. \end{aligned} \tag{1}$$

Here $\vec{h} \cong \hat{z}+(r/R)\hat{\theta}$ is a vector in the symmetry direction, i.e. the direction of the ignorable coordinate, and ψ is the component of the vector potential in this direction. It should be mentioned that in the tokamak approximation the ideal $m = 1$ mode is marginally stable.

We want to derive an equation for ψ within two-fluid theory. First it is noted from the general definition of the electric field $\vec{E}$ that the helical component is

$$E_h = \frac{\partial\psi}{\partial t} \; . \tag{2}$$

From the electron equation of motion, i.e. Ohm's law, neglecting electron inertia and viscosity and assuming infinite parallel electron heat conduction $\nabla_\parallel T_e = 0$,

$$\eta\vec{j} = \vec{E} + \vec{v}_e\times\vec{B} + \frac{1}{ne}\nabla p_e \; , \tag{3}$$

we find that

$$E_h = \eta j_h - \vec{v}_{e\perp}\cdot\nabla\psi \cong \eta j_z - \vec{v}_{e\perp}\cdot\nabla\psi, \tag{4}$$

$$j_z = \nabla^2\psi + \frac{2B_o}{R}$$

since $\vec{h}\cdot\nabla f = 0$ and $j_h \cong j_z \cong j_{\shortparallel} >> j_{\perp}$ in the tokamak ordering. From eq. (3) one obtains

$$\vec{v}_{e\perp} \cong \vec{v}_E + \vec{v}_{e*} \quad ,$$

$$\vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2} \cong \frac{\hat{z}\times\nabla\zeta}{B_o} \quad , \quad \vec{v}_{e*} = -\frac{\nabla P_e\times\hat{z}}{enB_o} \tag{5}$$

Combining eqs. (2), (4), (5) yields

$$\frac{\partial\psi}{\partial t} + \vec{v}_E\cdot\nabla\psi = \eta j_z - \frac{1}{ne}\nabla_{\shortparallel}p_e \quad . \tag{6}$$

(Note that because of $\vec{h}\cdot\nabla p_e = 0$, $\nabla_{\perp}p_e$ and $\nabla_{\shortparallel}p_e$ are in general not independent.) The perpendicular ion velocity is

$$\vec{v}_{i\perp} = \vec{v}_E + \vec{v}_{i*} + \vec{v}_p \quad , \tag{7}$$

$$v_{i*} \cong \frac{\nabla p_i\times\hat{z}}{enB_o} = \frac{\gamma_i T_i \nabla n\times\hat{z}}{enB_o} \quad ,$$

assuming adiabatic change of p_i , and $\vec{v}_p$ is the polarization drift. We only need to determine the incompressible part of $v_{i\perp}$, $\vec{u} = \vec{v}_E + \vec{v}_{i*} \equiv \hat{z}\times\nabla\phi$ (the stream function ϕ should be distinguished from the electrostatic potential ζ in $\vec{v}_E$, which will not enter the final equations). An equation for ϕ is obtained by applying the operator $\hat{z}\cdot\nabla\times$ to the ion equation of motion, which yields

$$\frac{\partial W}{\partial t} + (\vec{u}-\vec{v}_{i*})\cdot\nabla W - \hat{z}\cdot\nabla n\times\nabla\frac{u^2}{2} = \vec{B}\cdot\nabla j_z + \mu\nabla^2 W \quad , \tag{8}$$

$$W = \nabla\cdot n\nabla\phi$$

The term $\vec{v}_{i*}\cdot\nabla W$, displaying the effect of magnetic viscosity, as well as the collisional viscosity term $\mu\nabla^2 W$ originate from the ion stress tensor. With $\vec{v}_E = \vec{u}-\vec{v}_{i*}$ eq.(5) can be rewritten as

$$\frac{\partial\psi}{\partial t} + \vec{u}\cdot\nabla\psi = \eta j_z + \frac{(T_e+\gamma_i T_i)}{eB_o n}\,\hat{z}\cdot(\nabla n\times\nabla\psi), \tag{9}$$

again using $\nabla_{\shortparallel}T_e = 0$. The change of the electron density n ($n = n_e = n_i$ because of quasineutrality) is determined by the equation

$$\frac{\partial n}{\partial t} + \nabla\cdot\vec{v}_{e\perp}n = -\nabla_{\shortparallel}nv_{e\shortparallel} + \kappa\nabla^2 n\ , \tag{10}$$

which becomes

$$\frac{\partial n}{\partial t} + \vec{u}\cdot\nabla n = \frac{1}{e}\nabla_{\shortparallel}j_z + \kappa\nabla^2 n \tag{11}$$

on neglecting ion parallel flow and making use of the relation $v_*\cdot\nabla n = 0$

Let us introduce the units a = plasma radius, $B_\theta(a)$, n_o = central plasma density and hence the poloidal Alfvén speed $v_A = B_\theta(a)/\sqrt{m_i n_o}$. Now eqs. (8), (9), (11) take the dimensionless form

$$\frac{\partial W}{\partial t} + (\vec{u}-\vec{v}_{i*})\cdot\nabla W - \hat{z}\cdot\nabla n\times\nabla\frac{u^2}{2} = \hat{z}\cdot\nabla\psi\times\nabla j_z + \mu\nabla^2 W\ , \tag{12}$$

$$\frac{\partial\psi}{\partial t} + \vec{u}\cdot\nabla\psi = \eta j_z + \alpha\,\frac{(T_e+\gamma_i T_i)}{n}\,\hat{z}\cdot\nabla n\times\nabla\psi - E_o\ , \tag{13}$$

$$\frac{\partial n}{\partial t} + \vec{u}\cdot\nabla n = \alpha\hat{z}\cdot\nabla\psi\times\nabla j + \kappa\nabla^2(n - N)\ , \tag{14}$$

$$j_z = \nabla^2\psi + 2q(1)\quad ,\quad W = \nabla\cdot(n\nabla\phi)\ ,\quad \vec{u} = z\times\nabla\phi\ .$$

Here we have introduced E_o, an axial electric field (corresponding to the loop voltage in a tokamak), $E_o = \eta(r)j_o(r)$, and a source term $-\kappa\nabla^2 N$, N(r) = equilibrium density distribution, in order to prevent resistive or diffusive decay of the global configuration.

Equations (12) to (14) contain the dimensionless smallness parameters η, μ, κ, all measured in units of $v_A a$, where $\eta^{-1} = S$, the magnetic Reynolds number, and the parameter $\alpha = (c/\omega_{pi}a)(B_\theta(a)/B_o)$, measuring the compressibility of the ion flow (polarization drift). The temperatures $T_{e,i}$ are given in units of $B_\theta^2(a)$; hence $2(T_e+T_i)$ is the poloidal β. The magnitude of the diamagnetic effects is measured by $\alpha T_{e,i}$.

Let us briefly discuss the magnitude of the parameters η, μ, κ. For typical tokamak plasmas one has $\eta \sim 10^{-7} - 10^{-6}$. Expressing the collisional ion viscosity μ in terms of η

$$\mu \cong 0.1\ \beta \left(\frac{T_e}{T_i}\frac{m_i}{m_e}\right)^{1/2} \eta\ ,$$

we find $\mu/\eta \cong 0.03 - 0.1$. The neoclassical diffusion coefficient in the banana regime is

$$\kappa \cong \nu_e \rho_e^{\,2} q^2 (\frac{R}{r})^{3/2} \sim 10^2 \nu_e \rho_e^{\,2}$$

$$\cong 10^2 \beta\eta \lesssim \eta\ .$$

The experimentally observed diffusion coefficient, however, is much larger, at least by a factor of 10. Because of this magnitude plasma diffusion plays an important role for the m = 1 mode, as we shall discuss in the subsequent sections.

Equations (12) to (14) were solved numerically for various values of the parameters η, α, μ, κ. We choose a static equilibrium $\vec{u}_o = 0$ (which implies the presence of a certain radial electric field to compensate the ion diamagnetic drift). The equilibrium current profile is $j_o(r) = 2s^2(1+s^2)/(r^2+s^2)^2$, $s = 0.6$ and $q(1) = 3.4$, $q(0) = 0.9$, so that the resonant radius r_s, $q(r_s) = 1$, is $r_s = 0.2$. We assume equal electron and ion temperature profiles, $T_e = \gamma_i T_i \propto j_o^{\,2/3}$ with $T_e(r_s) = 0.84$. Two different numerical programms have been developed, one using a finite difference scheme in both the radial and poloidal directions, the other using a Fourier decomposition in θ.

3. Linear Theory

The linear properties of the m = 1 instability were recently investigated extensively, including even additional effects not accounted for in eqs. (12) to (14), such as toroidicity or kinetic effects. Here we should like to point out the influence of plasma diffusion, which has previously not been considered, and which significantly modifies the properties of the m = 1 mode. Figure 1 shows the growth rate γ as a function of κ for different values of η, μ and α, i.e. ω_{e*}/γ_T. To summarize the results, the function $\gamma(\kappa)$ changes character when ω_{e*}/γ_T is increased above a certain value. While for $\omega_{e*}/\gamma_T \lesssim 2$ γ grows monotonically with κ, for $\omega_{e*}/\gamma_T > 2.5$ there is a deep well, even a stable region for sufficiently small μ (Fig. 1b) or sufficiently large ω_{e*}/γ_T. The value of κ, where γ is minimal, is approximately $\kappa_o \cong 0.3\eta\ (\omega_{e*}/\gamma_T)^{5/2}$. There is a correlation with a change of the eigenfunction $\tilde{n}(r)$, Fig.2. For $\kappa << \kappa_o$ $\tilde{n}$ has an oscillatory structure around r_s as predicted from $\kappa = 0$ theory[8], while for $\kappa > \kappa_o$ these oscillations are smeared out over the entire volume inside the resonant surface. This behavior is analogous to the decrease and reincrease of the growth rate of $m \geq 2$ drift tearing modes observed in a previous investigation[13]. Moving across the transition region $\kappa \sim \kappa_o$ from small to large values of κ, the real part of the frequency ω increases from $-\omega_{i*}$ to $\omega_{e*} - \omega_{i*}$.

An analytical theory of the behavior for $\kappa << \kappa_o$ and $\kappa \to \infty$ can easily be given. In the first case the dispersion relation was derived in Refs. 8 and 9:

$$\Gamma \equiv \gamma - i\omega = \frac{\gamma_T^{\,3}}{|\omega_{i*}\omega_{e*}|} + i\omega_{i*} \tag{15}$$

with

$$\omega_{e,i*} = \frac{1}{r_s} v_{e,i*}(r_s) \quad , \quad \gamma_T = \gamma\ (\omega_*=0)$$

(Note that we use a static equilibrium, $\vec{u}_o = 0$, in contrast to Ref. 9, where the ions move with the diamagnetic velocity. The difference, however, only amounts to a Doppler shift ω_{i*}). Hence in contrast to

$m \geq 2$ drift tearing modes the $m = 1$ mode does not rotate with the electron fluid (which would have the frequency $\omega_{e*} - \omega_{i*}$). Since the perturbed magnetic field $\propto \tilde{\psi}(r_s) \sim \eta^{1/3}$ is small for $m = 1$ [8], the mode is not as strongly tied to the electrons as for $m \geq 2$ where $\tilde{\psi}(r_s)$ is finite ("constant ψ" property).

Now consider the opposite case of large κ, where the density perturbation $\tilde{n}$ is suppressed. Linearizing eqs. (12), (13) and assuming a time dependence $e^{\Gamma t}$, one obtains in the vicinity of the resonant radius, with $x = r-r_s$, $\psi_o'/r = (x/r_s)\psi_o''(r_s) = xF$, $n(r_s) = 1$,

$$(\Gamma - i\omega_{i*})\tilde{\phi}'' = ixF\tilde{\psi}'' , \tag{16}$$

$$(\Gamma + i\omega_*)\tilde{\psi} - ixF\tilde{\phi} = \eta\tilde{\psi}'' \tag{17}$$

with

$$\omega_* \equiv \omega_{e*} - \omega_{i*} .$$

Proceeding as in Refs. 4 and 8 one introduces the auxiliary function χ defined by $\chi' = x\tilde{\psi}''$. Integrating eq. (16) once, we obtain

$$\chi = x\tilde{\psi}' - \tilde{\psi} = -\frac{i(\Gamma-i\omega_{i*})}{F}\tilde{\phi}' . \tag{18}$$

Substitution of $\tilde{\psi}$ by χ in eq. (17) yields

$$\chi'' - \frac{2}{x}\chi' - (A + Bx^2)\chi = 0 \tag{19}$$

with

$$A = \frac{\Gamma+i\omega_*}{\eta} , \qquad B = \frac{F^2}{(\Gamma-i\omega_{i*})\eta} .$$

The solution of eq. (19) is

$$\chi = \chi_o \, e^{-Ax^2/2} \tag{20}$$

Insertion of eq. (20) into eq. (19) yields the dispersion relation $A^2 = B$ or

$$(\Gamma+i\omega_*)^2(\Gamma-i\omega_{i*}) = \eta F^2 = \gamma_T{}^3 \tag{21}$$

For $\omega_* = 0$ we have the well-known result $\Gamma = \gamma_T = \eta^{1/3}\, F^{2/3}$, while $\omega_* >> \gamma_T$ gives

$$\Gamma = \left(\frac{\gamma_T{}^3}{2\omega_{e*}}\right)^{1/2} - i\omega_* \; . \tag{22}$$

Thus for large κ the mode rotates with the electron fluid in contrast to the behavior at small κ, eq. (15), and the growth rate is significantly larger.

To obtain an estimate of the transition point κ_o we treat the diffusion term in the (linearized) density equation eq. (14) in the following approximate way, $\kappa\nabla^2\tilde{n} = -\kappa k^2\tilde{n}$ with $k^2 = \delta_s^{-2}$, δ_s the resonant layer width $\delta_s^2 = \eta/\gamma$ see eq. (25) in the subsequent section. Now the density perturbation becomes

$$\tilde{n} = \frac{i}{r}n'_o \frac{1}{\Gamma+\kappa\gamma/\eta}\tilde{\phi}$$

Inserting this expression into the (linearized) ψ - equation we find instead of eq. (17)

$$(\Gamma+i\omega_*)\tilde{\psi} - ixF(1+\frac{i\omega_*}{\Gamma+\kappa\gamma/\eta})\tilde{\phi} = \eta\tilde{\psi}''$$

We again obtain eq. (19) with a modified coefficient B. Approximating $\kappa\gamma/\eta$ by $\kappa\gamma(\kappa{=}0)/\eta$ the dispersion relations $A^2 = B$ again reduces to polynomial form

$$(\Gamma+i\omega_*)^2(\Gamma-i\omega_{i*})\left[\Gamma+\frac{\kappa}{\eta}(\frac{\gamma_T}{\omega_{e*}})^2\gamma_T\right] = \gamma_T^3\left[\Gamma+\frac{\kappa}{\eta}(\frac{\gamma_T}{\omega_{e*}})^2\gamma_T+i\omega_*\right]$$

which can easily be solved numerically. We find that at $\kappa = \kappa_o$ the most unstable mode switches from the branch $\omega \cong -\omega_{i*}$ to the branch $\omega \cong \omega_*$, where $\kappa_o \cong \eta(\omega_{e*}/\gamma_T)^{5/2}$.

4. Nonlinear Behavior

In this section we first describe numerical simulations of the nonlinear evolution of the m = 1 instability, identify the most important nonlinear process and then give a qualitative analytical interpretation. The main numerical result is that in the case when diamagnetic effects are important, roughly speaking for $\omega_{e*}/\gamma_T > 1$, the Kadomtsev theory of complete flux reconnection no longer applies. Instead the instability saturates at a finite amplitude (and island size). Figure 3 shows the evolution of the growth rate, the frequency and the mode amplitude Re $\{\tilde{\psi}\ (r_s,\ t)\}$, and snap shots of the convection pattern $\phi(r,\theta)$ in the linear phase and of $\phi(r,\theta)$, $\psi(r,\theta)$ in the saturated state for $\eta = 3\times10^{-6}$, $\alpha = 8\times10^{-3}$, corresponding to $\omega_{e*}/\gamma_T \cong 2.5$. The parameters $\kappa = 10^{-5}$, $\mu = 10^{-7}$ represent typical experimental conditions in tokamaks, where the collisional viscosity μ is smaller than η, while the effective diffusion coefficient is larger, as discussed in Sec. 2. The dependence of the saturation amplitude ψ_m, the saturation value of $\max\{|\tilde{\psi}(r)|\}$, on α, μ, κ is shown in Fig. 4. As already mentioned, ψ_m decreases on increasing α, i.e. ω_{e*}/γ_T. Somewhat surprisingly, one finds a strong dependence on μ for μ above a certain value, ψ_m increasing with μ. We also find that ψ_m depends sensitively on κ, in a similar way as the linear growth rate, Fig. 1.

In order to identify the dominant nonlinear process, we artificially freeze two of the three quantities to their initial values, the average current profile $j_o(r) = \langle j(r,\theta)\rangle$, the average density profile $n_o(r)$ and the vorticity distribution $W_o(r)$. We find that the quasilinear change $\delta W_o(r)$ is by far the most important effect in the regime $\omega_{e*}/\gamma_T > 1$. Neither δj_o nor δn_o may keep the mode from completely reconnecting. Inertia effects thus play a dominant role in saturating the m = 1 instability, in contrast to nonlinear drift tearing modes with $m \geq 2$, which are mainly stabilized by a modification of the current profile and where δW_o only leads to a reduction of the mode frequency[11].

To demonstrate the quasilinear character of the stabilization, we use the vorticity $k\delta W_o(r)$ as an initial equilibrium property, where $\delta W_o(r)$ is the distribution generated nonlinearly at the saturation time. We find that for $k > 1$ this equilibrium is stable with respect to an m = 1 perturbation, for $0 < k < 1$ the growth rate is reduced compared with the original static equilibrium, and for $k < 0$ the mode becomes more unstable. We also show that only the local velocity shear given by $\delta W_o'(r_s)$ is important for stabilisation (note that W_o = const only leads to a rigid rotation which cannot affect the instability).

Let us now outline a qualitative analytical theory explaining the numerical results. Averaging the vorticity equation (12) over θ yields

$$\frac{\partial W_o}{\partial t} \cong \langle \hat{z}\cdot\nabla\tilde{\psi}\times\nabla\tilde{j} \rangle + \mu W_o'' \tag{23}$$

First consider the regime of small viscosity, where the μ - term in eq.(23) is negligible. As long as the amplitude is not too large, $\tilde{\psi}(r)$ in the nonlinear term may be approximated by the linear eigenfunction. Integrating eq.(20) we obtain

$$\tilde{\psi} = \psi_s(e^{-Ax^2/2} - Ax\int_x^\infty e^{-Ay^2/2}\,dy) \quad , \quad A = \frac{\gamma + i(\omega_* - \omega)}{\eta} \tag{24}$$

Neglecting n_o' and using $\partial\delta W_o/\partial t = 2\gamma\delta W_o$, $\omega_* >> \gamma$, we find on integration of eq.(23) that

$$\begin{aligned}\delta\phi_o' &\cong \frac{1}{\gamma r_s}\,\mathrm{Im}\,\{\tilde{\psi}\tilde{\psi}''^*\}\\ &\cong \frac{|\psi_s|^2}{\gamma r_s}\Big[-\frac{\omega_* - \omega}{\eta}\,e^{-\gamma x^2/\eta}\\ &\qquad - x\,\frac{(\omega_* - \omega)^2}{\eta^2}\,\mathrm{Im}\{\,e^{-A^*x^2/2}\int_x^\infty e^{-Ay^2/2}\,dy\,\}\Big]\end{aligned} \tag{25}$$

From eq.(25) we compute $\delta W_o' \cong \delta\phi_o'''$ at $x = 0$

$$\delta W_o' \cong \frac{2}{r_s}\,\frac{\omega_* - \omega}{\eta^2}\,|\psi_s|^2 \tag{26}$$

It is interesting to compare eq. (25) with the corresponding result for $m \geq 2$[13]:

$$\delta\phi_o' \cong \frac{m}{r_s} \frac{1}{\gamma\eta} \Delta\omega \, |\psi_s|^2$$
$$\cong - \frac{m}{r_s} \frac{\gamma}{\eta\omega_*} \, |\psi_s|^2 \quad , \tag{27}$$

where $\Delta\omega = \omega - \omega_* \cong - \gamma^2/\omega_*$ is the small frequency mismatch between the mode and the electron fluid. For m = 1 this mismatch is much larger, $\Delta\omega \sim \omega_*$, which leads to the larger value of $\delta\phi_o'$ in eq. (25).

Since $\delta W_o(r)$ has been found numerically to be the most important stabilizing quantity, the only nonlinear term to be kept in the equations of the fluctuating quantities is $\vec{u}\cdot\nabla W$ in the W equation. Equation (16) then becomes

$$(\Gamma - i\omega_{i*})\tilde{\phi}'' - \frac{i}{r_s} \delta W'_o \tilde{\phi} \cong ix F \tilde{\psi}'' \ , \tag{28}$$

while the $\tilde{\psi}$ equation remains unchanged:

$$(\Gamma + i\omega_*)(\tilde{\psi} - i \frac{xF}{\Gamma} \tilde{\phi}) = \eta\tilde{\psi}'' \ . \tag{29}$$

As mentioned before, the main stabilizing contribution comes from $\delta W_o'(r_s)$ given by eq. (26). Thus in eq. (28) $\delta W_o'$ may be considered constant. But even with this simplification the system (28) and (29) is equivalent to a fourth-order differential equation (or an integro-differential equation), which has not been solved. We can, however, estimate the stabilizing effect of $\delta W_o'$ by inserting $\tilde{\psi}''$ from eq. (29) into the r.h.s. of eq. (28)

$$\tilde{\phi}'' - \left[\frac{i}{r_s} \delta W_o' + \frac{x^2F^2}{\eta} \frac{\Gamma + i\omega_*}{\Gamma}\right] \frac{1}{\Gamma - i\omega_{i*}} \tilde{\phi}$$
$$= i \frac{xF}{\eta} \frac{\Gamma + i\omega_*}{\Gamma - i\omega_{i*}} \tilde{\psi} \ . \tag{30}$$

Consider the imaginary part of the expression in brackets on the l.h.s. of eq. (30), which is the only term where $\delta W_o'$ enters.

Since

$$\mathrm{Im}\left\{\frac{\Gamma+i\omega_*}{\Gamma}\right\} \cong \frac{\gamma\omega_*}{\omega^2}$$

one has

$$\mathrm{Im}\{[\quad]\} = \frac{1}{r_s}\,\delta W_o' + \frac{x^2F^2}{\eta}\,\frac{\gamma\omega_*}{\omega^2} \tag{31}$$

To obtain an estimate of the magnitude of $\delta W_o'$ necessary to substantially change the growth rate, we set expression (31) equal to $x^2F^2\omega_*\gamma_o/\eta\omega^2$, where γ_o is the linear growth rate for $\delta W_o' = 0$. Hence we see that $\delta W_o' > 0$ exerts a stabilizing influence. The relation

$$\frac{1}{r_s}\,\delta W_o' = \frac{x_o^2F^2}{\eta}\,\frac{\gamma_o\omega_*}{\omega^2} \tag{32}$$

yields a rough value of the saturation amplitude ψ_m, x_o being some characteristic layer width. For not too small amplitude it is natural to choose x_o equal to the island half-width δ_I, $\delta_I^2 \cong 4|\psi_s/Fr_s|$. Inserting eq.(23) into eq.(32) we thus find

$$\psi_m \cong 2r_sF\eta\,\frac{\gamma_o\omega_*}{(\omega_*-\omega)\omega^2} \propto \frac{\gamma_o\eta}{\omega_*^2} \tag{33a}$$

Note that γ_o, too, depends on ω_* as given in eqs. (15) and (22). For very small amplitude the island becomes thinner than the original resistive layer width δ_s, $\delta_s^2 \sim \eta/\gamma_o$, which appears in eq.(25). In this case we have to choose $x_o \cong \delta_s$ in eq.(32) which yields the scaling

$$\psi_m \propto \frac{\eta}{\omega_*} \tag{33b}$$

The strong dependence on ω_* for relatively large saturation amplitude, (33a), as well as the weaker scaling with ω_* for small island size, (33b), are consistent with the simulation results, Fig.4a. Also the absolute values of ψ_m are in satisfactory agreement.

Let us now consider the opposite case, where in eq.(23) the viscous term is larger than the l.h.s.,

$$< \hat{z}\cdot\nabla\tilde{\psi}\times\nabla\tilde{j} > + \mu\delta W_o'' \cong 0$$

Integration yields

$$\delta W_o' \cong \frac{2}{r_s} \frac{\omega_* - \omega}{\mu\eta} |\psi_s|^2 \tag{34}$$

Performing the same analysis as before we obtain the scaling of the saturation amplitude ψ_m for $\delta_I > \delta_s$

$$\psi_m \propto \frac{\gamma_o \mu}{\omega_*^2} \tag{35a}$$

and for $\delta_I < \delta_s$

$$\psi \propto \frac{(\eta\mu)^{1/2}}{\omega_*^2} \tag{35b}$$

The increase of the saturation level with increasing μ, which is in qualitative agreement with the simulation results, Fig.4b, is due to the fact, that viscosity reduces the stabilizing nonlinear flows, eq.(34).

The diffusion coefficient κ does not appear explicitly in the estimates (33) and (35), since this effect has been ignored in the preceding analysis. Nevertheless the main dependence on κ is contained implicitly through the linear growth rate γ_o. As we have mentioned before, ψ_m has about the same κ-dependence as γ_o, which is clearly seen when comparing Figs.4c,d with the corresponding plots of γ in Fig.1.

The preceding quasi-linear analysis is limited to sufficiently small amplitude to justify use of the linear eigenfunction (24) in eqs.(25) and following. In fact the relations (33) and (35) only give the amplitude, at which the original exponential growth is quenched. Whether this implies full stabilization or only further growth on a slower time scale as in the case of $m \geqq 2$ tearing modes growing on the resistive time scale[14] cannot be decided on these grounds. The numerical simulations, however, strongly suggest true saturation at amplitudes consistent with the estimates (33), (35), at least for small values of μ. For larger μ continued slow growth cannot be excluded.

We should also mention, that for $\omega_*/\gamma_T \lesssim 1$ and also for sufficiently large values of μ or κ, the mode fully reconnects as predicted by Kadomtsev's theory.

5. Conclusions

We have presented a theory of the m = 1 kink mode including a number of nonideal effects in addition to finite resistivity, namely diamagnetic drifts, ion viscosity and plasma diffusion, which should be important in high-temperature tokamak plasmas. Concerning linear theory we find, in particular, a strong dependence of the growth rate on the plasma diffusion κ, the growth rate being reduced or even negative in a certain range of κ values, $\kappa \gtrsim \eta$, which is experimentally the most interesting parameter range. Nonlinearly we find saturation of the m = 1 instability in the high temperature regime $\omega_*/\gamma_T > 1$ in contrast to the theory of the resistive m = 1 mode as given by Kadomtsev predicting full reconnection. In addition, we observe a strong dependence of the saturation amplitude on both the viscosity and diffusion rate. The results of 2D numerical simulations can be explained qualitatively by a simple quasilinear theory. The dominant nonlinear effect is found to be an azimuthal shear flow $\delta W_o'$ around the resonant radius, in contrast to m $\geq$ 2 drift tearing modes, where this effect does not significantly change the saturation amplitude.

The consequences of our theory for high-temperature tokamak operation are significant. Since the m = 1 mode, if unstable at all, no longer leads to internal disruption but saturates at low amplitude, the safety factor on axis q(o) should not be limited to 1, but may become smaller, perhaps as low as 1/2. Experimental evidence of such behavior is still scarce. Most tokamak discharges clearly show sawtooth activity though operating in the regime $\omega_* \sim \gamma_T$, where saturation effects should start to show up. Possibly the effective ion viscosity is larger than the classical value $\mu \sim \nu_i \rho_i^2$, damping the nonlinear flows as outlined above. Absence of sawtooth oscillations, however, were recently observed in some neutral beam heated devices. In PLT, where temperatures are particularly high and hence conditions most favorable for saturation of the m = 1 mode, a number of discharges have been observed with persistent m = 1 oscillations of relatively low amplitude without internal disruptions. Similarly in recent experiments on the W-7a stellarator an increase of the plasma temperature has led into a regime with no visible sawtooth activity. Although these experimental observations seem to confirm our theoretical predictions, one should not overlook the difficulties encountered when interpreting the experimental data on neutral beam heated machines. For instance, one of the crucial parameters, γ_T, requires knowledge of the central current profile, which is only very poorly known, nor are there any measurements of the effective ion viscosity.

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References

1) M.N. Rosenbluth, P.Y. Dagazian, and P.H. Rutherford, Phys. Fluids 16, 1894 (1973)

2) M.N. Bussac, R. Pellat, D. Edery, and J.L. Soulé, Phys. Rev. Lett. 35, 1638 (1975)

3) W. Kerner, R. Gruber, and F. Troyon, Phys. Rev. Lett. 44, 536 (1980)

4) B. Coppi, R. Galvao, R. Pellat, M.N. Rosenbluth, and P.H. Rutherford, Fizica Plazmy 6, 961 (1976)

5) H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids 6, 459 (1963)

6) A. Sykes and J.A. Wesson, Phys. Rev. Lett. 37, 140 (1976); B.V. Waddell, M.N. Rosenbluth, D.A. Monticello, and R.B. White, Nucl. Fusion 16, 528 (1976)

7) B.B. Kadomtsev, Fizica Plazmy 1, 710 (1975)

8) B. Basu and B. Coppi, MIT report, PRR-76/38, 1976

9) B.V. Waddell, G. Laval, and M.N. Rosenbluth, Oak Ridge Nat. Lab. report, ORNL/TM-5968, 1977

10) G.L. Jahns, M. Soler, B.V. Waddell, J.D. Callen, and H.R. Hicks, Nucl. Fusion 18, 609 (1978)

11) D. Biskamp, Nucl. Fusion 19, 777 (1979)

12) N.R. Sauthoff, S.v.Goeler, D.R. Eames, and W. Stodiek, Proceedings of the IAEA Symposium on Current Disruptions in Toroidal Devices, Garching 1979, IPP 3/51, paper C5

13) D. Biskamp, Nucl. Fusion 18, 1059 (1978)

14) P.H. Rutherford, Phys. Fluids 16, 1903 (1973)

Figure Captions

Fig. 1 Linear growth rate γ as function of κ.

a) $\gamma(\kappa)$ for $\eta = 3x10^{-6}$, 10^{-6} ; $\omega_{e*}/\gamma_T = 2.4$, 3 ; $\mu = 3x10^{-7}$

b) $\gamma(\kappa)$ for $\eta = 3x10^{-6}$, $\omega_{e*}/\gamma_T = 3$ and $\mu = 10^{-7}$.

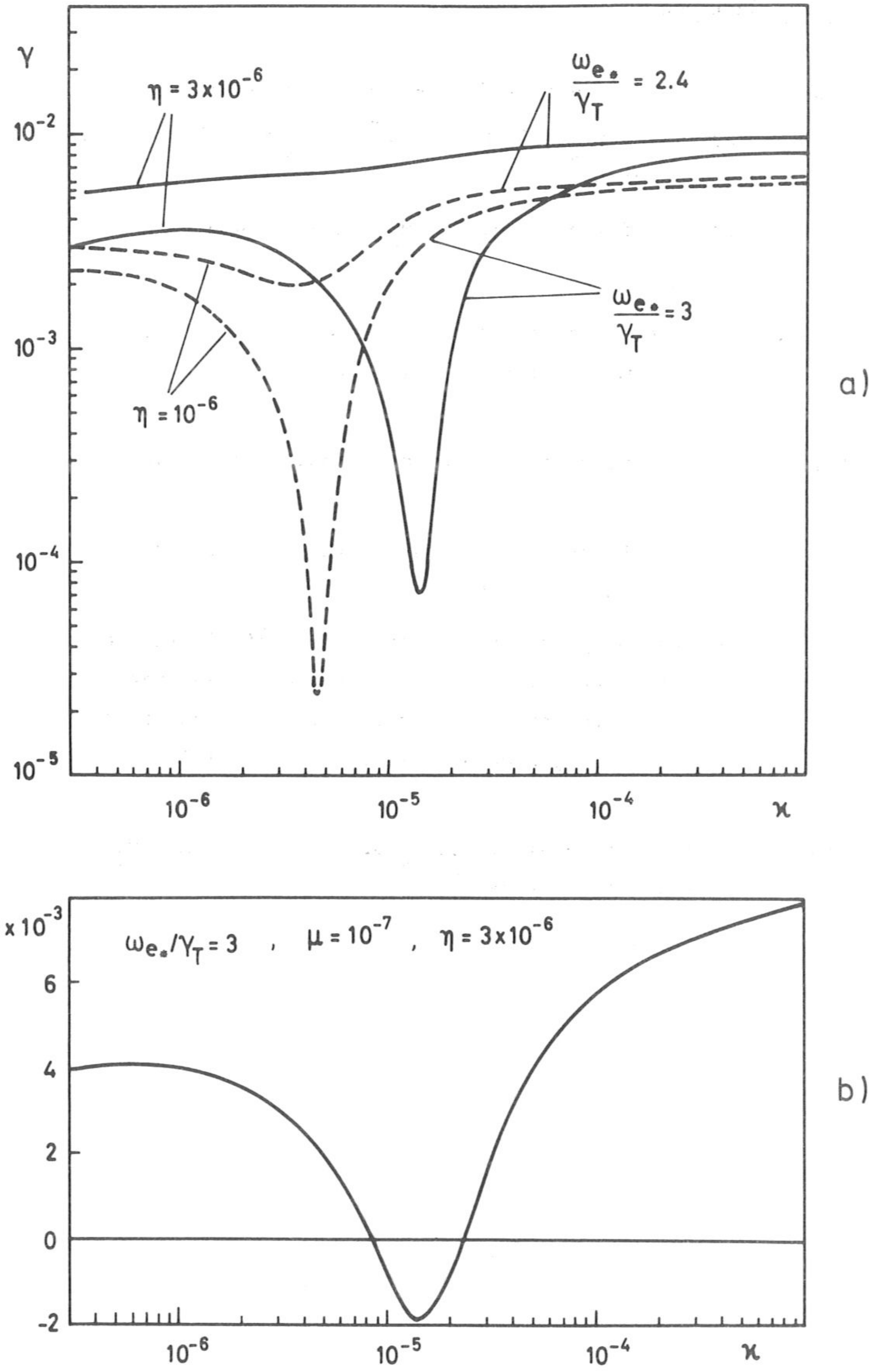
Fig. 2 Linear eigenfunctions $\tilde{n}_R = \mathrm{Re}\{\tilde{n}(r)\}$, $\tilde{j}_R$, $\tilde{\phi}_R$ for small and large value of κ.

Fig. 3 Nonlinear behavior of m = 1 mode for $\eta = 3x10^{-6}$, $\kappa = 10^{-5}$, $\mu = 10^{-7}$:

a) growth rate and frequency $\gamma(t)$, $\omega(t)$, and amplitude $\tilde{\psi}_R(r_s,t)$;

b) contour plots of the plasma flow $\phi(r,\theta)$ in the linear phase, ϕ_L , and in the saturated state, ϕ_s , and of the helical flux $\psi(r,\theta)$ in the saturated state, ψ_s , showing (small) magnetic island.

Fig. 4 Saturation amplitude ψ_m as function of ω_{e*}/γ_T , μ , κ for $\eta = 3x10^{-6}$.



γ
η = 3 x 10^-6
ω_e* / γ_T = 2.4
10^-2
10^-3
10^-4
10^-5
ω_e* / γ_T = 3
η = 10^-6
a)
10^-6
10^-5
10^-4
κ
γ x 10^-3
ω_e*/γ_T = 3 , μ = 10^-7 , η = 3x10^-6
6
4
2
0
-2
b)
10^-6
10^-5
10^-4
κ

Fig. 1

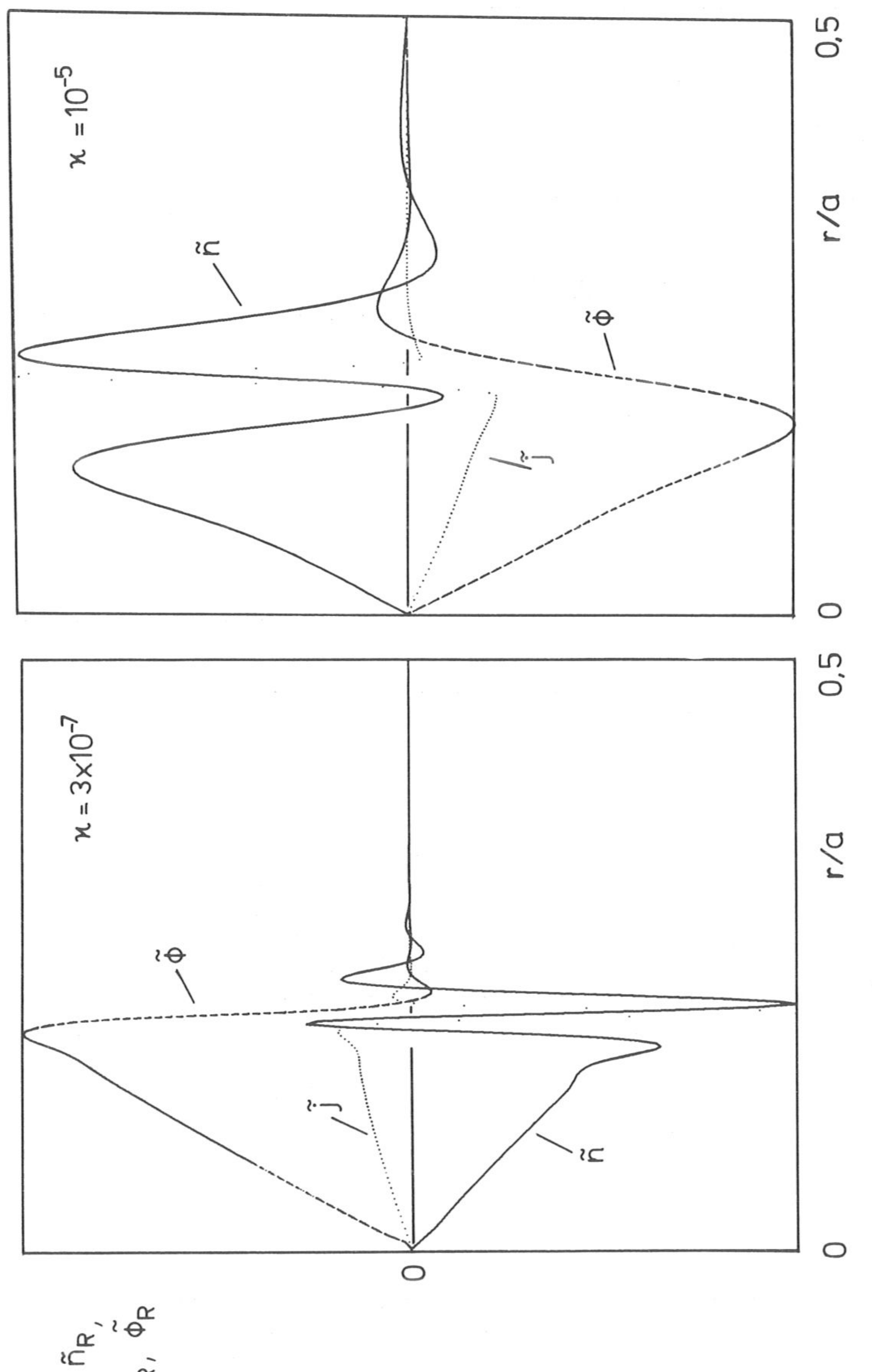


Fig. 2

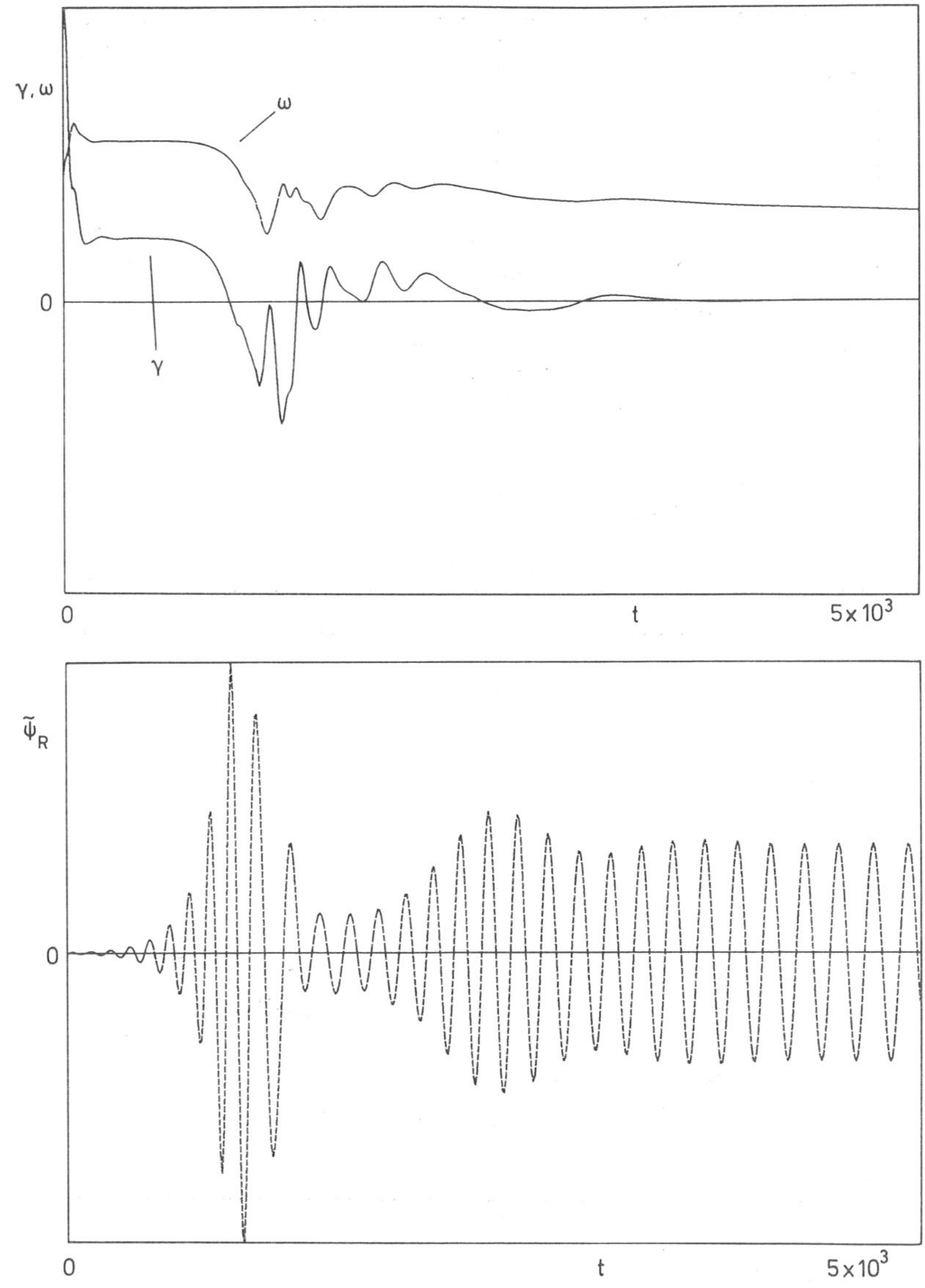


Fig. 3a

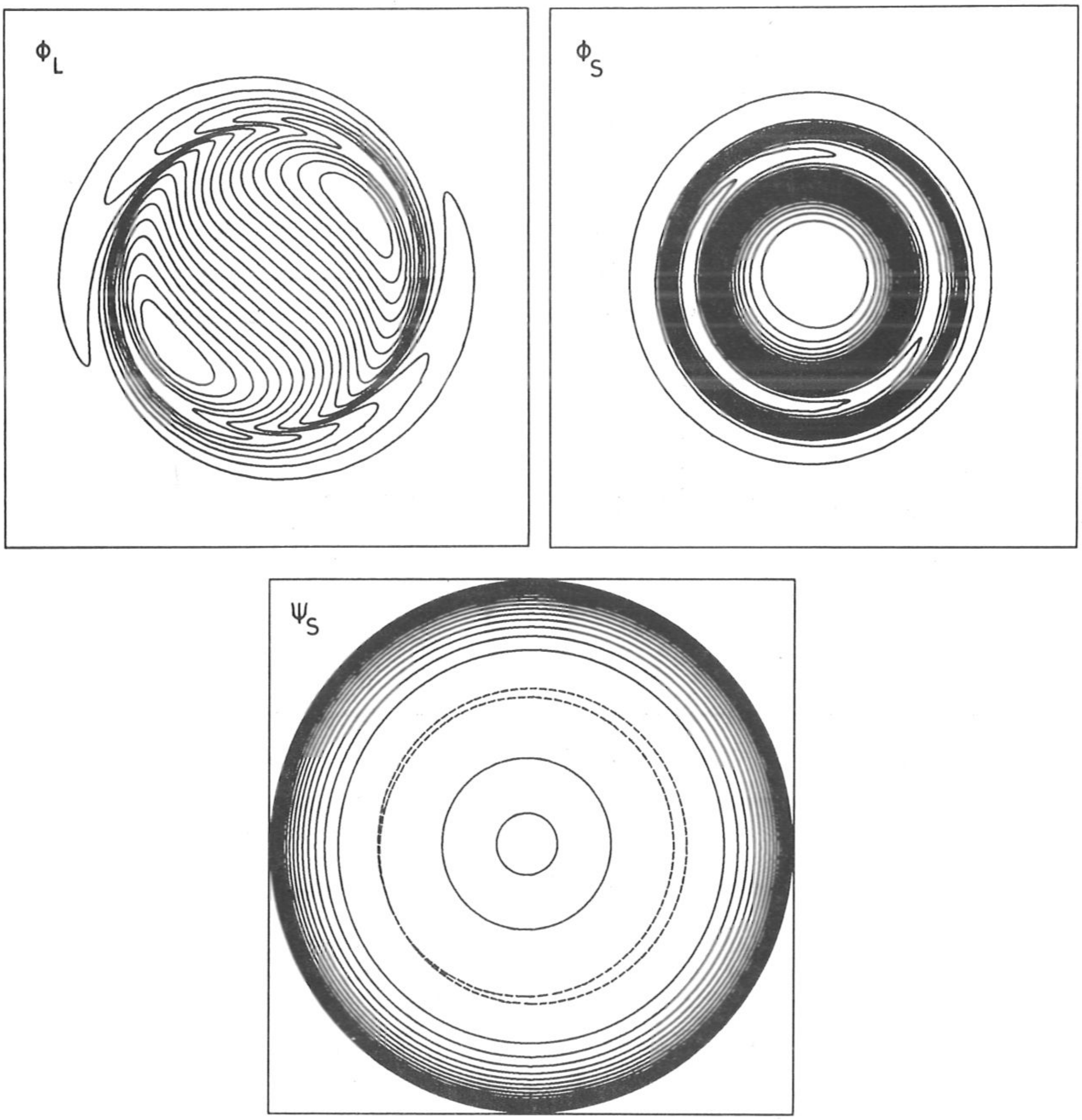
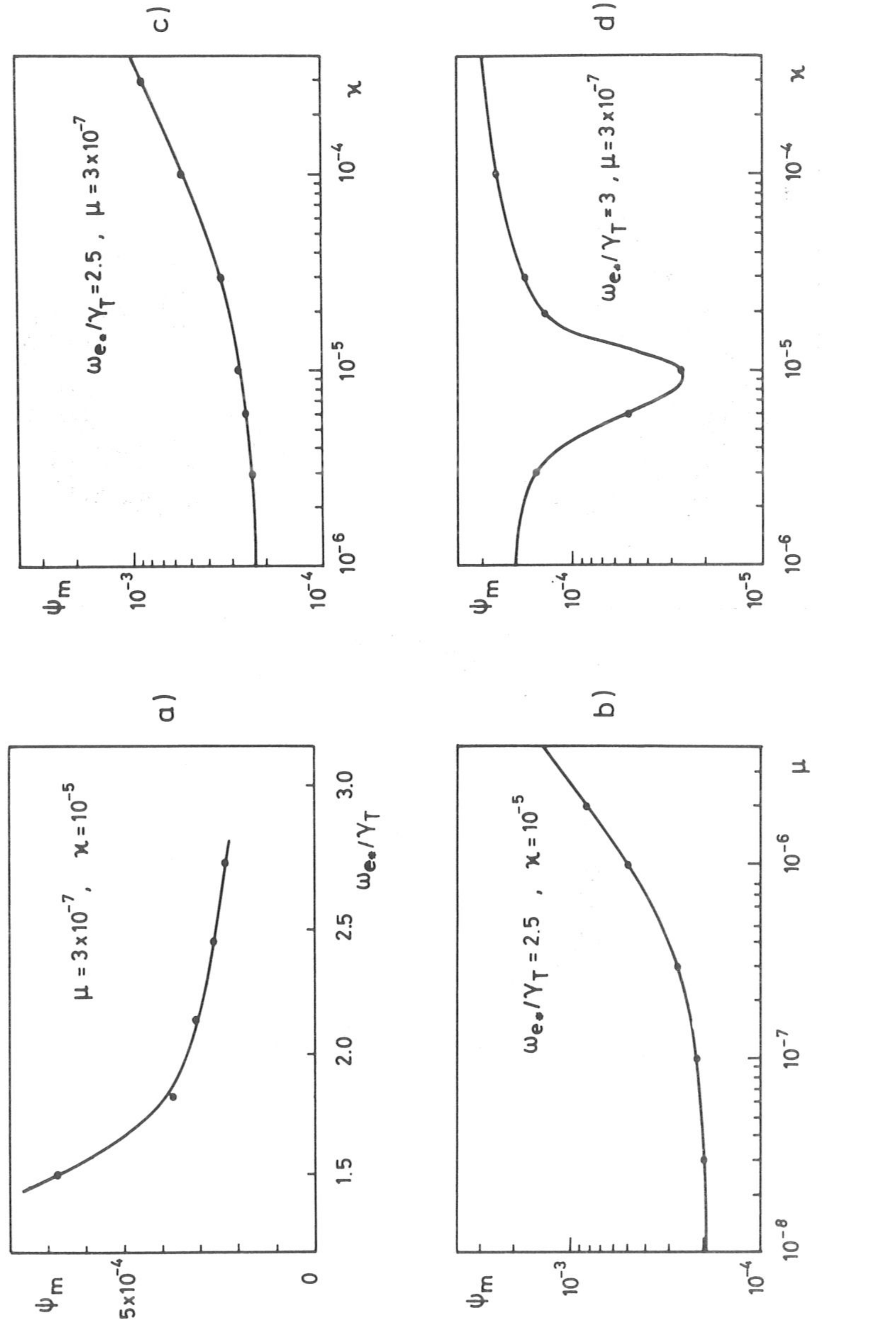


Fig. 3b



a)
ψ_m
5×10^{-4}
0
$\mu = 3\times10^{-7}$, $\varkappa = 10^{-5}$
1.5
2.0
2.5
3.0
ω_{e*}/γ_T
b)
ψ_m
10^{-3}
10^{-4}
$\omega_{e*}/\gamma_T = 2.5$, $\varkappa = 10^{-5}$
10^{-8}
10^{-7}
10^{-6}
μ
c)
ψ_m
10^{-3}
10^{-4}
$\omega_{e*}/\gamma_T = 2.5$, $\mu = 3\times10^{-7}$
10^{-6}
10^{-5}
10^{-4}
$\varkappa$
d)
ψ_m
10^{-4}
10^{-5}
$\omega_{e*}/\gamma_T = 3$, $\mu = 3\times10^{-7}$
10^{-6}
10^{-5}
10^{-4}
$\varkappa$

Fig. 4