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Energy Principle for 3-d Resistive
Instabilities in Shaped-cross-section
Tokamaks

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Abstract

An energy principle for 'helical' incompressible perturbations in shaped-cross-section plasmas is derived in the tokamak scaling ($\varepsilon \equiv ka \stackrel{\sim}{\sim} B_{\perp}/B_z << 1$). Two models for the resistivity are used. The resistivity is assumed to be transported either by the fluid or by the magnetic surfaces. In the first case generalized rippling and tearing modes are discovered, while in the latter case the rippling is cancelled in a self-consistent way. The Euler equation for the tearing modes generalizes the previously derived equation for two-dimensional perturbations. It is pointed out that the energy principle cannot be extended to higher orders in ε.

1.INTRODUCTION

The stability equation of some dissipative systems can be written in the form

$$N\ddot{\Psi} + M\dot{\Psi} + Q\Psi = 0 \qquad (1)$$

where Ψ is a complex state vector, N, M and Q are time independent Hermitian operators and, in addition, N and M are positive. As has been shown by Barston (1969), a necessary and sufficient condition for the stability of such systems is that Q is positive definite, i.e.

$$(\Psi, Q\Psi) > 0 \qquad (2)$$

for all Ψ. The scalar product is defined by

$$(\Psi,\Phi) = \int (\Psi^*)^T \Phi \, d\tau$$

where the star denotes the complex conjugate and T the vector transpose.

Unfortunately, not all interesting systems can be represented by an equation of type (1) as has been discussed by Tasso (1978). In particular, the operator Q generally contains both a Hermitian and an anti-Hermitian part, $Q = Q_h - iQ_a$ (Q_h and Q_a are Hermitian), which makes the stability analysis much more difficult. The basic reason is that the system may be overstable and that $\omega \neq 0$ at the marginal point. To see this, we consider a marginal mode $\Psi = \Psi_o e^{i\omega t}$ with real ω. Substituting this in (1) and taking the scalar product with Ψ_o, we get

$$- n\omega^2 + im\omega + q_h - iq_a = 0 \qquad (3)$$

where the small letters stand for the expectation values of the respective operators, $n = (\Psi_o, N\Psi_o)$ etc. All these are real numbers because of the hermiticity of the operators. The imaginary part of equation (3) gives

$$\omega = \frac{q_a}{m} \quad . \qquad (4)$$

When Q is Hermitian, one has $q_a = 0$ and the marginality occurs at $\omega = 0$. The mode equation (1) for the marginal state is therefore simply $Q\Psi = 0$, which makes it understandable that the stability can be decided by considering Q alone. If the anti-Hermitian part iQ_a does not vanish then also, in general, one has $q_a \neq 0$ and the marginal point shifts from the zero frequency. No simple stability criterion exists for such cases. One cannot hope to get rid of Q_a by a time independent coordinate or variable transformation since ω at the marginal point is a physical property of the system and cannot vanish in one representation if it is shown to be non-zero in some other representation by equation (4). One could try to substitute (4) in the real part of (3) and require the left-hand side to be positive for all Ψ. This would be a sufficient stability condition, but unfortunately this condition can always be violated for resistive plasmas. It seems that, apart from a special limiting case to be discussed, one has to face dealing with the full eigenvalue equation (1) with a complex ω in order to solve the stability problem.

In this paper we carry the analysis of resistive stability as far as seems to be possible with the aid of the energy principle (2). It turns out that one has to restrict oneself within tokamak scaling in order to get a Hermitian Q. The energy integrals and corresponding Euler equations are derived for plasmas with arbitrary cross-sections. It is shown that by assuming the resistivity to be transported by the magnetic surfaces the rippling mode is cancelled in a self-consistent way. The remaining stability equation for the tearing modes emerges as a natural generalization of the stability equation for two-dimensional perturbations previously derived by Tasso (1975). The present work also generalizes the previous results for the helical modes of circular-cross-section tokamaks (Tasso, 1977) and completes the test function approach for shaped cross-sections by Caldas and Tasso (1978). Our equation for the marginal state is essentially the same as the one recently obtained by Jensen and Chu (1979). The energy method, however, is more rigorous and allows the use of test functions and at the same time affords justification for the method of neighbouring equilibria.

2. EQUILIBRIUM AND PERTURBED EQUATIONS

We consider resistive systems in a static equilibrium. The equilibrium magnetic field $\underline{B}$ and current density $\underline{J}$ are given by (Tasso, 1975)

$$\underline{B} = \underline{B}_{\perp} + \underline{B}_{\parallel} = \underline{e}_z \times \nabla\psi + B_z \, \underline{e}_z \qquad (5)$$

$$\underline{J} = J(\psi)\underline{e}_z \qquad (6)$$

where $\underline{e}_z$ is the unit vector along the straight plasma. The poloidal magnetic flux ψ satisfies the equilibrium equation

$$\frac{1}{\mu_o} \nabla^2\psi = J(\psi) = - \frac{dp_o}{d\psi}$$

with an arbitrary pressure profile $p_o(\psi)$. Moreover, B_z and the electric field

$$\underline{E} = \eta_o(\psi) \; \underline{J}(\psi) \qquad (7)$$

where $\eta_o(\psi)$ is the resistivity, are constants in the equilibrium.

The linearized equations of motion around the equilibrium are

$$\rho_o \ddot{\underline{\xi}} + \nabla p_1 - \underline{j} \times \underline{B} - \underline{J} \times \underline{b} = 0 \qquad (8)$$

$$\underline{e} + \dot{\underline{\xi}} \times \underline{B} - \eta_1 \underline{J} - \eta_o \underline{j} = 0 \qquad (9)$$

$$\nabla \times \underline{e} = - \dot{\underline{b}} \qquad (10)$$

$$\nabla \cdot \underline{b} = 0 \qquad (11)$$

$$\mu_o \underline{j} = \nabla \times \underline{b} \qquad (12)$$

where $\underline{\xi}$ is the displacement vector, and $\underline{j}$, $\underline{b}$, $\underline{e}$, p_1 and η_1 denote the perturbed current density, magnetic field, electric field, pressure and resistivity, respectively. We restrict ourselves to incompressible perturbations

$$\nabla \cdot \underline{\xi} = 0 . \qquad (13)$$

For the resistivity we use two alternative physical models. We assume that the resistivity is transported either by the fluid

$$\eta_1 = - \underline{\xi} \cdot \nabla \eta_o \qquad (14)$$

or by the magnetic surfaces

$$(\underline{B} \cdot \nabla) \eta_1 + (\underline{b} \cdot \nabla) \eta_o = 0. \qquad (15)$$

The latter model is representative of hot plasmas where the heat conductivity along the magnetic field is large.

Equations (10), (11) and (13) are satisfied identically by the introduction of the vector potentials $\underline{A}$ and $\underline{U}$ and the scalar potential ϕ :

$$\underline{b} = \nabla \times \underline{A}$$
$$\underline{\xi} = \nabla \times \underline{U}$$
$$\underline{e} = -\dot{\underline{A}} - \nabla\phi .$$

With the particular choice of the gauge $\phi = -\underline{B}\cdot\dot{\underline{U}}$ equation (9) becomes

$$\dot{\underline{A}} - \nabla(\underline{B}\cdot\dot{\underline{U}}) + \underline{B} \times (\nabla\times\dot{\underline{U}}) + \eta_1\underline{J} + \eta_o\,\underline{j} = 0 . \qquad (16)$$

In this gauge B_z enters into the equation only in combination with the z derivative. Indeed, we have

$$-\nabla(\underline{B}\cdot\dot{\underline{U}}) + \underline{B} \times (\nabla\times\dot{\underline{U}}) = -\dot{\underline{U}} \times (\nabla\times\underline{B}) - (\underline{B}\cdot\nabla)\dot{\underline{U}} - (\dot{\underline{U}}\cdot\nabla)\underline{B}$$

and, since B_z is constant, the only term where B_z survives is $(\underline{B}\cdot\nabla)\dot{\underline{U}} = (\underline{B}_\perp \cdot \nabla_\perp + B_z \frac{\partial}{\partial z})\dot{\underline{U}}$.

The pressure can be eliminated from equation (8) by taking the curl

$$\nabla \times \rho_o \nabla \times \ddot{\underline{U}} - \nabla \times (\underline{j} \times \underline{B}) - \underline{B}(\nabla\cdot\underline{j}) - \nabla\times(\underline{J} \times \underline{b}) = 0 . \qquad (17)$$

Corresponding to the particular choice of the gauge in equation (16), we have here added zero in the form $-\underline{B}(\nabla\cdot\underline{j})$ in order to gain symmetry of the subsequent operators. We note that also in this equation B_z only appears through the magnetic differential operator $\underline{B}\cdot\nabla$.

3. OPERATORIAL EQUATION AND ENERGY INTEGRALS

Upon substitution of $\underline{j}$ from equation (16) in (17) this pair of equations can be written in the form (1), where the state vector is defined as

$$\Psi = \begin{pmatrix} \underline{U} \\ \underline{A} \end{pmatrix} .$$

The operators N and M are then

$$N = \begin{pmatrix} \nabla \times \rho_o \nabla\times & 0 \\ 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} -(\nabla\times\underline{B}\times - \underline{B}\nabla\cdot)\frac{1}{\eta_o}(\underline{B}\times\nabla\times - \nabla\underline{B}\cdot) & -(\nabla\times\underline{B}\times - \underline{B}\nabla\cdot)\frac{1}{\eta_o} \\ \frac{1}{\eta_o}(\underline{B}\times\nabla\times - \nabla\underline{B}\cdot) & \frac{1}{\eta_o} \end{pmatrix}$$

The fact that N and M are of the required type will shortly be proved.

The last term of equation (1) reads for our system

$$Q\Psi = \begin{pmatrix} (\underline{B}\cdot\nabla)\frac{\eta_1}{\eta_o}\underline{J} - (\underline{b}\cdot\nabla)\underline{J} + (\underline{J}\cdot\nabla)\underline{b} \\ \\ \frac{\eta_1}{\eta_o}\underline{J} + \dot{\underline{j}} \end{pmatrix}$$

where use has been made of the constancy of $\underline{B}$ along the z axis, $(\underline{J}\cdot\nabla)\underline{B} = 0$. The explicit form of the Q operator (not necessarily Hermitian) depends on the model for the resistivity. In the case of the fluid convection model (14) we have

$$Q^{(f)} = \begin{pmatrix} \underline{e}_z(\underline{B}\cdot\nabla)(\nabla J)\cdot\nabla x & -\underline{e}_z(\nabla J)\cdot\nabla x + (\underline{J}\cdot\nabla)\nabla x \\ \underline{e}_z(\nabla J)\cdot\nabla x & \frac{1}{\mu_o}\nabla x\nabla x \end{pmatrix} \quad (18)$$

while the surface convection model (15) leads to

$$Q^{(s)} = \begin{pmatrix} 0 & (\underline{J}\cdot\nabla)\nabla x \\ 0 & \underline{e}_z(\underline{B}\cdot\nabla)^{-1}(\nabla J)\cdot\nabla x + \frac{1}{\mu_o}\nabla x\nabla x \end{pmatrix} . \quad (19)$$

The inverse operator $(\underline{B}\cdot\nabla)^{-1}$ formally introduced here may not exist on some resonant surfaces, as will be seen later. In obtaining these equations we have used the relation $J\nabla\eta_o/\eta_o = -\nabla J$, which is due to the constancy of $J\eta_o$ (see equation (7)). Because of equations (5) and (6) the gradient

∇J is equal to $J'(\underline{B}_\perp \times \underline{e}_z)$, where $J' = \frac{dJ}{d\psi}$. The operator $(\nabla J)\cdot\nabla x$ appearing in (18) and (19) thus becomes

$$(\nabla J)\cdot\nabla x = J' \left[(\underline{B}_\perp\cdot\nabla)\underline{e}_z - (\underline{e}_z\cdot\nabla)\underline{B}_\perp \right].$$

A straightforward calculation shows that

$$(\dot{\Psi}, N\dot{\Psi}) = \int \rho_o |\nabla \times \underline{\dot{U}}|^2 \, d\tau \ , \qquad (20)$$

$$(\dot{\Psi}, M\dot{\Psi}) = \int \frac{1}{\eta_o} |\underline{\dot{A}} - \nabla(\underline{B}\cdot\underline{\dot{U}}) - (\nabla\times\underline{\dot{U}})\times\underline{B}|^2 \, d\tau \ , \qquad (21)$$

where $\underline{E}_r \equiv - \underline{\dot{A}} + \nabla(\underline{B}\cdot\underline{\dot{U}}) + (\nabla\times\underline{\dot{U}})\times\underline{B}$ is equal to the resistive electric field, $\underline{E}_r = \eta_1 \underline{J} + \eta_o \underline{j}$, owing to equation (16). The integrals (20) and (21) are real and positive quantities, implying that the real operators N and M are Hermitian and positive as required in equation (1). Physically, $(\dot{\Psi}, N\dot{\Psi})$ and $(\dot{\Psi}, M\dot{\Psi})$ represent the kinetic energy and the resistive dissipation of the system.

We also calculate the expectation value of the Q operator as follows:

$$(\Psi \ , Q^{(f)}\Psi) = \delta W^{(f)} + \delta\tilde{W}^{(f)}$$

where

$$\delta W^{(f)} = \int \{J'(\underline{U}^*\cdot\underline{e}_z)(\underline{B}\cdot\nabla)(\underline{B}_\perp\cdot\nabla)(\underline{U}\cdot\underline{e}_z)+J'[(\underline{A}^*\cdot\underline{e}_z)(\underline{B}_\perp\cdot\nabla)(\underline{U}\cdot\underline{e}_z)+$$

$$+(\underline{A}\cdot\underline{e}_z)(\underline{B}_\perp\cdot\nabla)(\underline{U}^*\cdot\underline{e}_z)] + \frac{1}{\mu_o}(\nabla x\underline{A}^*)\cdot(\nabla x\underline{A})\}\ d\tau$$

$$\delta\tilde{W}^{(f)} = \int \{-J'(\underline{U}^*\cdot\underline{e}_z)(\underline{B}\cdot\nabla)(\underline{e}_z\cdot\nabla)(\underline{B}_\perp\cdot\underline{U})-J'(\underline{A}^*\cdot\underline{e}_z)(\underline{e}_z\cdot\nabla)(\underline{B}_\perp\cdot\underline{U})+$$

$$+ J'(\underline{U}^*\cdot\underline{e}_z)(\underline{e}_z\cdot\nabla)(\underline{B}_\perp\cdot\underline{A})+J\underline{U}^*\cdot(\underline{e}_z\cdot\nabla)\nabla x\underline{A}\}\ d\tau$$

and

$$(\Psi,\ Q^{(s)}\Psi) = \delta W^{(s)} + \delta\tilde{W}^{(s)}$$

where

$$\delta W^{(s)} = \int \{J'(\underline{A}^*\cdot\underline{e}_z)(\underline{B}\cdot\nabla)^{-1}(\underline{B}_\perp\cdot\nabla)(\underline{A}\cdot\underline{e}_z)+\frac{1}{\mu_o}(\nabla x\underline{A}^*)\cdot(\nabla x\underline{A})\}\ d\tau$$

$$\delta\tilde{W}^{(s)} = \int \{J\ \underline{U}^*\cdot(\underline{e}_z\cdot\nabla)\nabla x\underline{A} - J'(\underline{A}^*\cdot\underline{e}_z)(\underline{B}\cdot\nabla)^{-1}(\underline{e}_z\cdot\nabla)(\underline{B}_\perp\cdot\underline{A})\}\ d\tau\ .$$

When singularities arise from the inverse operator $(\underline{B}\cdot\nabla)^{-1}$ these integrals are to be understood to mean appropriately defined principal parts. The δW parts of the expectation values are real, but the $\delta\tilde{W}$'s, in general, have both real and imaginary parts. Hence the Q operators are not Hermitian and one is not allowed to apply the energy principle. We note, however, that each term in the $\delta\tilde{W}$'s contains the derivative $(\underline{e}_z\cdot\nabla)$ along the ignorable direction (without being multiplied by B_z). Since the exact solutions can be taken to depend on z as e^{ikz}, the $\delta\tilde{W}$'s have k as a factor and in the limit $k \to 0$ they vanish leaving real expectation values, i.e. Hermitian Q operators.

4. TOKAMAK SCALING

A comparison of the terms in the δW's and $\delta\tilde{W}$'s shows that the latter may be ignored when $\varepsilon \equiv ka << 1$, where a is a characteristic dimension of the plasma in the cross-sectional plane. Symmetry of Q is obtained in zeroth order of ε. Equations (16) and (17) were written in a form where B_z enters only through the expression $\underline{B}\cdot\nabla = \underline{B}_\perp\cdot\nabla + B_z \frac{\partial}{\partial z}$. One is allowed to assume that at the same time as $ka \to 0$ B_z increases in such a way that $ka \approx B_\perp/B_z$ and $\underline{B}\cdot\nabla$ still remains of the order $O(\varepsilon^0)$. Then $\underline{A}$ and $\underline{U}$ also remain of the order $O(\varepsilon^0)$, and the $\delta\tilde{W}$'s of the order $O(\varepsilon^1)$, thus validitating the use of the energy principle in the zeroth order. In this tokamak scaling, $\varepsilon \equiv ka \approx B_\perp/B_z << 1$, the perturbation may be resonant with the magnetic field even though ka is small; the derivative $\underline{B}\cdot\nabla$ does not reduce to $\underline{B}_\perp\cdot\nabla$. It also follows from the tokamak scaling that the aspect ratio $L/2\pi a$ of the plasma (L is the length in the z direction) has to be large $L/2\pi a = n/\varepsilon >> 1$, where n is the mode number ($k = 2\pi n/L$), when modes other than the strictly two-dimensional ones (i.e. $n \neq 0$) are considered.

In the limit ka → 0 the magnetic energy term in the energy integrals δW reduces to the form

$$\frac{1}{\mu_0} |\nabla x \underline{A}|^2 = \frac{1}{\mu_0} |\nabla_\perp x \underline{A}_\perp|^2 + \frac{1}{\mu_0} |\nabla_\perp x (\underline{A} \cdot \underline{e}_z) \underline{e}_z|^2 \quad .$$

Since the transverse component $\underline{A}_\perp$ appears only in this expression, we can trivially perform a minimization of the δW's with respect to this variable to give $\nabla_\perp x \underline{A}_\perp = 0$. Only the z components of the vector potentials $U = \underline{U} \cdot \underline{e}_z$ and $A = \underline{A} \cdot \underline{e}_z$ then remain in the energy integrals, thus leading to a reduced problem in terms of the reduced state vector

$$\Psi = \begin{pmatrix} U \\ A \end{pmatrix} \quad .$$

The Q operators are now

$$Q^{(f)} = \begin{pmatrix} J'(\underline{B} \cdot \nabla)(\underline{B}_\perp \cdot \nabla) & -J'(\underline{B}_\perp \cdot \nabla) \\ J'(\underline{B}_\perp \cdot \nabla) & -\frac{1}{\mu_0} \nabla_\perp{}^2 \end{pmatrix} \tag{22}$$

$$Q^{(s)} = \begin{pmatrix} 0 & 0 \\ 0 & J'(\underline{B} \cdot \nabla)^{-1} (\underline{B}_\perp \cdot \nabla) - \frac{1}{\mu_0} \nabla_\perp{}^2 \end{pmatrix} \tag{23}$$

and the corresponding energy expressions are

$$\delta W^{(f)} = \int \left[J'U^*(\underline{B}\cdot\nabla)(\underline{B}_\perp\cdot\nabla)U - J'U^*(\underline{B}_\perp\cdot\nabla)A + J'A^*(\underline{B}_\perp\cdot\nabla)U - \frac{1}{\mu_o} A^*\nabla_\perp{}^2 A \right] d\tau \qquad (24)$$

$$\delta W^{(s)} = \int \left[J'A^*(\underline{B}\cdot\nabla)^{-1}(\underline{B}_\perp\cdot\nabla)A - \frac{1}{\mu_o} A^* \nabla_\perp{}^2 A \right] d\tau \quad . \qquad (25)$$

The operator (22) was previously discovered by Tasso (1975). He considered strictly two-dimensional perturbations but his derivation leading to (22) is also valid for three-dimensional perturbations in the tokamak scaling, virtually without any modifications.

It can easily be shown that the energy integral for ideal incompressible MHD equations can be written in the form

$$\delta W^{(id)} = \int \{ J' \left[(\underline{B}\cdot\nabla)U^* \right] (\underline{B}_\perp\cdot\nabla)U - \frac{1}{\mu_o} \left[(\underline{B}\cdot\nabla)U^* \right] \nabla_\perp{}^2 (\underline{B}\cdot\nabla)U \} \; d\tau$$

in the tokamak scaling. This is essentially the same as $\delta W^{(s)}$ formulated in terms of U rather than A, $A = (\underline{B}\cdot\nabla)U$, the difference being that in the resistive case A can be taken as finite everywhere.

5. STABILITY CONDITION AND DISCUSSION

In tokamak scaling we have obtained Hermitian Q operators (22) and (23) which can be used for a stability analysis by the energy principle (2). The stability is determined by the sign of the minimum value of $\delta W = (\Psi, Q\Psi)$ over all permissible states Ψ. We therefore extremize δW under the constraint that some appropriate norm $\|\Psi\|^2 = (\Psi, N\Psi)$ be constant, where N is a Hermitian positive definite operator. The resulting Euler equation reads

$$Q\Psi = \lambda N\Psi \tag{26}$$

where the Lagrange multiplier λ appears as an eigenvalue. The extremum value of δW is $\lambda(\Psi, N\Psi)$. The stability is thus decided by the sign of the lowest eigenvalue λ. The form of the extremum solution depends on the norm; the extremization of a homogeneous expression like $(\Psi, Q\Psi)$ makes sense only with respect to the norm. With regard to the stability of the system, however, the choice of the norm is immaterial since at the marginal point λ in (26) is zero in any case.

For our purposes it is convenient to choose $\|\Psi\|^2 = \int |A|^2 d\tau$. We then get for the fluid convection model $Q^{(f)}$ the pair of equations

$$(\underline{B}\cdot\nabla)(\underline{B}_\perp\cdot\nabla)U - (\underline{B}_\perp\cdot\nabla)A = 0$$

$$J'(\underline{B}_\perp\cdot\nabla)U - \frac{1}{\mu_o}\nabla_\perp^2 A = \lambda A \tag{27}$$

and solving for $(\underline{B}_\perp\cdot\nabla)U$ from the first equation and substituting the result in the second one yields

$$J'(\underline{B}\cdot\nabla)^{-1}(\underline{B}_\perp\cdot\nabla)A - \frac{1}{\mu_o}\nabla_\perp^2 A = \lambda A. \tag{28}$$

Exactly the same equation is directly obtained for the surface convection model. There is a difference, however, between the models. In the case of the fluid convection model, it is not guaranteed that the first equation (27), which represents extremization of $\delta W^{(f)}$ with respect to U, produces a minimum. In fact, as we show below, this is never quite true. $\delta W^{(f)}$ can be made negative by suitable test functions, giving rise to the rippling mode instability. The question of the nature of the extremum does not arise in the context of the surface convection model. The rippling is cancelled here in a self-consistent way.

In order to make the considerations more explicit, we have to specify a coordinate system. We choose Hamada-like coordinates ψ, Θ, ζ, where ψ is the equilibrium poloidal flux, $\zeta = 2\pi z/L$, and Θ is an angle-like variable defined on each magnetic surface by

$$d\Theta = 2\pi \frac{d\ell/B_\perp}{\oint d\ell/B_\perp}$$

where $d\ell$ is the differential arc length along the contour around the plasma. (The curve $\Theta = 0$ at the $z = 0$ plane can be taken to be any smooth curve that begins from the magnetic axis and intersects each magnetic surface only once.) While ψ and Θ coordinate curves are not, in general, orthogonal, these coordinates have the advantage that the operators $\underline{B}\cdot\nabla$ and $\underline{B_\perp}\cdot\nabla$ have constant coefficients on the magnetic surface, as discussed by, for example, Dewar et al. (1974):

$$\underline{B}\cdot\nabla = \frac{2\pi B_z}{Lq} \left(\frac{\partial}{\partial\Theta} + q \frac{\partial}{\partial\zeta} \right)$$

$$\underline{B_\perp}\cdot\nabla = \frac{2\pi B_z}{Lq} \frac{\partial}{\partial\Theta} \qquad (29)$$

where $q(\psi) = \frac{B_z}{L}\oint\frac{d\ell}{B_\perp}$ is the safety factor. Because of the tokamak scaling q is of the order of unity.

The existence of the rippling instability in the fluid convection model can now be shown in the same way as in previous work (Tasso, 1977; Caldas and Tasso, 1978). Choosing test functions $A \approx 0$, $U = u(\psi)\, e^{i(n\zeta - m\Theta)}$ equation (24) becomes

$$\delta W^{(f)} = \int J' \left(\frac{2\pi B_z}{Lq}\right)^2 m(nq-m)\, uu^* d\tau \; .$$

The factor nq-m changes sign at the resonant surface. Concentrating $|u|^2$ on the negative side of the integrand, the energy integral can always be made negative.

In the Euler equation (28) for the tearing modes one needs the inversion of the magnetic differential operator $\underline{B}\cdot\nabla$. By using equations (29) one readily sees that the expression $(\underline{B}\cdot\nabla)^{-1}(\underline{B}_{\perp}\cdot\nabla)A$ is equal to $A-\tilde{A}$, where $\tilde{A}$ satisfies the magnetic differential equation

$$\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\zeta}\right)\tilde{A} = iqnA \quad . \tag{30}$$

Both $\tilde{A}$ and A depend on ζ as $e^{in\zeta}$ and are periodic functions of θ with the period 2π. Equation (30) can be written as an ordinary differential equation along the characteristic curve (field line)

$$\frac{d}{d\alpha}\tilde{A} = iqnA \tag{31}$$

with

$$\theta = \theta_o + \alpha$$

$$\zeta = \zeta_o + q\alpha$$

where α is the curve parameter. Integrating (31) over the inverval $\alpha \in [0,2\pi]$ and making use of the periodicity in θ and harmonicity in ζ, we obtain

$$\tilde{A}(\psi,\theta) = \frac{\int_0^{2\pi} A(\psi,\theta+\alpha)\, e^{inq\alpha} d\alpha}{\int_0^{2\pi} e^{inq\alpha}\, d\alpha} \tag{32}$$

where the exponential dependence on ζ has been suppressed from the notation. $\tilde{A}$ is a weighted surface average of A. Using these results the Euler equation (28) becomes

$$-\frac{1}{\mu_o} \nabla_\perp^2 A + J'(A-\tilde{A}) = \lambda A. \tag{33}$$

This equation is to be solved subject to the boundary condition that A vanish at the conducting wall (or at the infinity). The solution and its gradient must be continuous everywhere. Since the denominator in (32) becomes zero on the resonant surfaces where qn is integer, the equation has a singularity at those surfaces. The solution A of equation (33) can, however, be continued across the singularity so that both A and ∇A are continuous by using an analytic expansion near the singular surface similar to that given by Furth et al. (1973). The gradient ∇A becomes infinite at the singularity but it may still be continuous in the sense of a 'principal value'.

Our procedure of solving equation (33) with continuous ∇A and looking at the sign of λ does not, in fact, differ from the more familiar method (in plane and cylindrical geometries) of solving the Euler equation with $\lambda = 0$ and looking at the sign of the discontinuity of the logarithmic derivative Δ' at the resonant surface. The apparent difference arises solely from different norms used in the minimization. The usual Δ' method can also be interpreted as resulting from equation (26) when the norm is defined to be the integral of $|A|^2$ over the resonant surface. Then $N = \delta(\psi-\psi_o)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and the corresponding eigenvalue is seen to be proportional to $-\Delta'$ in simple geometries. Here ψ_o refers to the resonant surface. As discussed after (26), the marginality condition is not affected by the choice of the norm. The eigenvalue at the marginal point is zero, $\lambda = \Delta' = 0$, and the solutions are identical. The discontinuity Δ' in the minimum solution, for a non-marginal system, is quite unrelated to the singularity of $\tilde{A}$ in equation (33) and could be shifted anywhere by changing the value of ψ_o in the norm operator N. The extremal solutions resulting from the choice where ψ_o is the ψ of the resonant surface seem physically appealing because they resemble the real modes which behave 'discontinuously' near the resonant surface. One has to remember, however, that the energy principle does not at any rate give the real modes correctly except for the marginal point.

In order to reveal more clearly the nature of the resonances it is useful to expand $A(\psi,\Theta)$ in Fourier series in Θ ('multihelical' representation):

$$A(\psi,\Theta) = \sum_m a_m(\psi)\, e^{-im\Theta}$$

Substituting in (32) one gets

$$A-\tilde{A} = \sum_m \frac{m}{m-nq}\, a_m(\psi)\, e^{-im\Theta}. \qquad (34)$$

When nq is integer, exactly the harmonic component $m = nq$ is resonant. The singularity is of the type $1/x$ ($x=\psi-\psi_o$; $nq(\psi_o) = m$). As seen from equation (34), $\tilde{A}$ does not couple different harmonic components. The coupling comes from $\nabla_\perp^2$ because the metric coefficients of our coordinate system depend on Θ (and ψ). In the case of circular cross-section, however, Θ is the ordinary angle of the cylindrical coordinates, and the harmonic components decouple completely. For each component we then find an equation identical to that given by Glasser et al. (1977) for a marginal MHD mode.

While the Fourier expansion has the advantage of clarity, it is not obvious that it provides the best method for the numerical solution of the full equation (33). In particular, if the deviation from circular symmetry is large, some other numerical scheme based directly on equations (32) and (33) may give more rapid convergence. On the other hand, the energy principle allows one to use test functions, and for this purpose the Fourier expansion is well suited. For instance, a necessary condition for the stability is that (25) is positive for any single harmonic component. Testing this reduces the problem to a one-dimensional one similar to that in the circular cross-section case, with the only distinction that the 'radial part' of $\nabla_{\perp}^2$ is modified by the geometry of the problem.

In the case of two-dimensional perturbations (n = 0) the expression (32) reduces to the ordinary surface average $\bar{A} = \oint A \frac{d\ell}{B_{\perp}} / \oint \frac{d\ell}{B_{\perp}}$, and the Euler equation derived by Tasso (1975) is regained. Equation (33) is a natural extension of this equation for helical perturbations.

Equations (33) and (34) were recently also found by Jensen and Chu (1979) using a more heuristic neighbouring equilibria approach. By showing that in the tokamak scaling the Q operator is approximately Hermitian and consequently at the marginal point $\omega = 0$ we have given justification for the use of these equations. We have also shown that in first and higher orders in ε the energy principle is not applicable. The antisymmetric part of Q causes the frequency ω at the marginal point to shift from zero. With a finite value of ε one has to face the solution of the full eigenvalue problem with complex eigenvalues. The stability criterion in such cases most likely depends on the resistivity, finite Larmor radius effects, gyroviscosity and other physical effects.

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