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On the Mercier Criterion in

Guiding Centre Theory

A. Salat

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Abstract

Collisionless two-fluid guiding centre equations are used to investigate stability of toroidal plasma equilibria with respect to localized MERCIER-type modes. The anisotropic equilibria are characterized by $\sigma(V,B) = 1-(P_{\parallel}-P_{\perp})/B^2$, where V is the volume inside the flux surfaces. It is found that, in general, no localized MERCIER-type eigenmodes exist. For $\sigma = \sigma(B)$ such modes exist but no explicit stability condition can be given. For σ = const in the localization region and no trapped particles the necessary stability condition for axisymmetric equilibria agrees with the MERCIER criterion except that the pressure is replaced by $(P_{\parallel} + P_{\perp})/(2\sigma)$.

1. Introduction

MERCIER (1960) derived a necessary stability condition for axisymmetric toroidal plasmas with respect to displacements which are localized around a rational magnetic surface and are almost constant along the field lines. Subsequently the criterion was shown to be necessary and sufficient with respect to such modes in general toroidal geometry (GREENE and JOHNSON, 1962, MERCIER and LUC, 1974). For axisymmetric equilibria PAO (1975) derived the criterion from an eigenmode analysis.

It is now investigated whether a MERCIER-type mode ansatz may also be used in guiding centre theory (GCT) to derive an anologous stability condition. The GCT (GRAD, 1961) allows for kinetic effects parallel to the field lines and for pressure anisotropy but keeps only $\vec{E} \times \vec{B}$ drifts and assumes vanishing guiding centre radii. No collisions are included. GCT-type energy principles were derived by KRUSKAL and OBERMAN (1958) and by ANDREOLETTI (1963). Instead of these principles we use here an eigenmode approach similar to that of PAO (1974, 1975).

Stability and mode analyses of GCT plasmas have hitherto been mainly performed in simple geometries. ALEKSIN and YASHIN (1961), PAO (1974) and CHOE, TATARONIS and GROSSMAN (1977) have investigated modes in straight cylindrical (screw pinch) geometry where in MHD the MERCIER criterion changes to the SUYDAM criterion (SUYDAM, 1958). Stability in bumpy θ-pinch and helical equilibria was investigated by, for example, VAHALA and VAHALA (1977) and WEITZNER (1976) for the cases of small bumpiness and helicity, respectively. Here, we initially consider arbitrary toroidal plasma geometry. Later on, consideration is restricted to axisymmetric plasmas which are symmetric with respect to the equatorial plane.

The presentation is as follows: Secion 2 gives the basic equations. In Section 3 plasma equilibria are described as far as necessary and particular coordinates are introduced. Section 4 presents the linearized eigenmode equations. In Section 5 a MERCIER-type localization ansatz is made and the existence of such modes is discussed. When the pressure anisotropy coefficient $\sigma = 1-(P_{\parallel} - P_{\perp})/B^2$ is a function of B only ($B = |\vec{B}|$) and the plasma is axisymmetric, the eigenmode equations are further discussed in Section 6. For σ = const in the localization region and with the further assumption that trapped particles may be neglected a necessary MERCIER-type stability condition is finally derived and compared with the MHD case in Section 7. Conclusions are given in Section 8.

2. Basic Equations

We consider a fully ionized collisionless plasma which is described by the following guiding centre equations (GRAD, 1961):

$$\rho \frac{d\vec{U}}{dt} = \left[\vec{J} \times \vec{B}\right] - \nabla \cdot \vec{\vec{P}} \quad , \tag{2.1}$$

$$\vec{J} = \text{curl}\ \vec{B} \quad , \tag{2.2}$$

$$\vec{\vec{P}} = P_\perp \vec{\vec{1}} + \frac{P_\parallel - P_\perp}{B^2} \vec{B}\vec{B} \quad , \tag{2.3}$$

$$\frac{\partial \vec{B}}{\partial t} = \text{curl}\left[\vec{U} \times \vec{B}\right] \quad . \tag{2.4}$$

The anisotropic pressure $P_\parallel$, $P_\perp$ and the mass density ρ consist of the contributions from electrons and ions:

$$\begin{aligned} P_{\parallel,\perp} &= P^+_{\parallel,\perp} + P^-_{\parallel,\perp} \quad , \\ \rho &= \rho^+ + \rho^- \quad , \end{aligned} \tag{2.5}$$

which, in turn, are defined as moments of the distribution functions $F^\pm$ of the guiding centres:

$$\begin{aligned} \left[P^\pm_\parallel , P^\pm_\perp\right] &= \int_{-\infty}^{+\infty} dv \int_0^\infty d\mu \left[v^2, \mu B\right] F^\pm(v,\mu,\vec{r},t) \quad , \\ \rho^\pm = m^\pm n^\pm &= \int_{-\infty}^{+\infty} dv \int_0^\infty d\mu\ F^\pm \quad , \end{aligned} \tag{2.6}$$

where v is the velocity of the guiding centres along the field lines, and μ their magnetic moment. The $F^\pm$ are determined from the kinetic equations

$$\frac{\partial F^{\pm}(v,\mu,\vec{r},t)}{\partial t} + \nabla\cdot\left[(\vec{u} + v\vec{\beta})F^{\pm}\right] +$$

$$+ \frac{\partial}{\partial v}\left\{\left[\left(\frac{e}{m}\right)^{\pm}E_{\shortparallel} + \vec{\beta}\cdot\nabla\left(\frac{u^2}{2} + \mu B\right) + v\vec{\kappa}\cdot\vec{u}\right]F^{\pm}\right\} = 0 \ , \tag{2.7}$$

where $e^{\pm}$ and $m^{\pm}$ are the charge and mass of the ions and electrons, respectively, and

$$\vec{\beta} = \frac{\vec{B}}{B} \ , \qquad \vec{\kappa} = (\vec{\beta}\cdot\nabla)\vec{\beta} \ , \tag{2.8}$$

$$\vec{u} = \vec{U} - \vec{\beta}(\vec{U}\cdot\vec{\beta}) = \left[\vec{\beta}\times\left[\vec{U}\times\vec{\beta}\right]\right] \ . \tag{2.9}$$

The electric field parallel to the magnetic field lines,

$$E_{\shortparallel} = -\vec{\beta}\cdot\nabla\Phi \ , \tag{2.10}$$

is given by

$$E_{\shortparallel} \sum_{+,-} \left(\frac{e^2 n}{m}\right)^{\pm} = \sum_{+,-} \left(\frac{e}{m}\right)^{\pm} \vec{\beta}\cdot\nabla\cdot\vec{\vec{P}}^{\pm} \tag{2.11}$$

or by the assumption of quasineutrality, $\sum_{+,-}(en)^{\pm} = 0$, which we shall adapt here.

For later purposes it is useful to write the equation of motion (2.1) in two alternative forms (NORTHROP and WHITEMAN, 1964):

$$\rho\,\frac{d\vec{U}}{dt} = -\nabla\left(P_{\perp} + \frac{B^2}{2}\right) + (\vec{B}\cdot\nabla)\sigma\vec{B} \tag{2.12}$$

$$= \left[\vec{K}\times\vec{B}\right] - \nabla P_{\shortparallel} + \frac{P_{\shortparallel} - P_{\perp}}{B^2}\nabla B \ , \tag{2.13}$$

where

$$\vec{K} = \text{curl}\ \sigma\vec{B} \tag{2.14}$$

and

$$\sigma = 1 - \frac{P_{\parallel} - P_{\perp}}{B^2} \tag{2.15}$$

($\sigma > 0$ is assumed to avoid fire hose instability; see, for example, KADISH (1966)).

3. Equilibria and coordinates

For static equilibria we have from equ. (2.13)

$$\left[\vec{K} \times \vec{B}\right] = \nabla P_{\parallel} - \frac{P_{\parallel} - P_{\perp}}{B^2} \nabla B \ . \qquad (3.1)$$

A discussion of anisotropic equilibria which satisfy equ. (3.1) is given by, for example, SPIES and NELSON (1974). We assume that the plasma has arbitrary toroidal geometry with nested joint flux surfaces of $\vec{B}$ and $\vec{K}$. It is also assumed that $P_{\parallel}$ is given as an arbitrary function of the flux surfaces and the absolute values of $\vec{B}$:

$$P_{\parallel} = P_{\parallel}(V,B) \ , \qquad (3.2)$$

where V is the volume contained inside each flux surface.

It follows from equs. (3.1) and (3.2) that

$$P_{\perp} = P_{\parallel} - B \frac{\partial P_{\parallel}}{\partial B} \qquad (3.3)$$

if $\vec{B}\cdot\nabla B \neq 0$ (which excludes circular straight cylinders, i.e. screw pinch geometry). Hence, $P_{\perp}(V,B)$ and $\sigma(V,B)$ are completely determined from $P_{\parallel}(V,B)$. For the pressure balance one obtains

$$\left[\vec{K} \times \vec{B}\right] = P_{\parallel}'(V,B)\nabla V \ . \qquad (3.4)$$

For the derivative with respect to V the following notational convention is adopted: $A' = (\partial/\partial V)|_B A$, $\dot{A} = (\partial/\partial V)|_{r^2,r^3} A$, where r^2,r^3 are arbitrary nonsingular coordinates on the flux surfaces. For isotropic pressure, $P_\parallel = P_\perp$, one has $\partial P_\parallel/\partial B = 0$, $\sigma = 1$ and $\vec{K} = \vec{J}$.

It is easily seen that static equilibrium distribution functions are given by

$$F^{\pm}(v,\mu,\vec{r}) = B\tilde{F}^{\pm}(\varepsilon^{\pm},\mu,\vec{r}) , \qquad (3.5)$$

where $\varepsilon^{\pm} = v^2/2 + \mu B + (e/m)^{\pm}\Phi$ and $\tilde{F}^{\pm}$ are arbitrary functions of $\varepsilon^{\pm},\mu,\vec{r}$, provided that $\vec{B}\cdot\nabla|_{\varepsilon^{\pm}}\tilde{F}^{\pm} = 0$. By taking moments of these distribution functions it follows that equ. (3.3) is valid separately for each species. With $\vec{B}\cdot\nabla\cdot\vec{\vec{P}}^{\pm} = \vec{B}\cdot\nabla P_\parallel^{\pm} + (P_\perp - P_\parallel)^{\pm}\vec{B}\cdot\nabla B/B$ one then obtains $\vec{B}\cdot\nabla\cdot\vec{\vec{P}}^{\pm} = 0$ so that according to equs. (2.10), (2.11) $E_\parallel = 0$, $\Phi = \text{const}$ along $\vec{B}$. One may set $\Phi = 0$ without loss of generality. Toroidal equilibria without electric field exist in guiding centre theory because only $\vec{E} \times \vec{B}$ drifts common to all species are retained.

For the calculation of axisymmetric equilibria it is useful to specify $r^2 = \tilde{\theta}$, $r^3 = \tilde{\phi}$, where $\tilde{\theta}$ is an arbitrary poloidal variable and $r,\tilde{\phi},z$ are cylindrical coordinates with $\tilde{\phi}$ equal to the ignorable angle. Explicit representation of $\vec{B}$ and $\vec{K}$ in terms of the poloidal fluxes $\chi(V)$, $I(V)$ of $\vec{B}$ and $\vec{K}$ is then possible:

$$\vec{B} = \frac{1}{2\pi}\left\{ \left[\nabla\tilde{\phi} \times \nabla\chi\right] + \Lambda(\chi,\tilde{\theta})\nabla\tilde{\phi} \right\} ,$$
$$\vec{K} = \frac{-1}{2\pi}\left\{ \left[\nabla\tilde{\phi} \times \nabla\sigma\Lambda\right] - r^2 \operatorname{div}\frac{\sigma\,\nabla\chi}{r^2}\nabla\tilde{\phi} \right\} , \qquad (3.6)$$

with $\sigma\Lambda = I(V_1) - I(V)$. The flux surfaces $\chi = \text{const}$ are determined by

$$\operatorname{div}\left(\frac{\sigma}{r^2}\nabla\chi\right) + \frac{1}{r^2}\Lambda\frac{d\sigma\Lambda}{d\chi} = -4\pi^2\frac{1}{\dot{\chi}}P_\parallel'(V,B) . \qquad (3.7)$$

In general we shall use covariant notation with coordinates r^α, $\alpha = 1,2,3$, where $r^1 = V$, and where r^m, $m = 2,3$, are arbitrary coordinates on the flux surfaces. The summation convention is that Greek indices indicate summation over all indices, while Latin indices indicate summation over surface coordinates only. Some of the usual definitions and relations are (LAUGWITZ, 1960)

$$A^\alpha = \vec{A}\cdot\nabla r^\alpha \quad , \quad A_\alpha = \vec{A}\cdot\vec{e}_\alpha \ , \quad \vec{e}_\alpha = \frac{\partial\vec{r}}{\partial r^\alpha} \ ,$$

$$g_{\alpha\beta} = \vec{e}_\alpha\cdot\vec{e}_\beta \ , \quad A_\alpha = g_{\alpha\beta}A^\beta \ ,$$

$$\left[\vec{A}\times\vec{B}\right]_\alpha = \frac{1}{h}\,\varepsilon_{\alpha\beta\gamma}A^\beta B^\gamma \ ,$$

$$\mathrm{div}\,\vec{A} = h\,\frac{\partial}{\partial r^\alpha}\left(\frac{1}{h}\,A^\alpha\right) \ , \tag{3.8}$$

$$(\mathrm{curl}\,\vec{A})^\alpha = h\,\varepsilon^{\alpha\beta\gamma}\,\frac{\partial A_\gamma}{\partial r^\beta} \ ,$$

$$h = \left[\nabla r^1 \times \nabla r^2\right]\cdot\nabla r^3 = \sqrt{\det\,(g^{\alpha\beta})} \quad .$$

Coordinates derived from SPIES-NELSON coordinates are particularly useful for the analysis of localized eigenmodes. For anisotropic equilibria SPIES and NELSON (1974) proved the existence of coordinates r^α with the properties: $r^1 = V$, r^2 and r^3 are poloidal-like and toroidal-like coordinates with periodicity unity and

$$\begin{aligned} B^1 &= 0 \ , \quad B^2 = h\dot{\chi} \quad , \quad B^3 = h\dot{\psi} \ , \\ K^1 &= 0 \ , \quad K^2 = h\dot{I} \quad , \quad K^3 = h\dot{J} \ , \end{aligned} \tag{3.9}$$

where $\psi(V)$, $J(V)$ are the toroidal fluxes of $\vec{B}$, $\vec{K}$. The Jacobian $h(V,r^2,r^3)$ is normalized by $\int_0^1\int_0^1 dr^2dr^3h^{-1} = 1$. The pressure balance is

$$h(\ddot{I}\dot{\psi} - \ddot{J}\dot{\chi}) = P_{\shortparallel}'(V,B) . \qquad (3.10)$$

The safety factor is defined as $q = B^3/B^2 = \dot{\psi}/\dot{\chi}$. For isotropic pressure one gets $h = 1$ and the coordinates change to HAMADA coordinates.

Let $V = V_o$ be a rational surface where $q_o = q(V_o) = M/N$. The coordinates θ,ϕ are defined by $\theta = r^2$, $\phi = r^3 - q_or^2$, which implies that

$$\begin{aligned} B^\theta &= h\dot{\chi} , \quad B^\phi = (q - q_o)h\dot{\chi} , \\ K^\theta &= h\dot{I} , \quad K^\phi = (q - q_o)h\dot{I} - \frac{P_{\shortparallel}'}{\dot{\chi}} \end{aligned} \qquad (3.11)$$

and that

$$\begin{aligned} \vec{B}\cdot\nabla &= h\dot{\chi}\left[\frac{\partial}{\partial\theta} + (q - q_o)\frac{\partial}{\partial\phi}\right] , \\ \vec{K}\cdot\nabla &= h\dot{I}\left[\frac{\partial}{\partial\theta} + (q - q_o)\frac{\partial}{\partial\phi}\right] - \frac{P_{\shortparallel}'}{\dot{\chi}}\frac{\partial}{\partial\phi} . \end{aligned} \qquad (3.12)$$

For $q = q_o$ θ is a coordinate along $\vec{B}$ and ϕ counts the field lines. Periodicity conditions are

$$a(\theta,\phi) = a(\theta,\phi + 1) = a(\theta + N,\phi) = a(\theta + 1,\phi - \frac{M}{N}) . \qquad (3.13)$$

4. Eigenmode equations

The system of eigenmode equations is obtained by linearizing equs. (2.1) to (2.9) around static equilibria of the type discussed in the last section. The time dependence of the linearized quantities is assumed to be of the form $e^{-i\omega t}$. The first-order quantities $\vec{B}^{(1)}$, $P_{\shortparallel,\perp}^{(1)\pm}$, $F^{\pm(1)}$, $\Phi^{(1)}$ will be denoted by $\vec{b}$, $p_{\shortparallel,\perp}^{\pm}$, $f^{\pm}$, Φ. The displacement vector $\vec{\xi}$ is defined by $\vec{U}^{(1)} = \frac{\partial\vec{\xi}}{\partial t} = -i\omega\vec{\xi}$. Its surface component is decomposed into the components x and y along the equilibrium fields $\vec{B}$ und $\vec{K}$: $\vec{\xi} = x\,\vec{B} + y\,\vec{K} + \vec{\xi}_{\text{normal}}$. In covariant notation

$$\xi^m = x\,B^m + y\,K^m \; . \tag{4.1}$$

From equs. (2.4), (3.4) and div $\vec{B} = 0$ one obtains

$$b^\alpha = B^m \frac{\partial\xi^\alpha}{\partial r^m} - \xi^\beta \frac{\partial B^\alpha}{\partial r^\beta} - B^\alpha \,\mathrm{div}\,\vec{\xi} \tag{4.2}$$

so that

$$b^1 = D\xi^1 \tag{4.3}$$

and, with equ. (4.1),

$$b^m = B^m Dx + K^m Dy + T^m y - \dot{B}^m \xi^1 - B^m \,\mathrm{div}\,\vec{\xi} \; . \tag{4.4}$$

Here and in the following we shall use the definitions

$$D = \vec{B}\cdot\nabla = B^m \frac{\partial}{\partial r^m} \; , \qquad E = \vec{K}\cdot\nabla = K^m \frac{\partial}{\partial r^m} \; . \tag{4.5}$$

The vector $\vec{T}$ is defined by $\vec{T} = D\vec{K} - E\vec{B}$, which, using div $\vec{B}$ = div $\vec{K} = 0$ and equs. (2.15), (3.3), (3.4), may be expressed as

$$\vec{T} = \text{curl}\left[\vec{K} \times \vec{B}\right] = \left[\nabla P_{\shortparallel}' \times \nabla V\right] = \frac{\partial P_{\shortparallel}'}{\partial B}\left[\nabla B \times \nabla V\right]$$
$$= B\left[\nabla\sigma \times \nabla B\right] \quad . \qquad (4.6)$$

For isotropic plasmas one gets $\vec{T} = 0$.

Let

$$p^{*} = (P_{\perp} + \frac{1}{2} B^{2})^{(1)} = p_{\perp} + \vec{B}\cdot\vec{b} \; . \qquad (4.7)$$

In the equation of motion (2.12) p^{*} is the only linearized quantity which has a derivitive in the radial direction. Using equs. (4.3), (4.4), the linearized covariant component of equ. (2.12) with index 1 yields

$$\frac{\partial p^{*}}{\partial r^{1}} = (D\sigma B_{1} + a)X + (D\sigma K_{1}D + bD + \tau_{1})\, y$$
$$+ \rho\omega^{2}(B_{1}x + K_{1}y + g_{11}\xi^{1})$$
$$+ (D\sigma g_{11}D + A + c_{1}D)\,\xi^{1} \qquad (4.8)$$
$$+ (DB_{1} + s_{1})\,\sigma^{(1)} \quad ,$$

where

$$a = P_{\shortparallel}' + \sigma B_{m}\dot{B}^{m} , \qquad b = \sigma K_{m}\dot{B}^{m} \; ,$$

$$A = -D\sigma g_{1m}\dot{B}^{m} - \sigma g_{mn}\dot{B}^{m}\dot{B}^{n} + \dot{B}^{m}\left(\frac{\partial\sigma B_{m}}{\partial r^{1}} - \frac{\partial\sigma B_{1}}{\partial r^{m}}\right)$$
$$-\dot{\sigma}B_{m}\dot{B}^{m} \; , \qquad (4.9)$$

$$c_1 = \dot{\sigma}B_1 , \quad s_1 = \frac{1}{2}(B_m\dot{B}^m - \dot{B}_m B^m) \tag{4.9}$$

$$\tau_1 = T_m(\sigma B^m)^{\cdot} - T^m\left(\frac{\partial \sigma B_m}{\partial r^1} - \frac{\partial \sigma B_1}{\partial r^m}\right) + D\sigma T_1 ,$$

and

$$X = Dx - \operatorname{div} \vec{\xi}. \tag{4.10}$$

Here and in the following the operators D and E act on all quantities to the right of them, except if enclosed in square brackets [ ] . The only other radial derivative in the eigenmode equations occurs in the definition of div $\vec{\xi}$ which using equs. (3.8) and (4.1), may be solved for $\frac{\partial \xi^1}{\partial r^1}$:

$$\frac{\partial \xi^1}{\partial r^1} = -X - Ey - h\left(\frac{\partial}{\partial r^1}\frac{1}{h}\right)\xi^1 . \tag{4.11}$$

The remaining covariant components of equ. (3.8), when contracted with $\vec{B}$ and $\vec{K}$, yield

$$\begin{aligned}
&(D\sigma\vec{B}\cdot\vec{B} + a_1)X + (D\sigma\vec{B}\cdot\vec{K}D + b_1 D + D\tau_2 + \tau_3)\, y \\
&+ \rho\omega^2(\vec{B}\cdot\vec{B}x + \vec{B}\cdot\vec{K}\, y) = \\
&= (-D\sigma B_1 D + Da + c_2 + c_3 D - \rho\omega^2 B_1)\,\xi^1 \\
&+ (-D\,\vec{B}\cdot\vec{B} + s_2)\,\sigma^{(1)} + Dp^*
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
&(D\sigma\vec{B}\cdot\vec{K} + a_2)X + (D\sigma\vec{K}\cdot\vec{K}D + b_2 D + D\tau_4 + \tau_5)\, y \\
&+ \rho\omega^2(\vec{B}\cdot\vec{K}x + \vec{K}\cdot\vec{K}y) = \\
&= (-D\sigma K_1 D + Db + c_4 + c_5 D - \rho\omega^2 K_1)\,\xi^1 \\
&+ (-D\,\vec{B}\cdot\vec{K} + s_3)\,\sigma^{(1)} + E\,p^*
\end{aligned} \tag{4.13}$$

The coefficients are defined as follows

$$a_1 = [D\sigma]\ \vec{B}\cdot\vec{B}\ , \qquad a_2 = [E\sigma]\ \vec{B}\cdot\vec{B} - \sigma\ \vec{T}\cdot\vec{B}\ ,$$

$$b_1 = [D\sigma]\ \vec{B}\cdot\vec{K}\ , \qquad b_2 = [E\sigma]\ \vec{B}\cdot\vec{K} - \sigma\ \vec{T}\cdot\vec{K}\ ,$$

$$c_2 = [D\sigma]\ B_m\dot{B}^m - DP'_{\shortparallel}\ , \qquad c_3 = -[D\sigma]\ B_1\ , \tag{4.14}$$

$$c_4 = [E\sigma]\ B_m\dot{B}^m - \sigma\, T_m\dot{T}^m\ , \qquad c_5 = -[E\sigma]\ B_1 + \sigma T_1\ ,$$

$$s_2 = \frac{1}{2}\, D\,\vec{B}\cdot\vec{B}\ , \qquad s_3 = \frac{1}{2}\, E\,\vec{B}\cdot\vec{B} + \vec{T}\cdot\vec{B}\ ,$$

$$\tau_2 = \sigma\ \vec{B}\cdot\vec{T}\ , \qquad \tau_3 = [D\sigma]\ \vec{B}\cdot\vec{T}\ ,$$

$$\tau_4 = \sigma\ \vec{K}\cdot\vec{T}\ , \qquad \tau_5 = [E\sigma]\ \vec{B}\cdot\vec{T} - \sigma\vec{T}\cdot\vec{T}\ .$$

The l.h. sides of equs. (4.12), (4.13) constitute first-order differential operators along $\vec{B}$ for X, and second-order operators for y.

It remains to determine $\sigma^{(1)}$ and div $\vec{\xi}$ as functions of the other linearized quantities. Equations (2.15), (4.7) yield

$$\sigma^{(1)} = \frac{1}{B^2}\left[p_\perp - p_{\shortparallel} + 2(1-\sigma)(p^* - p_\perp)\right]\ , \tag{4.15}$$

where the partial pressures

$$p_{\shortparallel}^{\pm} = \int_{-\infty}^{+\infty} dv\ v^2 \int_0^{\infty} d\mu\ f^{\pm}\ ,$$

$$p_\perp^{\pm} = \int_{-\infty}^{+\infty} dv \int_0^{\infty} d\mu\mu\ (B^{(1)}F^{\pm} + Bf^{\pm}) \tag{4.16}$$

yield $p_{\shortparallel,\perp} = p^+_{\shortparallel,\perp} + p^-_{\shortparallel,\perp}$, with $B^{(1)} = \vec{B}\cdot\vec{b}/B = (p^* - p_\perp)/B$.

From the equation of continuity which follows from equ. (2.7) one obtains

$$\operatorname{div} \vec{\xi} = \frac{-1}{\rho} \left(\vec{\xi} \cdot \nabla \rho + \sum_{+,-} \int_{-\infty}^{+\infty} dv \int_{0}^{\infty} d\mu f^{\pm} \right) . \qquad (4.17)$$

Quasineutrality of the displacement requires that

$$\sum_{+,-} \left(\frac{e}{m}\right)^{\pm} \int_{-\infty}^{+\infty} dv \int_{0}^{\infty} d\mu f^{\pm} = 0 . \qquad (4.18)$$

The functions $f^{\pm}$ are determined from equ. (2.7). With the definitions $\varepsilon = v^2/2 + \mu B$ and $f^{\pm} = B\tilde{f}^{\pm}$ one gets

$$\left(\frac{\partial}{\partial t} + \frac{v}{B} D\right) \tilde{f}^{\pm}(\varepsilon, \mu, \vec{r}, t) =$$

$$= C_1 F^{\pm} + \vec{C}_2 \cdot \nabla|_v F^{\pm} + C_3^{\pm} \frac{\partial F}{\partial v} , \qquad (4.19)$$

where

$$C_1 = -\frac{1}{B} \left[\frac{\partial}{\partial t} (\operatorname{div} \vec{\eta} + \vec{\kappa} \cdot \vec{\eta}) + v \operatorname{div} \vec{\beta}^{(1)} \right] ,$$

$$\vec{C}_2 = -\frac{1}{B} \left(\frac{\partial \vec{\eta}}{\partial t} + v \vec{\beta}^{(1)} \right) , \qquad (4.20)$$

$$C_3^{\pm} = \frac{1}{B} \left[\left(\frac{e}{m}\right)^{\pm} \frac{1}{B} D\Phi + \mu (\vec{\beta}^{(1)} \cdot \nabla B + \vec{\beta} \cdot \nabla B^{(1)}) - v \frac{\partial \vec{\kappa} \cdot \vec{\eta}}{\partial t} \right]$$

and

$$\vec{\eta} = \left[\vec{\beta} \times \left[\vec{\xi} \times \vec{\beta} \right] \right] \qquad (4.21)$$

The ansatz

$$f^{\pm} = \zeta F^{\pm} - \vec{\eta} \cdot \nabla|_v F^{\pm} + \left[\left(\frac{e}{m}\right)^{\pm} \Phi + \mu B \zeta \right] \frac{\partial F^{\pm}}{\partial \varepsilon} + B\tilde{g} , \qquad (4.22)$$

where

$$\zeta \equiv - (\mathrm{div}\ \vec{\eta} + \vec{\kappa}\cdot\vec{\eta}) = \frac{1}{B}\ (\vec{\eta}\cdot\nabla B + \frac{p^{*} - p_{\perp}}{B}) \tag{4.23}$$

separates $f^{\pm}$ into an adiabatic part and an ω- dependent part $B\tilde{g}$, which after some algebraic manipulation is found to satisfy the equation

$$(-i\omega + \frac{v}{B}D)\ \tilde{g}^{\pm} = i\omega\Psi^{\pm}\ \frac{\partial\tilde{F}^{\pm}}{\partial\varepsilon}\ , \tag{4.24}$$

where

$$\Psi^{\pm} = (\frac{e}{m})^{\pm}\Phi + \mu B\zeta + \vec{\kappa}\cdot\vec{\eta}\ v^{2} \tag{4.25}$$

and $v^{2} = 2(\varepsilon - \mu B)$. The second half of equ. (4.23) follows from the definition of the curvature $\vec{\kappa}$ and equs. (4.2), (4.7), (4.21). Terms containing $\vec{\eta}$ may be expressed as functions of $\vec{\xi}$ by the relations

$$\begin{aligned} &\eta^{1} = \xi^{1}\ , \quad \vec{\kappa}\cdot\vec{\eta} = \kappa_{1}\xi^{1} + \vec{\kappa}\cdot\vec{K}\ y\ , \\ &\vec{\eta}\cdot\nabla B = \xi^{1}(\dot{B} - B_{1}DB) + y(\vec{K} - \frac{\vec{B}\cdot\vec{K}}{B^{2}}\ \vec{B})\cdot\nabla B\ . \end{aligned} \tag{4.26}$$

The component x of $\vec{\xi}$ in the direction along $\vec{B}$ does not occur in the microscopic equations.

In the formation of density and pressure moments from $f^{\pm}$ it is useful to have relations between the quantities $A^{\pm}_{m,n}$ and $B^{\pm}_{m,n}$, defined by

$$\begin{aligned} A^{\pm}_{m,n} &= \int_{-\infty}^{+\infty} dv \int_{0}^{\infty} d\mu(\mu B)^{m}\ v^{2n}\ F^{\pm}(\varepsilon,\mu,\vec{r}), \\ B^{\pm}_{m,n} &= \int_{-\infty}^{+\infty} dv \int_{0}^{\infty} d\mu(\mu B)^{m}\ v^{2n}\ \frac{\partial F^{\pm}}{\partial\varepsilon}\ . \end{aligned} \tag{4.27}$$

Direct verification and partial integration yields

$$B^{\pm}_{m,n} = (-m + B\frac{\partial}{\partial B})\, A^{\pm}_{m-1,n} = -(2n-1)\, A^{\pm}_{m,n-1} , \tag{4.28}$$

where the second equality is only valid for $n > 0$. Quasineutrality of the equilibrium implies that

$$\sum_{+,-} (\frac{e}{m})^{\pm} A^{\pm}_{o,o} = 0 . \tag{4.29}$$

With equs. (4.28), (4.29) quasineutrality of the linearized displacements requires that

$$\Phi \cdot \sum_{+,-} (\frac{e}{m})^{2\pm} B^{\pm}_{o,o} + \sum_{+,-} (\frac{e}{m})^{\pm} \int_{-\infty}^{+\infty} dv \int_{o}^{\infty} d\mu\, B\tilde{g}^{\pm} = 0 \tag{4.30}$$

Similarly, the pressure moments are determined by

$$\begin{bmatrix} p_{\parallel} \\ p_{\perp} \end{bmatrix} = -\xi^{1} \begin{bmatrix} P'_{\parallel} \\ P'_{\perp} \end{bmatrix} + \frac{p^{*} - p_{\perp}}{B} \begin{bmatrix} \partial P_{\parallel}/\partial B \\ \partial P_{\perp}/\partial B \end{bmatrix} + \\ + \sum_{+,-} \int_{-\infty}^{+\infty} dv \int_{o}^{\infty} d\mu \begin{bmatrix} v^{2} \\ \mu B \end{bmatrix} B\tilde{g}^{\pm} \tag{4.31}$$

and div $\vec{\xi}$ is given by

$$-\rho \operatorname{div} \vec{\xi} = (\vec{\xi} \cdot \nabla B + \frac{p^{*} - p_{\perp}}{B}) \frac{\partial \rho}{\partial B} + \Phi \sum_{+,-} (\frac{e}{m})^{\pm} B^{\pm}_{o,o} + \\ + \sum_{+,-} \int_{-\infty}^{+\infty} dv \int_{o}^{\infty} d\mu\, B\tilde{g}^{\pm} . \tag{4.32}$$

In order to obtain eigenmodes and eigenvalues ω, the following problems would have to be solved step by step. The differential equation for $\tilde{g}^{\pm}$ along $\vec{B}$ is solved as a path integral over source terms containing $p_\perp$, Φ, y, ξ^1 , p^* . The solution, inserted into the quasi-neutrality condition (4.30) and the equation (4.31) for the pressure $p_\perp$ yields two coupled integral equations of the second kind for Φ and $p_\perp$. Their solution and, from equs. (4.31), (4.32), (4.13), also $p_\parallel$, div $\vec{\xi}$ and $\sigma^{(1)}$ may be expressed as convolutions over y, ξ^1, p^* (MICHLIN, 1962). In the next step the integro-differential equations (4.12), (4.13) for x and y are solved as functionals of ξ^1, p^*, and the solutions are inserted into the equs. (4.8), (4.11) with the radial derivatives for ξ^1, p^* . The solution of these equations together with periodicity and boundary conditions finally determines the eigenvalues.

It is obvious that no general explicit solution is possible. In the next section a particular class of eigenmodes will be further investigated.

5. Localized Mercier-type eigenmodes

In MHD the Mercier criterion may be obtained by considering modes (MERCIER and LUC, 1974; PAO, 1975) which are confined in a small region ΔV around a rational surface $V = V_o$,

$$\left| \frac{\Delta V}{V_o} \right| = \varepsilon \ll 1 \ , \tag{5.1}$$

and have frequencies $\omega = O(\varepsilon)$. With the definition $\omega = \varepsilon w$ this amounts to $w = O(1)$. (No confusion should arise here with the previously defined $\varepsilon = v^2/2 + \mu B$.) Introducing the scaled radial coordinate $s = (V - V_o)/\varepsilon$, with $\partial/\partial V = (\partial/\partial s)/\varepsilon$, the localization is expressed by $s = O(1)$.

All nonequilibrium quantities are expanded in powers of ε, starting (arbitrarily) with ε^{-1}:

$$\begin{aligned} x &= \frac{1}{\varepsilon} x_{-1} + x_o + x_1 + \cdots , \\ \sigma^{(1)} &= \frac{1}{\varepsilon} \sigma_{-1} + \sigma_o + \varepsilon\sigma_1 + \cdots , \end{aligned} \tag{5.2}$$

etc. It follows from equs. (4.8), (4.11) that $p^* = p_o^* + \varepsilon p_1^* + \cdots$ and $\xi^1 = \xi_o + \varepsilon\xi_1 + \cdots$ cannot have a $O(\varepsilon^{-1})$ contribution owing to $\frac{\partial p^*_{-1}}{\partial s} = 0$ and $p^* = 0$ on the boundary of the localization domain.

The operators D and E are expanded as follows:

$$\begin{aligned} D &= h\dot{\chi} \left[\frac{\partial}{\partial\theta} + \varepsilon \dot{q} s \frac{\partial}{\partial\phi} + \cdots\right] , \\ E &= h\dot{I} \left[\frac{\partial}{\partial\theta} + \varepsilon \dot{q} s \frac{\partial}{\partial\phi} + \cdots\right] - \frac{P_{\shortparallel}'}{\dot{\chi}} \frac{\partial}{\partial\phi} \end{aligned} \tag{5.3}$$

In lowest order one obtains from equs. (4.12), (4.13)

$$(D_o\sigma\vec{B}\cdot\vec{B} + a_1)X_{-1} + (D_o\sigma\vec{B}\cdot\vec{K}D_o + b_1D_o + D_o\tau_2 + \tau_3)y_{-1}$$

$$= (D_o\,\vec{B}\cdot\vec{B} + s_2)\sigma_{-1} \,, \tag{5.4}$$

$$(D_o\sigma\vec{B}\cdot\vec{K} + a_2)\,X_{-1} + (D_o\sigma\vec{K}\cdot\vec{K}\,D_o + b_2D_o + D_o\tau_4 + \tau_5)\,y_{-1}$$

$$= (D_o\vec{B}\cdot\vec{K} + s_3)\,\sigma_{-1} \,,$$

where $D_o = h\dot{\chi}\frac{\partial}{\partial\theta}$. Equation (4.24) yields $D_o\tilde{g}^{\pm}_{-1} = 0$ so that $\tilde{g}^{\pm}_{-1} = \tilde{g}^{\pm}_{-1}(V,\phi)$. This function is determined by going to the next order in ε, some details of which will be touched on in Section 6. Here, it suffices to know that following the precedure described in the last section $\sigma_{-1}(y_{-1})$ is eventually determined as a convolution over y_{-1} (although no explicit representation of the kernel exists in general).

As a result, equs. (5.4) are a coupled system of integro-differential equations for X_{-1} and y_{-1} with a free parameter w which enters the equations only because $\sigma_{-1} = \sigma_{-1}(w,y_{-1})$. As in the theory of differential equations with periodic coefficients (KAMKE, 1977), it may be shown that the periodicity condition in θ for X_{-1} and y_{-1} here also fixes the eigenvalues $w_{n\sigma}$ and eigenfunctions $X_{-1} = X_{n\sigma}(\theta)$, $y_{-1} = y_{n\sigma}(\theta)$. This spectrum does not exist in MHD where $\sigma^{(1)} \equiv 0$. Its eigensolutions have nothing in common with MERCIER-type modes where eigenvalues are determined by a radial matching procedure (PAO (1974), (1975)). Here, the $w_{n\sigma}$ spectrum will not be considered any further.

In order to recover MERCIER-type modes, it is obviously necessary that w should not enter at this stage so that it is not "prematurely" fixed. As a first requirement, this restricts one to modes y_{-1} and equilibria such that $\sigma_{-1} = 0$. The only periodic solution of the resulting differential equations (5.4) with periodic coefficients is, however, in general, the trivial solution $X_{-1} = y_{-1} = 0$. To dominant order, this is a degenerate mode with displacements only along $\vec{B}$, $x_{-1} \neq 0$, but not a MERCIER mode with $y_{-1} \neq 0$.

In order to obtain MERCIER modes, as a second condition, the coefficients of y_{-1} in equs. (5.4) have to vanish. This implies

$$\vec{T} = \left[\nabla\sigma \times \nabla B\right] = 0 \tag{5.5}$$

so that $\tau_i = 0$, $i = 1, \cdot\cdot 5$. In this case the "trivial" periodic solution is

$$X_{-1} \equiv h\dot{\chi} \frac{\partial x_{-1}}{\partial\theta} - (\mathrm{div}\ \vec{\xi})_{-1} = \frac{\partial y_{-1}}{\partial\theta} = 0 \ , \tag{5.6}$$

which allows x_{-1} , $y_{-1} \neq 0$ and is the analogue of the MHD case for which $\partial x_{-1}/\partial\theta = (\mathrm{div}\ \vec{\xi})_{-1} = \partial y_{-1}/\partial\theta = 0$ (MERCIER and LUC, 1974).

Equation (5.5) implies the requirement $\sigma(V,B) = \sigma(B)$ and is equivalent to

$$P_{\shortparallel}(V,B) = P_1(V) + P_2(B) \tag{5.7}$$

with arbitrary functions P_1 and P_2.

The pressure balance becomes $\left[\vec{K} \times \vec{B}\right] = \nabla P_1$, which is completely analogous to the MHD equation. This correspondence together with $\sigma_{-1} = 0$ makes plausible the existence of MHD-like eigenmodes in this case.

On the other hand, for general $\sigma = \sigma(V,B)$, the plasma anisotropy seems to make the plasma so "stiff" that either σ_{-1} is nonzero or the frequency of the eigenmodes is higher, $\omega = 0(\varepsilon^{o})$, or the anisotropy does not allow x, y, ξ^{1} to be of different order, in contrast to the localization assumption x, y $= 0(\varepsilon^{-1})$, $\xi^{1} = 0(\varepsilon^{o})$.

For the present purposes this restricts one in the following to equilibria with $\sigma = \sigma(B)$. In order to cope more easily with the requirement $\sigma_{-1} = 0$, we shall, in addition, restrict ourselves to axisymmetric (tokamak) equilibria which are symmetric with respect to the equatorial plane.

6. Modes in axisymmetric $\sigma(B)$ equilibria

For axisymmetric equilibria ϕ is an ignorable coordinate and the ansatz $\sim e^{2\pi in\phi}$ for the linearized quantities can be made. Equations (4.24) can then be written as

$$\frac{\partial \tilde{g}^{\pm}}{\partial\theta} + \varepsilon\lambda g = \varepsilon\, G^{\pm} ,$$

$$\lambda = \lambda_o + \varepsilon\lambda_1 + \cdots = (2\pi n\dot{q}s - \frac{wB}{v\dot{\chi}})\, i + \varepsilon i\pi n\ddot{q}s^2 + \cdots , \qquad (6.1)$$

$$G^{\pm} = \frac{iwB}{v\dot{\chi}}\, \psi^{\pm}\, \frac{\partial \tilde{F}^{\pm}}{\partial\varepsilon}$$

$$= \frac{1}{\varepsilon}\, G^{\pm}_{-1} + G^{\pm}_{o} + \cdots .$$

In order to determine $\tilde{g}^{\pm}_{-1}$, $\tilde{g}^{\pm}_{o}$, etc., a distinction between trapped and untrapped particles has to be made. Boundary conditions for untrapped particles are $\tilde{g}^{\pm}(\theta) = \tilde{g}^{\pm}(\theta + N)$, while for trapped particles $\tilde{g}^{\pm}(\theta_i$, sign v) = $\tilde{g}^{\pm}(\theta_i$, - sign v) states that no particles should be lost at the turning points θ_i , i = 1,2 . (Generalization to more than one pair of turning points is obvious.) From $v^2 = 2[\varepsilon - \mu B(\theta)]$ there follow the domains (I) untrapped: $0 < \mu < \varepsilon/B_{max}$, and (II) trapped: $\varepsilon/B_{max} < \mu < \varepsilon/B$, where B_{max} is the maximum of $B(\theta)$. The turning points are determined by $\mu B(\theta_i) = \varepsilon$.

One easily obtains for untrapped and trapped particles

$$\tilde{g}^{\pm}_{-1u} = \frac{\langle G^{\pm}_{-1}\rangle}{\langle\lambda_o\rangle} \; ; \; \tilde{g}^{\pm}_{-1t} = \frac{\langle G^{\pm}_{-1}\rangle_t}{\langle\lambda_v\rangle_t} , \qquad (6.2)$$

where

$$\langle A\rangle = \frac{1}{N}\int_o^N d\theta\; A(\theta) , \qquad \langle A\rangle_t = \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} d\theta A(\theta) \qquad (6.3)$$

and $\lambda_v = \frac{-iwB}{v\dot{\chi}}$.

From the assumed up/down symmetry of the equilibria it follows that terms with y_{-1} vanish identically in $<G^{\pm}_{-1}>$ and $<G^{\pm}_{-1}>_t$. This follows from equs. (4.23) to (4.26) and the fact that $\vec{\kappa}\cdot\vec{K}$, $\vec{K}\cdot\nabla B$ and $\vec{B}\cdot\nabla B$ are antisymmetric, while $\vec{K}\cdot\vec{B}$ and y_{-1} are symmetric. (y_{-1} is constant.) The integral equations for $p_{\perp,-1}$ and Φ_{-1} , obtained by inserting equs. (6.2) into equs. (4.30), (4.31) are therefore purely homogeneous. The trivial solution $p_{\perp,-1} = \Phi_{-1} = 0$ is acceptable (at least for $\beta << 1$, $s \neq 0$ it is even the only solution; see Appendix A) and agrees with the MHD ordering. The result so far is then $X_{-1} = \partial y_{-1}/\partial\theta = p_{\parallel,-1} = p_{\perp,-1} = \Phi_{-1} = \sigma_{-1} = 0$.

From equs. (4.11), (4.8) one obtains

$$\frac{\partial\xi_o}{\partial s} = -Ey_{-1} = \frac{\dot{P}_1}{\dot{\chi}}\frac{\partial y_{-1}}{\partial\phi} \qquad (6.4)$$

and $\partial p^*_o/\partial s = 0$. This implies that $\partial\xi_o/\partial\theta = p^*_o = 0$.

The next order in ε yields

$$\frac{\partial p^*_1}{\partial s} = (D_o\sigma B_1 + a)\,X_o + (D_o\sigma K_1 + b)\,(Dy)_o + A\xi_o + (D_oB_1 + s_1)\,\sigma_o \; . \qquad (6.5)$$

Averaging yields

$$\frac{\partial <p^*_1>}{\partial s} = <aX_o> + <b(Dy)_o> + <A>\xi_o + <s_1\sigma_o> \; . \qquad (6.6)$$

Here one has

$$(Dy)_o = \dot{\chi}\left(\frac{\partial y_o}{\partial\theta} + \dot{q}s\,\frac{\partial y_{-1}}{\partial\phi}\right) \tag{6.7}$$

so that

$$\dot{\chi}\dot{q}s\,\frac{\partial y_{-1}}{\partial\phi} = \langle (Dy)_o\rangle \ . \tag{6.8}$$

As "basic" quantities we consider ξ_o and $\langle p_1^*\rangle$, in terms of which X_o , $(Dy)_o$ and σ_o have to be determined. $\partial y_{-1}/\partial\phi$ then follows from equ. (6.8). To $O(\varepsilon^o)$ the equs. (4.12), (4.13) are

$$\begin{aligned}
&(D_o\sigma\vec{B}\cdot\vec{B} + a_1)\,X_o + (D_o\sigma\vec{B}\cdot\vec{K} + b_1)\,(Dy)_o = \\
&= (D_o a + c_2)\,\xi_o + (-D_o\vec{B}\cdot\vec{B} + s_2)\,\sigma_o \ , \\
&(D_o\sigma\vec{B}\cdot\vec{K} + a_2)\,X_o + (D_o\sigma\vec{K}\cdot\vec{K} + b_2)\,(Dy)_o = \\
&= (D_o b + c_4)\,\xi_o + (-\,D_o\vec{B}\cdot\vec{K} + s_3)\,\sigma_o \ .
\end{aligned} \tag{6.9}$$

In order to determine σ_o equ. (6.1) has to be solved to $O(\varepsilon^o)$. With $\langle G^{\pm}_{-1}\rangle = \langle G^{\pm}_{-1}\rangle_t = 0$ one gets

$$\begin{aligned}
\tilde{g}^{\pm}_{ou} &= \int_o^{\theta} d\theta' G^{\pm\,'}_{-1} - \frac{1}{\langle\lambda_o\rangle}\left(\langle\lambda_o \int_o^{\theta} d\theta' G^{\pm\,'}_{-1}\rangle - \langle G^{\pm}_o\rangle\right) \ , \\
\tilde{g}^{\pm}_{ot} &= \int_{\theta_1}^{\theta} d\theta' G^{\pm\,'}_{-1} - \frac{1}{\langle\lambda_v\rangle_t}\left(\langle\lambda_q \int_{\theta_1}^{\theta} d\theta' G^{\pm\,'}_{-1}\rangle_t - \langle G^{\pm}_o\rangle_t\right) \ ,
\end{aligned} \tag{6.10}$$

where $\lambda_q = 2\pi in\dot{q}s$. The first terms do not contribute to density and pressure moments, respectively, since they are antisymmetric in sign v. When this is inserted into the quasineutrality condition (4.30) and the equations (4.31) for $p_{\parallel}$, $p_{\perp}$, the following coupled systems of integral equations for ϕ_o, $p_{\perp o}$ is obtained, together with an equation for $p_{\parallel o}$:

$$\Phi_o \int_o^\infty d\varepsilon \int_o^{\varepsilon/B} d\mu \, \frac{1}{v} \sum_{+,-} \left(\frac{e}{m}\right)^{2\pm} \frac{\partial F^\pm}{\partial \varepsilon} = \int_o^\infty d\varepsilon \int_o^{\varepsilon/B_{max}} d\mu \, \frac{1}{v} \frac{\Omega}{<\frac{B}{v}>} \sum_{+,-} \left(\frac{e}{m}\right)^{\pm} \frac{\partial F^\pm}{\partial \varepsilon}$$

$$\cdot \left\{ <\frac{B}{v}\Psi_o^\pm> + 2\dot{q}s \frac{\partial}{\partial \phi} \left(\frac{1}{<\frac{B}{v}>} <\frac{B}{v} \int_o^\theta d\theta' \, \frac{B'}{v'} \Psi'_{-1}> \; - < \int_o^\theta d\theta' \frac{B'}{v'} \Psi'_{-1} > \right) \right\}$$

$$+ \int_o^\infty d\varepsilon \int_{\varepsilon/B_{max}}^{\varepsilon/B} d\mu \, \frac{1}{v} \frac{1}{<\frac{B}{v}>_t} \sum_{+,-} \left(\frac{e}{m}\right)^{\pm} \frac{\partial F^\pm}{\partial \varepsilon} \left\{ <\frac{B}{v}\Psi_o^\pm>_t - 2\dot{q}s \frac{\partial}{\partial \phi} < \int_{\theta_1}^{\theta} d\theta' \frac{B'}{v'} \Psi'_{-1}>_t \right\} \cdot$$

(6.11)

$$\begin{bmatrix} p_{\shortparallel o} \\ \\ p_{\perp o} \end{bmatrix} = -\xi_o \begin{bmatrix} P'_{\shortparallel} \\ \\ P'_{\perp} \end{bmatrix} - \frac{p_{\perp o}}{B^2} \begin{bmatrix} \partial P_{\shortparallel}/\partial B \\ \\ \partial P_{\perp}/\partial B \end{bmatrix} - 2 \int_o^\infty d\varepsilon \int_o^{\varepsilon/B_{max}} d\mu$$

$$\cdot \frac{1}{v} \begin{bmatrix} v^2 \\ \\ \mu B \end{bmatrix} \cdot \frac{\Omega}{<\frac{B}{v}>} \sum_{+,-} \frac{\partial F^\pm}{\partial \varepsilon} \{ \cdot \cdot \cdot \}$$

$$- 2 \int_o^\infty d\varepsilon \int_{\varepsilon/B_{max}}^{\varepsilon/B} d\mu \, \frac{1}{v} \begin{bmatrix} v^2 \\ \\ \mu B \end{bmatrix} \frac{1}{<\frac{B}{v}>_t} \sum_{+,-} \frac{\partial F^\pm}{\partial \varepsilon} \{\cdots\} ,$$

where the brackets $\{\cdots\}$ in the second equation are identical with those in the first one, and

$$\Omega = \frac{w^2 <\frac{B}{v}>^2}{w^2 <\frac{B}{v}>^2 - (2\pi n \dot{\chi} \dot{q} s)^2} . \qquad (6.12)$$

Ψ_{-1} and $\Psi_o^\pm$ are defined in equs. (4.23 to 4.26) as functions of y_{-1} and ξ_o, y_o, Φ_o, $p_{\perp o}$, respectively.

The integral equations only depend on w^2, as do the macroscopic equations (4.8) to (4.13). For w real and $\dot{q} \neq 0$, Ω may diverge, which corresponds to the wave-particle resonance at $(i\omega - v \frac{\vec{B}\cdot\nabla}{B}) \tilde{f}^{\pm} = 0$. For unstable modes, $w^2 < 0$, on the other hand, Ω is finite.

The existence and properties of solutions of the integral equations are discussed in Appendix A. It is obvious, however, that, in general, no explicit solution may be given. This is also valid for $w^2 \to 0$ since the trapped-particle term is independent of w^2. Furthermore, even if σ_o were a given function no explicit solution of the coupled first-order system of differential equations (6.9) for $X_o, (Dy)_o$ is available in general.

There is an additional complication which will become apparent in the next section: The $O(\varepsilon^1)$ equations which have to be taken into account introduce coupling to σ_1, y_1, etc. It is far from obvious how the coupling (ultimately between all orders) may be truncated.

One is thus forced to the conclusion that a further analytic discussion of stability with respect to the localized modes is not possible in general. There is one particular case, however, for which an explicit stability criterion can be derived. This case will be investigated in the next section.

7. Stability criterion for special σ = const equilibria

Let us consider plasmas in which the number of trapped particles (at the resonant surface) may be neglected. In order to obtain criteria for marginal stability, we consider $0 > w^2 \to 0$. With $\Omega \to 0$ the solution to equs. (6.11), (6.12) becomes trivial:

$$p_{\perp o} = -\frac{1}{\Gamma}\xi_o P_\perp' = \frac{-1}{\Gamma}\vec{\xi}_o\cdot\nabla P_1 ,$$
$$p_{\shortparallel o} = \Gamma p_{\perp o} , \qquad \Phi_o = 0 , \tag{7.1}$$

with $\Gamma = 1 + \frac{1}{B}\frac{dP_\perp}{dB}$. ($\Gamma > 0$ is required to avoid mirror instability (KADISH, 1966).) With equs. (4.15), (5.7) σ_o may be expressed as

$$\sigma_o = -\frac{p_{\perp o}}{B}\frac{d\sigma}{dB} . \tag{7.2}$$

The differential equations (6.9) for X_o, $(Dy)_o$ may be explicitely integrated in the case $a_1 = a_2 = b_1 = b_2 = 0$. According to equs. (4.14) this requires

$$\sigma = \text{const} \tag{7.3}$$

(provided $\vec{B}\cdot\nabla B \neq 0$, $\vec{K}\cdot\nabla B \neq 0$), i.e. if the plasma anisotropy is constant on the rational surface. This will be assumed in the following. Equation (7.3) implies that

$$P_\shortparallel = P_1(r^1) - \frac{c}{2}B^2 , \qquad P_\perp = P_1(r^1) + \frac{c}{2}B^2 ,$$
$$\sigma = \Gamma = 1 + c , \tag{7.4}$$

where c is an arbitrary constant, provided $c > -1$, $|c| \lesssim 2P_1/B^2$, in order to satisfy $\sigma > 0$, $P_{\shortparallel,\perp} \gtrsim 0$. Equation (7.3) also implies that

$c_i = 0$, $i = 1, \cdot\cdot 5$, and $\sigma_o = 0$ even although the equilibrium is allowed to be anisotropic.

One finally obtains

$$\begin{aligned} &\sigma \vec{B}\cdot\vec{B}\, X_o + \sigma\, \vec{B}\cdot\vec{K}\, (Dy)_o - a\, \xi_o = C_1 \;, \\ &\sigma \vec{B}\cdot\vec{K}\, X_o + \sigma\, \vec{K}\cdot\vec{K}\, (Dy)_o - b\, \xi_o = C_2 \;, \end{aligned} \tag{7.5}$$

where the constants C_1, C_2 have to be determined from the next order in ε. From equs. (4.12), (4.13) one obtains to $O(\varepsilon^1)$

$$\begin{aligned} &\dot{\chi}\dot{q}s \frac{\partial}{\partial\phi} \langle \sigma\vec{B}\cdot\vec{B}\, X_o + \sigma\vec{B}\cdot\vec{K}\, (Dy)_o - a\, \xi_o \rangle = \langle s_2\sigma_1 \rangle \;, \\ &\dot{\chi}\dot{q}s \frac{\partial}{\partial\phi} \langle \sigma\vec{B}\cdot\vec{K}\, X_o + \sigma\vec{K}\cdot\vec{K}\, (Dy)_o - b\, \xi_o \rangle = \langle s_3\sigma_1 \rangle + \\ &- \frac{\dot{P}_1}{\dot{\chi}} \frac{\partial}{\partial\phi} \langle p_1^* \rangle \;, \end{aligned} \tag{7.6}$$

where most $O(\varepsilon^1)$ quantities have disappeared by the averaging, but not σ_1. In Appendix B, however, it is shown that σ_1 is up-down symmetric, and since s_2, and s_3 are antisymmetric, the terms $\langle s_2\sigma_1 \rangle$, $\langle s_3\sigma_1 \rangle$ vanish. (Note that in the more general case, $\sigma = \sigma(B)$, more $O(\varepsilon^1)$ quantities such as $\langle a_1X_1 \rangle$ etc. would remain in the equations.)

Comparison of equs. (7.6) with equs. (7.5), averaged over θ, determines

$$C_1 = 0 \;, \quad C_2 = - \frac{\dot{P}_1}{\dot{\chi}\dot{q}s} \langle p_1^* \rangle \,. \tag{7.7}$$

The determinant d of equs. (7.5) is

$$d = \vec{B}\cdot\vec{B}\; \vec{K}\cdot\vec{K} - (\vec{B}\cdot\vec{K})^2 = [\vec{K} \times \vec{B}\,]^2 = (\nabla P_1)^2 \,. \tag{7.8}$$

When the solutions X_o, $(Dy)_o$ are inserted into equs. (6.8), (6.6), (6.4) one finally obtains

$$s^2 \frac{d\xi_o}{ds} = c_{11}\, s\, \xi_o + c_{12} < p_1^* > ,$$

$$s \frac{d<p_1^*>}{ds} = c_{21}\, s\, \xi_o + c_{22} < p_1^* > , \tag{7.9}$$

i.e. $s^2 d^2 \xi_o / ds^2 + 2s\, d\xi_o/ds + M\, \xi_o = 0$ with $M = c_{11}c_{22} - c_{12}c_{21} - c_{11}$.

Here

$$c_{11} = \lambda < \frac{\vec{B}\cdot\vec{Q}}{\sigma d} > , \quad c_{12} = -\lambda^2 < \frac{\vec{B}\cdot\vec{B}}{\sigma d} > ,$$

$$c_{21} = < \frac{\vec{Q}\cdot\vec{Q}}{\sigma d} > + < A > , \quad c_{22} = - c_{11} \tag{7.10}$$

and

$$\lambda = - \frac{\dot{P}_1}{\dot{\chi}\dot{q}^2} , \quad \vec{Q} = a\, \vec{K} + b\, \vec{B} ,$$

$$< A > = < -\sigma g_{mn}\, \dot{B}^m \dot{B}^n + \sigma\, \dot{B}^m \left(\frac{\partial B_m}{\partial r^1} - \frac{\partial B_1}{\partial r^m}\right) . \tag{7.11}$$

The ansatz $\xi_o \sim s^\nu$ yields $\nu = \frac{1}{2}(-1 \pm \sqrt{1-4M})$. The condition

$$1 - 4M > 0 \tag{7.12}$$

ensures that the solution is nonoscillating and serves as a necessary criterion for stability. (See PAO (1974) for a discussion of the matching procedure across the singular layer.)

It is straightforward to express the covariant and contravariant vector components in the criterion by means of quantities involving

$\vec{B}^2$, $\vec{B}\cdot\vec{J}$, $(\vec{K} = \sigma\vec{J})$, and covariant field line curvature κ_1. As a result, the stability condition is

$$\frac{\dot{\chi}^4}{4<S>}\left(\dot{q} + \frac{2}{\dot{\chi}^2}<R\cdot S>\right)^2 - 2\kappa_1 \frac{\dot{P}_1}{\sigma} - \dot{q}\dot{\chi}^2<R> - <R^2 S> \; > 0 \; , \qquad (7.13)$$

where $R = \vec{J}\cdot\vec{B}/B^2$ and $S = B^2/(\nabla V)^2$. This is identical with MERCIER's criterion, except that the hydrodynamic pressure P(V) is replaced by an effective pressure $P_o(V)$,

$$P_o = \frac{1}{\sigma} P_1(V) \; , \qquad (7.14)$$

where P_1 was defined in equs. (5.7), (7.4).

A posteriori, this simple result is not surprizing: For σ = const the anisotropic equilibria are identical to MHD equilibria with the same magnetic field configuration but with P replaced by P_o. In the linearized equations the plasma motion behaves as if it were isotropic, $\sigma_{-1} = \sigma_o = 0$, with no influence of $\sigma_1 \neq 0$, $p_{\shortparallel o} \neq p_{\perp o}$. In spite of the fact that, in general, $(\text{div}\,\vec{\xi}_{-1}) \neq 0$, unlike in MHD, its actual value does not matter either, because only the combination $X = Dx - \text{div}\,\vec{\xi}$ enters into the derivation of the criterion.

From equ. (7.4) one can deduce which direction of plasma anisotropy is favourable. Let us compare plasmas which are marginal with respect to the criterion (7.13), i.e. have the same $P_{o\,marg}$ (and the same field configuration). The plasmas, however, may differ in their $P_\shortparallel/P_\perp$ ratio. If we introduce $\beta = 2(P_\shortparallel + 2P_\perp)/(3B^2)$ as a figure of merit, then

$$\beta = \frac{2P_o}{B^2}\left(1 + c + c\,\frac{B^2}{6P_o}\right) \; . \qquad (7.15)$$

This shows that among all plasmas with the same P_o , those with $c > 0$, i.e. $P_{\parallel} < P_{\perp}$, are better, within the framework of the present theory.

8. Conclusions

The guiding centre eigenmode equations for toroidal plasmas are investigated. Eigenmodes which are localized around a mode-rational surface and are almost constant along $\vec{B}$ (MERCIER-type modes) are shown not to exist for general anisotropic equilibria with $\sigma \equiv 1 - (P_{\shortparallel} - P_{\perp})/B^2 = \sigma(V,B)$ (V is the volume inside a flux surface). Such eigenmodes do exist for $\sigma = \sigma(B)$. No closed stability condition is obtained, however, because the mathematical problems involved do not have explicit solutions and because no obvious truncation procedure is available for the hierarchy of equations which result from expansion in the localization parameter $\varepsilon << 1$.

In the rather special case when σ = const in the localization region and when trapped particles may be neglected the difficulties may be overcome and a necessary stability condition for axisymmetric equilibria is obtained. It is identical to MERCIER's criterion, except that the scalar pressure is replaced by an effective pressure $P_o(V) = (P_{\shortparallel} + P_{\perp})/ /2\sigma)$. With the above-mentioned restrictions this implies, for example, that marginally MERCIER stable MHD equilibria can be loaded with higher plasma-β if $P_{\perp} > P_{\shortparallel}$ and still be marginal according to guiding centre theory.

The results obtained here do not exclude the existence of MERCIER-type stability criteria for general $\sigma(V,B)$ equilibria. Properly tailored test modes, when applied to guiding centre energy principles, could yield the desired result. These modes would not, however, be eigenmodes of the system.

Appendix A

The integral equations (6.11) for Φ_o, $p_{\perp o}$ are of the form

$$\phi(\theta) = \lambda \oint d\theta' \, K(\theta,\theta')\phi(\theta') + r(\theta) \ . \tag{A1}$$

with $\lambda = 1$ formally. Inspection shows that $K(\theta,\theta')$ diverges logarithmically at $\theta = \theta'$ but, with $w^2 < 0$

$$M^2 = \oint\oint d\theta d\theta' \ |K(\theta,\theta')|^2 < \infty \tag{A2}$$

is finite. For Φ_o and $p_{\perp o}$ M is $0(1)$ and $0(\beta = \frac{P}{B^2})$, respectively. Since for all eigenvalues λ_i one has

$$|\lambda_i| \geq \frac{1}{M} \tag{A3}$$

(MICHLIN, 1962), it follows that at least for $\beta << 1$ the equation for $p_{\perp o}$ always has a unique solution, which may be obtained, for instance, recursively as a NEUMANN series.

The equation for Φ_o, however, has an eigenvalue $\lambda_{i_o} = 1$ with eigenfunction $\Phi_{oo}(\theta) = \text{const}$, at $s = 0$. Since, if (A2) is satisfied, eigenvalues do not accumulate, there is always a finite region around, and excluding, $s = 0$ where the equation for Φ_o also has a unique solution. At $s = 0$ existence of an inhomogeneous solution requires that the inhomogeneous term be orthogonal to Φ_{oo}. This yields a side condition on $y_o(\theta, s = 0)$ of the form

$$\oint d\theta \, c_1(\theta) \, y_o(\theta) = \xi_o \ , \tag{A4}$$

where $c_1(\theta)$ is antisymmetric.

Appendix B

Equations (7.5) show that X_o, $(Dy)_o$ are up-down symmetric. According to equ. (6.5) p_1^* is also symmetric, provided that B_1, K_1 are antisymmetric. The latter may be proved by, for example, using the explicit representation of HAMADA coordinates, available for axisymmetric equilibria (LORTZ and NÜHRENBERG, 1974). Therefore p_1^* does not contribute to σ_1 in $< s_{2,3}\sigma_1 >$. The same holds for ξ_1, which according to equ. (7.2), disappears from σ_1 if σ = const. (The adiabatic ξ^1 terms behave identically in all orders.)

For free particles the non-adiabatic term of $O(\varepsilon^1)$ is governed by

$$\tilde{g}_1^{\pm} = - \int_o^{\theta} d\theta' \lambda_o' \tilde{g}_o'^{\pm} + \int_o^{\theta} d\theta' G_o'^{\pm} + c_1 \tag{B1}$$

where c_1 is constant and $\lambda_o, \tilde{g}_o^{\pm}$, $G_o^{\pm}$ are defined in equs. (6.1), (6.10). It follows that for $\omega^2 \to 0$ terms which contribute to even moments of $\tilde{g}_1^{\pm}$ are symmetric. This completes the proof that for the conditions of Section 7 $< s_{2,3}\sigma_1 > = 0$ is valid.

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References

Aleksin V. and Yashin V. (1961) JETP 13, 787.

Andreoletti J. (1963) Comptes Rendus Acad.Sc. 256, 251.

Choe J., Tataronis J.A. and Grossman W. (1977) Plasma Physics 19, 117.

Grad H. (1961) Electromagnetics and Fluid Dynamics of Gaseous Plasma (Proc. Symposium), Polytechnic Press, Brooklyn, New York.

Greene J.M. and Johnson J.L. (1962) Phys. Fluids 5, 510.

Kadish A. (1966) Phys. Fluids 9, 514.

Kamke E. (1977) Differentialgleichungen, I., G.B. Teubner, Stuttgart.

Kruskal M.D. and Oberman C.R. (1958) Phys. Fluids 1, 275.

Laugwitz D. (1960) Differentialgeometrie, B.G. Teubner Verlagsgesellschaft, Stuttgart.

Lortz D. and Nührenberg J. (1974), 5th Int. Congr. on Plasma Phys. and Contr. Nucl. Fus. Research, IAEA, Vienna, 1975.

Mercier C. (1960) Nucl. Fusion 1, 47.

Mercier C. and Luc H. (1974) Lectures in Plasma Physics: The Magnetohydrodynamic Approach to the Problem of Plasma Confinement in Closed Magnetic Configurations (Commission of the European Communities, Luxembourg).

Michlin S.G. (1962) Vorlesungen über lineare Integralgleichungen, VEB Deutscher Verlag der Wissenschaften, Berlin.

Northrop T.G. and Whiteman K.J. (1964) Phys. Rev. Lett. 12, 639.

Pao Y. (1974) Nucl. Fusion 14, 25.

Pao Y. (1975) Phys. Fluids 18, 1192.

Spies G.O. and Nelson D.B. (1974) Phys. Fluids 17, 1879.

Suydam R. (1958) 2nd. Int. Conf. Peaceful Uses At. Energy (Proc. Conf. Geneva), 31, 157.

Vahala G. and Vahala L. (1977) J. Plasma Phys. 18, 317.

Weitzner H. (1976) Phys. Fluids 19, 420.