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STOCHASTIC STABILITY OF MHD EQUILIBRIA

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Abstract

The stochastic stability of stationary magnetohydrodynamic or plasma equilibria under the influence of stationary random fluctuations is studied using the direct Liapunov method. The destabilizing effect of fluctuations is demonstrated.

The stability of deterministic stationary magneto-hydrodynamic systems was intensively studied in a Lagrangian formulation (e.g. /1, 2/). Recently, it was demonstrated (/3-8/) that for a large class of such systems the linearised equations of motion in the Lagrangian framework take the form

$$A \ddot{\xi}(t) + (B+C) \dot{\xi}(t) + D \xi(t) = 0 , \quad t \geq 0 , \qquad (1)$$

where $\xi(t)$ is a generalized Lagrangian variable expressing the deviation from the stationary equilibrium state $\xi(0) = 0$ and A, B, C and D are time-independent operators. The n-dimensional complex vector $\xi(t)$ is defined in a complex Hilbert space L_2, the parameter $t \in T$. Sufficient and necessary conditions for stability of system (1) have been established, e.g. /3, 4, 5, 6, 7, 8/. Let us reformulate these results. Let us assume A to be a positive definite Hermitian operator having a left inverse. The system (1) can then be written in a canonical form

$$\dot{\zeta}(t) = Q \zeta(t) \quad , \quad \zeta = (\xi , \dot{\xi})^T \qquad (2)$$

where $Q = \begin{pmatrix} O & U \\ -A^{-1}D & -A^{-1}(B+C) \end{pmatrix}$. U is the unit operator.
The null solution $\zeta(0) = 0$ of (2) is said to be stable in the sense of Liapunov if and only if for the Liapunov

functional $V_D(\xi, t)$, positive definite and satisfying the conditions: $V_D(0,t) = 0$, $V_D(\xi,t)$ being continuous in ξ and t and possessing first derivatives with respect to ξ and t, the time derivative along trajectories (2) reads (e.g. /10/)

$$\frac{d}{dt} V_D(\xi,t) \leq 0 \quad . \tag{3}$$

Let us take for $V_D(\xi,t)$ the quadratic functional $(\xi, S\xi)$, where S is a Hermitian operator, e.g.
$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, $S_1 = D$, $S_2 = A$. Then, if C is anti-Hermitian, one has

$$V_D(\xi,t) = (\xi, D\xi) + (\dot{\xi}, A\dot{\xi});$$

$$\frac{d}{dt} V_D(\xi,t) = -\left[(B\dot{\xi}, \dot{\xi}) + (\dot{\xi}, B\dot{\xi})\right]. \tag{4}$$

Thus for A positive definite and Hermitian, C anti--Hermitian, the system (2) is stable if D is a Hermitian positive definite operator and B is Hermitian and positive semi-definite. If B is positive definite, the system (2) is asymptotically stable.

Let us now assume that the stable deterministic magnetohydrodynamic (or plasma) system (1) is perturbed by a stochastic variation /12/ of one or more parameters, such as density or magnetostatic field, or that in a system (1) are excited electromagnetic fluctuations due to turbulence. For simplicity, let us suppose that the fluctuations represent a stationary random process, normally distributed with zero mean and constant spectral density. In this case, the system (2) is replaced by the following Itô equation /9/:

$$d\xi(t) = Q\xi(t)dt + G(\xi, t)dW_t, \quad \xi(0) = 0, \; t \geq t_o \; . \tag{5}$$

Here W_t is the n dimensional Wiener process. The non--anticipating operator $G(\xi, t)$ contains the intensity parameters σ_j of the Wiener process. The space $L_2 = L_2(\Omega, \sigma, P)$ is now a complete linear normed space of random complex-valued vectors $\xi(t)$ on the probability space $\{\Omega, \sigma, P\}$ satisfying $E|\xi|^2 < \infty$.

The corresponding inner product is defined as $E(\xi, \eta) = E\int_L d\tau < \tilde{\xi}^* \cdot \tilde{\eta} >$, where integration is performed over the space L in L_2 occupied by the fluid. E is the expectation operator, $t \in T$, T being a linear index set. The stability of the process (5) may be studied similarly to the deterministic case with the help of a stochastic Liapunov functional. We limit the discussion here to the definition of the stability in probability /9, 10/. The equilibrium $\xi_o(t_o)$ is said to be stable in probability if for a given $\varepsilon, \varepsilon' > 0$

there exists a $\delta(\varepsilon, \varepsilon', t_o)$ such that $\| \xi_o \| < \delta$ implies

$$P \left\{ \sup_{t_o \leq t \leq \infty} | \xi(t, \xi_o, t_o) | > \varepsilon' \right\} < \varepsilon \qquad (6)$$

Let us define the stochastic Liapunov functional $V_s(\xi, t)$, continuous, positive definite and twice continuously differentiable with compact support. Then /9, 10/ the condition (6) is satisfied if

$$E \left[\frac{d}{dt} V_s(\xi, t) \right] \leq 0, \qquad (7)$$

i.e. if $V_s(\xi, t)$ is a positive supermartingale. Taking again for the process $V_s(\xi, t)$ the form $V_s(\xi, t) = (\xi, S\xi)$ we obtain using the Itô's formula /12/

$$\frac{d}{dt} V_s(\xi, t) = (Q\xi, S\xi) + (\xi, SQ\xi) + \\ + \frac{1}{2} \int_L d\tau \sum_{\alpha, \beta, \gamma = 1}^{n} G_{\alpha\gamma}(\xi) G_{\beta\gamma}(\xi) \frac{\partial}{\partial \xi_\alpha} \frac{\partial}{\partial \xi_\beta} \langle \xi, S\xi \rangle . \qquad (8)$$

The relation (8) may be simplified in the case of the scalar Wiener process $W_t^{(1)}$. Then the second term on the R.H.S. of eq. (5) becomes $G(\xi, t) dW_t = R\xi(t) dW_t^{(1)}$.

Defining the operator S in the same way as in the deterministic case and assuming that A is a diagonal matrix, we have for real ξ :

$$\frac{d}{dt} V_s(\xi, t) = -2 \; (\dot{\xi}, B\dot{\xi}) +$$

$$+ \; \sigma^2 \; ([R_1\xi + R_2\dot{\xi}], A[R_1\xi + R_2\dot{\xi}]). \qquad (9)$$

Here we have decomposed the operator R as $R = \sigma \begin{pmatrix} 0 & 0 \\ R_1 & R_2 \end{pmatrix}$.

The last term in the relation (9) is positive definite. Thus, the stability condition (7) is not generally satisfied for all t and all σ. The relation (9) illustrates the destabilizing effect of fluctuations in (5). The stability threshold

$$2 \, E \, \| (B\dot{\xi}(s), \dot{\xi}(s)) \| = \sigma^2 \, E \, \| ([R_1\xi(s) +$$

$$+ R_2\dot{\xi}(s)], A[R_1\xi(s) + R_2\dot{\xi}(s)]) \| \qquad (10)$$

may be used for defining the time interval $s-t_o$ during which the system (5) remains stable in probability. Using in eq. (10) the Schwarz inequality one has

$$\| B\dot{\xi}(s) \| \; \| \dot{\xi}(s) \| \geq \| (B\dot{\xi}(s), \dot{\xi}(s)) \|$$

$$\leq \frac{1}{2} \sigma^2 \| R_1\xi(s) + R_2\dot{\xi}(s) \|^2 \; \| A \| \qquad (11)$$

from which a rough estimate for s may be obtained for a particular MHD system with the help of dimensional analysis. For a bounded $V_s(\zeta, t) < m$, the estimate for the upper bound of the probability in eq. (6) is $\frac{1}{m} V_s(\zeta, t)$, /9/. Owing to the definition of the inner product, the conditions (3) and (7) define the stability in large. Eq. (9) also shows that equilibria of ideal nondissipative magnetohydrodynamics /11/ are always unstable against stochastic perturbations.

Application of the theory presented to dissipative MHD equilibria /13/ and discussion of the important problem of selfconsistency will be given in a forthcoming paper.

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