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ON THE STABILITY OF DISSIPATIVE
MHD EQUILIBRIA

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Abstract

The global stability of stationary equilibria of dissipative MHD is studied using the direct Liapunov method. Sufficient and necessary conditions for stability of the linearized Euler-Lagrangian system with the full dissipative operators are given. The case of the two-fluid isentropic flow is discussed.

There has been a considerable amount of work concerning the stability of static and stationary magnetohydrodynamic equilibria in the framework of the so-called Energy Principles based on the Lagrangian formulation [1, 2]. It was recently demonstrated for a large class of such equilibria that linearized equations of motion in the Euler-Lagrangian description take the form

$$A \frac{\partial^2}{\partial t^2} \xi(t) + [B+C] \frac{\partial}{\partial t} \xi(t) + [D+E] \xi(t) = 0, \quad t \geq 0, \quad (1)$$

where $\xi(t)$ is the n-dimensional vector of Lagrange displacements from the equilibrium state $\xi = 0$. The time independent operators A, B, C, D and E, the vector $\xi(t)$ and the corresponding inner product $\langle \xi, \eta \rangle = \int dV \sum_j \xi_j^* \eta_j$ are defined in the complex Hilbert space L_2 with the norm $\| \cdot \|$. The operators A, B and D are Hermitian and A is non-negative. The operators C and E are anti-Hermitian. The parameter $t \in T$.

Equation (1) encompasses several limiting cases. For $B = C = E = 0$ eq. (1) corresponds to the static equilibrium of ideal MHD [3] and for $B, C \neq 0$ and $E = 0$ we have the case with static equilibrium including viscosity and resistivity, e.g. [4, 5, 6, 7]. However, the most interesting case of stationary flow in the dissipative MHD is characterized by $E \neq 0$.

In the previous studies [3,4,5,6,7,8,9,10] necessary and sufficient conditions for exponential stability of the system (1) were given for $E = 0$. It was also demonstrated [10, 11] that in order to keep in eq. (1) all essential dissipative effects in the operators B and C and simultaneously neglect the operator E, one has to accept several restrictions on the validity of eq. (1).

In this note we demonstrate sufficient and necessary conditions for stability in the sense of Liapunov for the full system (1).

The stability of the system (1) may be studied using the Liapunov direct method. Let us denote [12] a positive definite Liapunov functional $V(\tilde{\xi}, t)$, continuously differentiable in t and $\tilde{\xi}$ and satisfying the following conditions:

$V(\tilde{\xi}, t)$ is defined in a set $\Omega, \Omega: \|\tilde{\xi}\| < a$ for all $t \geq 0$,
 $V(0, t) = 0$ for $t \geq 0$, $V(\tilde{\xi}, t)$ dominates a certain $V(\tilde{\xi})$,
 $V(\tilde{\xi}) \leq V(\tilde{\xi}, t)$ for all $\tilde{\xi}$ in Ω and all $t \geq 0$.

Moreover, it is assumed that $V(0, t) = 0$ for $t \geq 0$. Then if the time derivative of $V(\tilde{\xi}, t)$ taken along the trajectories (1)

is negative semi-definite, the null solution $\tilde{\xi} = 0$ of the system (1) is stable; if the derivative is negative definite, the null solution is asymptotically stable. Owing to the validity of the converse theorem [13] these conditions are sufficient and necessary.

A suitable functional $V(\tilde{\xi}, t)$ for the system (1) with $E \neq 0$ has not yet been found. If A is invertible it is convenient, although not necessary, to rewrite eq. (1) in the form

$$\ddot{\xi} + [M+N] \dot{\xi} + [P+Q] \xi = 0 \quad (2)$$

using the mapping K , $\dot{\xi} = \frac{\partial \xi}{\partial t}$, $\xi = K \eta$, $K^*AK = I$, $M = K^*BK$, $N = K^*CK$, $P = K^*DK$, $Q = K^*EK$. Let us further define a time dependent linear unitary transform $L(t)$

$$\xi(t) = L(t) \eta(t). \quad (3)$$

Let $L(t)$ be bounded in the time interval $[0, \infty)$, have a bounded continuous derivative $\dot{L}(t)$ and have a lower bound $0 < m < \|L(t)\|$ for $t \geq 0$. We seek the isometry $L(t)$ such that eq. (2) is transformed into

$$\eta'' + [\hat{M} + \hat{N}] \dot{\eta} + \hat{P} \eta = 0, \quad \hat{Q} = 0. \quad (4)$$

Let us impose the following constraints on $L(t)$:

$$L^*(t)L(t) = L^*(0)L(0) = I, \hat{M} = \hat{M}^*, \hat{P} = \hat{P}^*, \hat{N} = -\hat{N}^*, \hat{Q} = 0. \quad (5)$$

Then the compatibility conditions lead to a differential equation for $L(t)$:

$$\dot{L}(t) = \left\{ -(M+N)^\dagger (Q+H) + [I - (M+N)^\dagger (M+N)] R \right\} L(t) \equiv T L(t)$$

where $(M+N)^\dagger$ is in the general case the Moore-Penrose generalized inverse, which is unique and is defined in the corresponding subspace of L_2 . The arbitrary operators H and R are defined from the conditions (5), H being Hermitian. The transformed operators in eq. (4) are then

$$\begin{aligned} \hat{M} &= L^* M L, \quad \hat{N} = L^* (2T + N) L, \quad \hat{P} = L^* (P + TT + S) L \\ S &\equiv \frac{1}{2} (MT - TM) + \frac{1}{2} (NT + TN) \end{aligned} \quad (6)$$

Having specified the system (4) we can construct the Hermitian functional $\mathcal{V}(\eta, t)$ e.g. in the form as given in [14]. Then the sufficient and necessary conditions for stability (asymptotic stability) of the system (4) and at the same time of the system (1) are

$$\mathcal{V}(\eta, \dot{\eta}, t) = \langle \eta, L^* (P + TT + S) L \eta \rangle + \langle \dot{\eta}, \dot{\eta} \rangle > 0 \quad (7)$$

$$\begin{aligned} \dot{\mathcal{W}}(\eta, \dot{\eta}, t) \equiv \frac{d}{dt} \mathcal{V}(\eta, \dot{\eta}, t) &= -2 \langle \dot{\eta}, L^* M L \dot{\eta} \rangle + \\ &+ \langle \eta, L^* [(P+S)T - T(P+S)] L \eta \rangle \leq 0 \end{aligned}$$

The conditions (7) are verified also if $\dot{\eta}(\tau_1) = 0$ for $\eta(\tau_1) \neq 0$ if

$$\begin{aligned} \langle \xi, (P+TT+S)\xi \rangle > 0 \quad , \quad \langle \xi, [T(P+S) - (P+S)T]\xi \rangle \geq 0 \\ \langle (\xi' - T\xi), M(\xi' - T\xi) \rangle \geq 0 . \end{aligned} \tag{8}$$

The stabilizing or destabilizing influence of dissipative effects cannot be estimated without a detailed analysis from eq. (7) owing to the rather complicated structure of the operator T. In a particular case when the operator T commutes simultaneously with M, N and P the system (1) becomes reducible and all transformed operators (6) are time independent. This condition represents additional constraints to eq. (5). However, these conditions can only be satisfied for some particular cases, e.g. for some mechanical systems with matrix coefficients [15]. In the case $E = 0$ (static equilibrium or nondissipative fluid) in eq. (1), $L(t) = L^*(t) = I$, $T \equiv 0$, $V = V(\xi, \xi')$ and the stability conditions (7) read

$$\langle \xi, P\xi \rangle > 0 \quad , \quad \langle \xi', M\xi' \rangle \geq 0 . \tag{9}$$

These conditions correspond to earlier results. In contrast to the conditions (7) the operator N is here irrelevant for stability. If, furthermore, $M = 0$, as is the case of

ideal MHD, $\frac{d}{dt} \mathcal{V}(\xi, \xi^*) = 0$ and the system (1) cannot be asymptotically stable in the sense of Liapunov. Owing to the definition of the inner product, the conditions (7), (8) and (9) define the stability in the large.

We give an example of the Euler-Lagrange equations (2) in the case of dissipative two-fluid MHD. This model respects dissipative effects such as viscosity and resistivity ($\tilde{\eta}$) as well as Hall and finite Larmor radius effects. The corresponding MHD equations (α, β mean particle species) are

$$m_\alpha n_\alpha \frac{d}{dt} \underline{v}_\alpha = n_\alpha e_\alpha (\underline{E} + \underline{v}_\alpha \times \underline{B}) - \nabla p_\alpha - \nabla \cdot \underline{\Pi}_\alpha - \tilde{\eta} \frac{1}{c} e_\alpha^2 n_\alpha (n_\alpha \underline{v}_\alpha - n_\beta \underline{v}_\beta)$$

$$\frac{\partial}{\partial t} n_\alpha + \nabla \cdot n_\alpha \underline{v}_\alpha = 0$$

$$\frac{d}{dt} (p_\alpha n_\alpha^{-\gamma}) = 0.$$

Assuming stationary equilibrium isentropic flow, $\underline{v}_\alpha \neq 0$, using in the corresponding Maxwell equations the gauge $\underline{\Phi} = 0$ and introducing the Lagrangian displacements [2, 3, 16]

$$\underline{\xi} = \left[(m_i n_i)^{-1/2} \underline{\xi}_i, (m_e n_e)^{-1/2} \underline{\xi}_e, c (\mu_0)^{1/2} \underline{A} \right]^T$$

we obtain for the operators M, N, P and Q in eq. (2) (operators A, B, C and partly D are given in [10]):

$$M_{11} = M_i, M_{22} = M_e, M_{\alpha} \xi_{\alpha}^{\circ} = \tilde{\eta} \frac{e^2}{c} \frac{n_{\alpha}}{m_{\alpha}} \xi_{\alpha}^{\circ} + m_{\alpha}^{-1} n_{\alpha}^{-1/2} \nabla \cdot \Pi_{\alpha}^H (\xi_{\alpha}^{\circ} n_{\alpha}^{-1/2}),$$

$$M_{21} = M_{12} = -\tilde{\eta} \frac{e^2}{c} \left(\frac{n_e n_i}{m_e m_i} \right)^{1/2}, N_{11} = N_i, N_{22} = N_e,$$

$$N_{\alpha} \xi_{\alpha}^{\circ} = 2 n_{\alpha}^{1/2} (\underline{v}_{\alpha} \cdot \nabla) \xi_{\alpha}^{\circ} n_{\alpha}^{-1/2} + \frac{e_{\alpha}}{m_{\alpha}} \underline{B}_0 \times \xi_{\alpha}^{\circ} + m_{\alpha}^{-1} n_{\alpha}^{-1/2} \nabla \cdot \Pi_{\alpha}^{AH} (\xi_{\alpha}^{\circ} n_{\alpha}^{-1/2}),$$

$$N_{13} = -N_{31} = c e \mu_0^{1/2} \left(\frac{n_i}{m_i} \right)^{1/2}, N_{23} = -N_{32} = -c e \mu_0^{1/2} \left(\frac{n_e}{m_e} \right)^{1/2},$$

$$P = P_N + P_D, \quad \delta_{\alpha} \xi_{\alpha} \equiv (\underline{v}_{\alpha} \cdot \nabla) \xi_{\alpha} n_{\alpha}^{-1/2} - n_{\alpha}^{-1/2} (\xi_{\alpha} \cdot \nabla) \underline{v}_{\alpha},$$

$$P_{N,11} = P_i, P_{N,22} = P_e, P_{N,33} A = c^2 \nabla \times \nabla \times \underline{A},$$

$$P_{\alpha} \xi_{\alpha} = n_{\alpha}^{1/2} (\underline{v}_{\alpha} \cdot \nabla) (\underline{v}_{\alpha} \cdot \nabla) \xi_{\alpha} n_{\alpha}^{-1/2} + \frac{e_{\alpha} n_{\alpha}^{1/2}}{m_{\alpha}} \underline{B}_0 \times \delta_{\alpha} \xi_{\alpha} \\ - \frac{e_{\alpha}}{m_{\alpha}} (\xi_{\alpha} \cdot \nabla) [\underline{E}_0 + \underline{v}_{\alpha} \times \underline{B}_0] - m_{\alpha}^{-1} n_{\alpha}^{-1/2} \nabla (n_{\alpha}^{-1/2} \xi_{\alpha} \cdot \nabla P_{\alpha}) - m_{\alpha}^{-1} n_{\alpha}^{-1/2} \nabla [\underline{v}_{\alpha} P_{\alpha} (\nabla \cdot \xi_{\alpha} n_{\alpha}^{-1/2})] \\ + m_{\alpha}^{-1} n_{\alpha}^{-1/2} \nabla P_{\alpha} (\nabla \cdot \xi_{\alpha} n_{\alpha}^{-1/2}) + m_{\alpha}^{-1} n_{\alpha}^{-1} (\xi_{\alpha} \cdot \nabla) \nabla P_{\alpha},$$

$$P_{N,13} A = -c e \mu_0^{1/2} \left(\frac{n_i}{m_i} \right)^{1/2} \underline{v}_i \times \nabla \times \underline{A},$$

$$P_{N,23} A = c e \mu_0^{1/2} \left(\frac{n_e}{m_e} \right)^{1/2} \underline{v}_e \times \nabla \times \underline{A},$$

$$P_{N,31} \xi_i = c e \mu_0^{1/2} m_i^{-1/2} [\underline{v}_i (\nabla \cdot \xi_i n_i^{1/2}) - n_i \delta_i \xi_i],$$

$$P_{N,32} \xi_e = -c e \mu_0^{1/2} m_e^{-1/2} [\underline{v}_e (\nabla \cdot \xi_e n_e^{1/2}) - n_e \delta_e \xi_e],$$

$$(P_D + Q)_{11} = (P_D + Q)_{ii}, (P_D + Q)_{22} = (P_D + Q)_{ee},$$

$$(P_D + Q)_{12} = (P_D + Q)_{ie}, (P_D + Q)_{21} = (P_D + Q)_{ei}, \quad \nabla \cdot \Pi_{\alpha} \equiv \nabla \cdot \Pi_{\alpha} (\underline{v}_{\alpha}),$$

$$(P_D + Q)_{\alpha\alpha} \xi_{\alpha} = m_{\alpha}^{-1} n_{\alpha}^{-1/2} [(\nabla \cdot \Pi_{\alpha}) (\nabla \cdot \xi_{\alpha} n_{\alpha}^{-1/2}) + n_{\alpha}^{-1/2} (\xi_{\alpha} \cdot \nabla) (\nabla \cdot \Pi_{\alpha}) + \nabla \cdot \Pi_{\alpha} (\delta_{\alpha} \xi_{\alpha})]$$

$$+ \tilde{\eta} \frac{e^2}{c m_{\alpha}} [-n_{\alpha}^{1/2} \underline{v}_{\alpha} (\nabla \cdot \xi_{\alpha} n_{\alpha}^{1/2}) + n_{\alpha}^{3/2} \delta_{\alpha} \xi_{\alpha} + (\xi_{\alpha} \cdot \nabla) (n_{\alpha} \underline{v}_{\alpha} - n_{\beta} \underline{v}_{\beta})]$$

$$+ \frac{e^2}{c m_{\alpha}} n_{\alpha}^{1/2} [\tilde{\eta} (\nabla \cdot \xi_{\alpha} n_{\alpha}^{-1/2}) - n_{\alpha}^{-1/2} (\xi_{\alpha} \cdot \nabla) \tilde{\eta}] (n_{\alpha} \underline{v}_{\alpha} - n_{\beta} \underline{v}_{\beta}),$$

$$(P_D + Q)_{\alpha\beta} \xi_{\beta} = \tilde{\eta} \frac{e^2}{c} \left(\frac{n_{\alpha}}{m_{\alpha} m_{\beta}} \right)^{1/2} n_{\beta}^{-1} \underline{v}_{\beta} (\nabla \cdot \xi_{\beta} n_{\beta}^{1/2}) - \delta_{\beta} \xi_{\beta}.$$

Here, we have separated the dominant non-dissipative terms into the Hermitian operator P_N and dissipative terms, Hermitian and anti-Hermitian are collected in the operators P_D+Q . The term $\nabla \cdot \Pi_{\approx d} \left(\begin{matrix} \xi \\ \sim d \end{matrix} \right)$ was also divided into the Hermitian and anti-Hermitian parts [17]. We assume a perfectly conductive wall at the plasma boundary.

The demonstrated criteria for stability are valid for any continuous medium with is described by the Lagrangian system (1). The criteria (7) represent the first step towards elaboration of practically useful stability conditions. Nevertheless the use of test functions, suitable for a given geometry will lead to conditions accessible for numerical analysis.

It should be stressed that the conditions (7) are valid for a subclass of square integrable accessible displacements of the linearized system (1). The Lagrangian formulation is also limited to isentropic flow.

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Appendix

It is possible to demonstrate directly the converse theorem in the case when the Liapunov functional $V(\eta, \dot{\eta}, t)$ and its derivative $W(\eta, \dot{\eta}, t)$ are quadratic forms in variables η and $\dot{\eta}$. Taking the identity

$$V(\eta, \dot{\eta}, t) = V(\eta, \dot{\eta}, t_0) + \int_{t_0}^t dt W(\eta, \dot{\eta}, t)$$

and using eq. (7) we have

$$\begin{aligned} \langle \eta, \hat{P}\eta \rangle + \langle \dot{\eta}, \dot{\eta} \rangle &= \langle \eta, \hat{P}\eta \rangle_{t_0} + \langle \dot{\eta}, \dot{\eta} \rangle_{t_0} - \\ &- \int_{t_0}^t dt \left\{ 2\langle \dot{\eta}, \hat{M}\dot{\eta} \rangle + \langle \eta, L^* [T(P+S) - (P+S)T] L \eta \rangle \right\} \end{aligned}$$

Then, in analogy to the case $E = 0$ [18], we conclude: if the second condition (7) is satisfied and if for some $t = t_1$ the following holds

$$\langle \dot{\eta}, \dot{\eta} \rangle_{t_1} = 0 \quad , \quad \langle \eta, \hat{P}\eta \rangle_{t_1} < 0$$

then for any $t > t_1$

$$\langle \eta, \hat{P}\eta \rangle_t < 0 \quad , \quad \langle \eta, \hat{P}\eta \rangle_t \neq 0$$

and the system (4) cannot return to the equilibrium state.

The solution to the system (7) with respect to the functional $V(\eta, \dot{\eta}, t)$

$$\begin{aligned} & \langle \xi, [P+S+TT] \xi \rangle + \langle [\dot{\xi}-T\xi], [\dot{\xi}-T\xi] \rangle = \\ & = \left\{ \langle \xi, [P+S+TT] \xi \rangle + \langle [\dot{\xi}-T\xi], [\dot{\xi}-T\xi] \rangle \right\}_{t_0} \exp \int_{t_0}^t dt \frac{W(\xi, \dot{\xi})}{V(\xi, \dot{\xi})} \end{aligned}$$

illustrates the growth of the Lagrangian displacements $\xi(t)$ in case of instability.

The conditions (7) do not include the particular case $M + N = 0$ and the case when A is singular. This case will be discussed elsewhere.

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