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Correct and Incorrect Constraints in the Theory of Flux Conserving Tokamaks

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Abstract

Prescribing, in addition to the total poloidal magnetic flux, the toroidal flux and the plasma pressure as functions of the poloidal flux leads to a critical value of the plasma beta above which the tokamak equilibrium problem becomes unphysical: A degenerate separatrix (i.e. a flux surface across which the magnetic field reverses) emerges from the magnetic axis, the entropy density diverges on this separatrix, and the flux attains values outside its prescribed range. This difficulty disappears if one prescribes, instead of the pressure, the plasma adiabate, as is appropriate for simulating the temporal evolution of an equilibrium due to plasma heating.

In axial symmetry [$\partial/\partial\theta = 0$, where (r, θ, z) are cylindrical coordinates], the magnetohydrostatic equations,

$$\underset{\sim}{J} \times \underset{\sim}{B} = \nabla P, \quad \underset{\sim}{J} = \text{curl}\, \underset{\sim}{B}, \quad \text{div}\, \underset{\sim}{B} = 0, \tag{1}$$

($\underset{\sim}{B}$ is the magnetic field, $\underset{\sim}{J}$ the current density, and P the plasma pressure) are solved with

$$\underset{\sim}{B} = \nabla\theta \times \nabla\psi + I\,\nabla\theta \tag{2}$$

if the equation

$$r^2\, \text{div}\left(\frac{1}{r^2}\nabla\psi\right) + \frac{dP}{d\psi} + I\,\frac{dI}{d\psi} = 0 \tag{3}$$

is satisfied, where P and I are functions of ψ ($2\pi\psi$ is the poloidal magnetic flux, and $2\pi I$ is the poloidal current). To specify a unique solution of Eq. (3), one must impose constraints. The purpose of this note is to discuss various possible constraints and their relation to the problem of determining a sequence of equilibria through which the plasma passes a result of heating.

The conventional approach to Eq. (3) is to impose the boundary condition $\psi = 0$ at the boundary of a toroidal domain and to adopt the constraint of given functions $P(\psi)$ and $I(\psi)$. Even though the resulting quasilinear elliptic problem is mathematically well-posed in many cases, it is unphysical for several reasons: Firstly, since the range of the solution $\psi(r,z)$ is not known a priori, it is also not known in what domain the functions $P(\psi)$ and $I(\psi)$ must be given. Secondly, there is no simple correspondence to actual experimental situations in which a parameter is varied; for instance, heating of the plasma, besides increasing P, changes I in such a complex manner that it is not useful to prescribe this quantity. Thirdly, there is no guarantee that the flux surfaces ψ = const form singly nested toroids; for instance, if one starts from an equilibrium with this desired property and then increases $P(\psi)$ keeping $I(\psi)$ fixed, a separatrix eventually occurs so that the plasma beta is limited to undesirably low values.

The concept of flux conserving equilibrium families was introduced [1] to avoid these disadvantages. The crucial idea is to prescribe, instead of the poloidal current, the toroidal magnetic flux Φ as a function of ψ. From Eq. (2), this flux is related to the poloidal current through

$$2\pi I = K_1 \frac{d\Phi}{d\psi} \dot{\psi} . \tag{4}$$

Here, the dot denotes the derivative with respect to V, the volume of the interior of a flux surface, and

$$K_1 = \langle 1/4\pi^2 r^2 \rangle^{-1}, \tag{5}$$

where

$$\langle \cdots \rangle = \oint\!\!\oint d^2S \cdots / |\nabla V| \tag{6}$$

is the usual flux surface average. Combining Eqs. (3) and (4) yields

$$\mathrm{div}\left(\frac{1}{r^2}\nabla\psi\right) + \frac{dP}{d\psi} + \frac{K_1}{4\pi^2 r^2}\frac{d\Phi}{d\psi}\left(K_1 \frac{d\Phi}{d\psi}\dot{\psi}\right)^{\cdot} = 0 . \tag{7}$$

Owing to the presence of the variable V, this is a generalized differential equation [2].

In most previous studies of flux conserving equilibria [1,3,4], Eq. (7) was considered with given functions $P(\psi)$ and $\Phi(\psi)$. The following arguments indicate why this constraint is superior to the conventional one: Firstly, the average of Eq. (7),

$$(K_2\dot{\psi})^{\cdot} + \frac{dP}{d\psi} + \frac{d\Phi}{d\psi}\left(K_1\frac{d\Phi}{d\psi}\dot{\psi}\right)^{\cdot} = 0, \tag{8}$$

with

$$K_2 = \langle |\nabla V|^2 / r^2 \rangle, \tag{9}$$

being a second order ordinary differential equation for $\psi(V)$, requires that two values of ψ be prescribed (e.g. $\psi(0) = 0$ and $\psi(V_0) = \psi_0$, where V_0 is the total volume). Iterating between Eqs. (7) and (8) then suggests that the problem is well posed [2]. Thus, the functions $P(\psi)$ and $\Phi(\psi)$ may be given in the prescribed domain $0 \leq \psi \leq \psi_0$, at least as long as the solution $\psi(V)$ is monotonic. Secondly, generating a family of equilibria by letting P be an increasing function of time while keeping $\Phi(\psi)$ fixed, one hopes to describe situations in which the plasma is heated fast enough for its resistivity to be ignored (this implies conservation of flux, and hence of $\Phi(\psi)$), and at the same time slowly enough for its inertia to be ignored (this implies that it passes through a sequence of equilibria). Thirdly, since the inverse rotation number

$$q = \frac{1}{2\pi} \frac{d\Phi}{d\psi} \tag{10}$$

diverges at an "ordinary separatrix" (i.e. a flux surface which crosses itself) unless it is identically zero (we exclude this exceptional case of a purely poloidal magnetic field because it does not correspond to a tokamak), such a separatrix cannot occur as long as the function $\Phi(\psi)$ is smooth and not a constant. This suggests that the plasma beta can be raised to arbitrarily high values without causing a change of topology.

In this note we show that there is, nevertheless, a limitation upon beta, but that no such limitation occurs if the plasma adiabate

$$\alpha = P\dot{\psi}^{-\gamma} \qquad (\gamma = 5/3) \tag{11}$$

(which is closely related to the entropy density) is given instead of the pressure. In other words, $\alpha(\psi)$ may be prescribed arbitrarily, but $P(\psi)$ may not. Specifically, assuming that $p(\psi) = P(\psi)/P(0)$ is fixed with $dp/d\psi \leq 0$ and $p(\psi_0) = 0$, we show that $\psi(V)$ increases monotonically if $P(0)$ is sufficiently small, but that $\psi(V)$ drops to negative values away from the axis before it increases to ψ_0 towards the boundary if $P(0)$ is too big. In the latter case, the range of $\psi(V)$, and hence also the domain of $P(\psi)$ and $\Phi(\psi)$, is a priori unknown (as it is in the conventional approach), the pressure attains its maximum where ψ attains its minimum (implying a change of topology because some pressure surfaces are disconnected), the magnetic field reverses across the minimum of ψ, and the entropy, along with the adiabate, diverges there. Clearly, $\nabla\psi$ vanishes identically at the flux surface of minimum ψ. Therefore, by analogy with an ordinary separatrix at which $\nabla\psi$ vanishes only along a curve, we call this surface a "degenerate separatrix". Thus, the constraint of flux conservation fails to prevent the presence of a separatrix; it merely forces a separatrix to be degenerate.

To substantiate our claim, we put the averaged equation (8) into Lagrangian form,

$$(\partial L/\partial \dot{\psi})^{\cdot} - \partial L/\partial \psi = 0, \tag{12}$$

with Lagrangian

$$L(\dot{\psi}, \psi, V) = \frac{M}{2}\dot{\psi}^2 - P(\psi), \tag{13}$$

$$M(\psi, V) = K_2(V) + K_1(V)\left(\frac{d\Phi}{d\psi}\right)^2. \tag{14}$$

Thus, the boundary-value problem for Eq. (8) is equivalent to determining the trajectory of a particle with mass M (depending on the position ψ and the time V) which moves in a potential $P(\psi)$, and which starts from the position $\psi = 0$ at time $V = 0$ and reaches the position $\psi = \psi_0$ at time $V = V_0$. Since $dP/d\psi \leq 0$, the particle goes downhill, with $P(0)$ measuring the steepness of the slope. If $P(0)$ is small, the particle must be pushed initially towards its final position in order not to reach this too late; hence $\psi(V)$ increases monotonically. If, on the other hand, $P(0)$ is too big, the particle reaches its final position too early unless it is pushed initially in the opposite direction; hence $\psi(V)$ has a negative minimum. The only exception occurs if $dP/d\psi = 0$ for $\psi = 0$ (the pressure profile is flat on axis) because then the particle stays at its initial position forever if its initial velocity is zero, regardless of the value of $P(0)$.

In order actually to calculate the critical value of $P(0)$, one needs the quantity M . Since this depends upon the geometry of the flux surfaces, determining it requires consideration of the full equilibrium equation (7) rather than just its average (8). However, this problem simplifies in the limit of large aspect ratio. Thus, introducing the major radius R such that $V_0 = 2\pi R A$, where A is the area of the poloidal cross-section of the domain, and assuming that the inverse aspect ratio $\varepsilon \sim A^{1/2}/R$ is small, we note that $K_2/K_1 = O(\varepsilon^2)$ and that $K_1 \approx (2\pi R)^2$. Hence, for small ε,

$$M \approx (4\pi^2 R q)^2 \tag{15}$$

and the Lagrangian (13) has no explicit dependence upon V , implying that the Hamiltonian, $H = \mathring{\psi}\partial L/\partial\mathring{\psi} - L$, is a constant. Introducing, instead of $\psi(V)$, the unknown function $\phi(V)$, we then have

$$H = 2\pi^2 R^2 \mathring{\phi}^2 + P(\phi). \tag{16}$$

Upon integration,

$$V = 2\pi R \int_0^{\phi} d\phi / \sqrt{2(H - P(\phi))}, \tag{17}$$

where the constant H must be determined from the boundary condition at $V = V_0$,

$$V_0 = 2\pi R \int_0^{\Phi_0} d\Phi / \sqrt{2(H - P(\Phi))} \tag{18}$$

with $\Phi_0 = \Phi(\psi_0)$. The critical pressure is now obtained by putting $\dot{\Phi}(0) = 0$. Equation (16) then yields $H = P(0)$, and Eq. (18), when solved for $P(0)$, becomes

$$P(0) = \frac{\Phi_0^2}{2A^2} \left(\int_0^1 \frac{d\varphi}{\sqrt{1 - p(\varphi)}} \right)^2, \tag{19}$$

where $\varphi = \Phi/\Phi_0$. The corresponding critical value of the volume-averaged pressure turns out to be

$$\frac{1}{V_0} \int_0^{V_0} dV P = \frac{\Phi_0^2}{4A^2} \int_0^1 \frac{d\varphi}{\sqrt{1 - p(\varphi)}} \int_0^1 \frac{d\varphi \, p(\varphi)}{\sqrt{1 - p(\varphi)}} . \tag{20}$$

Since, from Schwarz' inequality, $\Phi_0^2/A^2 \leq \int dV B^2/V_0$, Eq. (20) imposes an upper bound upon ß, the ratio of the volume-averaged thermal and magnetic pressures. This bound depends solely upon the profile $p(\varphi)$, and it may take any values between zero and infinity. To give an example, we put $p(\varphi) = (1 - \varphi)^a$ with $a > 0$ to find

$$\beta \leq \frac{2^{4/a} [\Gamma(1/a) \Gamma(1 + 1/a)]^2}{2 a^2 \Gamma(2/a) \Gamma(2 + 2/a)} . \tag{21}$$

For $a \to 0$ (flat pressure profile) this tends to infinity, as we have already seen in more generality; for $a \to \infty$ (peaked pressure profile) it tends to zero like $1/a$; for $a = 1$ (linear pressure profile) it equals 4/3. Thus, the critical ß appears to be rather high for simple profiles. This is the reason why no difficulties were incountered in numerical calculations [3].

The preceding discussion shows that prescribing $p(\psi)$ is not adequate for solving the equilibrium problem at high beta. As one would expect, a more adequate constraint is that suggested by the actual physical problem. Thus, considering the evolution of an equilibrium due to heating, we introduce an energy source with power density Q . Assuming an ideal gas, we then write the energy balance of ideal magnetohydrodynamics as

$$\rho^{\gamma} \frac{d}{dt} p \rho^{-\gamma} = (\gamma - 1) Q, \tag{22}$$

where ρ is the mass density, and d/dt is the Lagrangian time derivative following the fluid motion. Assuming that the plasma is in an axially symmetric equilibrium at every instant of time now implies that ρ and Q , along with p , are functions of ψ and t , and that d/dt is the derivative at fixed ψ . Using Eq. (11) to eliminate p in favor of $\dot{\alpha}$, and using the mass conservation

$$\frac{d}{dt}\rho/\dot{\psi} = 0, \tag{23}$$

we then write Eq. (22) as

$$\frac{d\alpha}{dt} = (\gamma - 1)\dot{\psi}^{-\gamma} Q. \tag{24}$$

This equation (which was already used in Refs. 5 and 6) is now coupled to Eq. (7), with ρ eliminated in favor of α :

$$\operatorname{div}\left(\frac{1}{r^2}\nabla\psi\right) + \frac{1}{\dot{\psi}}(\alpha\dot{\psi}^{\gamma})^{\cdot} + \frac{K_1}{4\pi^2 r^2}\frac{d\Phi}{d\psi}\left(K_1\frac{d\Phi}{d\psi}\dot{\psi}\right)^{\cdot} = 0. \tag{25}$$

We note that α is a natural variable because it is conserved if Q vanishes and because introducing any other variable one would also introduce the time derivative of $\dot{\psi}$.

Equations (24) and (25) govern the evolution of a plasma which is heated either by the source Q or by adiabatic compression. In the first case, one must solve them with $\alpha(\psi)$ given initially; numerically, this can be done by alternating between Eq. (25) [to compute $\psi(r,z)$ for given $\alpha(\psi)$] and Eq. (24) [to advance $\alpha(\psi)$ in time]. In the second case, Q is zero but the domain is a given function of time such that V_0 decreases; Eq. (24) then implies that $\alpha(\psi)$ is fixed, and it remains to solve Eq. (25) at every instant of time. Since Eq.(25) must be solved with given $\alpha(\psi)$ in either case, prescribing the adiabate appears to be the natural approach to the equilibrium problem. The constraint of given $\alpha(\psi)$ is crucially different from that of given $P(\psi)$ because in the former case α is kept finite (as it must be because the entropy remains finite in reality), while in the latter case α diverges at a degenerate separatrix. However, this does not prove by itself that a degenerate separatrix cannot occur; we can merely conclude that P vanishes at such a separatrix and becomes negative in its interior. Since P must not be negative, it is important to show that $\psi(V)$ is monotonic. This amounts to showing that the solution of the average of Eq. (25),

$$(K_2\dot{\psi})^{\cdot} + \frac{1}{\dot{\psi}}(\alpha\dot{\psi}^{\gamma})^{\cdot} + \frac{d\Phi}{d\psi}\left(K_1\frac{d\Phi}{d\psi}\dot{\psi}\right)^{\cdot} = 0, \qquad (26)$$

is monotonic.

We conjecture that this is so because Eq. (26) implies that $\ddot{\psi} = 0$ wherever $\dot{\psi} = 0$, but we have not succeeded in giving a proof, except in the two limiting cases of large aspect ratio (this describes the final state of a compressionally heated plasma) or large α (this describes the final state of a source-

-heated plasma). In these limits, the Lagrangian of Eq. (26)

$$L=\frac{1}{2}M\dot{\psi}^2+\frac{1}{\gamma-1}\alpha\dot{\psi}^\gamma \tag{27}$$

has no explicit dependence upon V (as before, M becomes independent of V for large aspect ratio; for large α, the term proportional to M may be neglected). Hence the Hamiltonian

$$H=\frac{1}{2}M\dot{\psi}^2+\alpha\dot{\psi}^\gamma \tag{28}$$

is a constant. For $\dot{\psi}\geq 0$, H is an increasing function of $\dot{\psi}$ which vanishes for $\dot{\psi}=0$, and which tends to infinity for $\dot{\psi}\to\infty$, thus having a monotonic inverse $\dot{\psi}=f(H,\psi)$ with $f=0$ for $H=0$ and $f\to\infty$ for $H\to\infty$. Upon integration,

$$V=\int_0^\psi d\psi/f(H,\psi), \tag{29}$$

where H , because of the monotonicity of f , is uniquely determined by the boundary condition

$$V_0=\int_0^{\psi_0} d\psi/f(H,\psi). \tag{30}$$

Hence the boundary-value problem has a unique solution whose monotonicity is obvious. In the case of large α a further conclusion is possible because f can be written down explicitly. Equations (29), (30) and (11) then yield

$$P = \left(\frac{1}{V_0}\int_0^{\psi_0} d\psi\, \alpha^{1/\gamma}\right)^{\gamma}. \tag{31}$$

Thus, the pressure becomes a large constant, this being consistent with our previous finding that P may become large without causing a separatrix only if the pressure profile is flat on axis. We note that Eqs. (28) - (31), in general, are invalid in a small boundary layer. For instance, P vanishes at the boundary if α does, in contradiction with Eq. (31); the formal reason is that the term proportional to M in Eq. (27) may not be neglected where α vanishes.

Finally, as an illustration, we consider the evolution due to a constant energy source of a large aspect ratio shearless equilibrium with zero temperature (Q and q are constants, and α vanishes initially). Since M is approximately constant in this case, Eqs. (24) and (26) imply that $\dot{\psi}$ is an approximate constant, too, while α remains approximately independent of ψ but increases linearly in time. To determine the flux surfaces, one must consider Eq. (25). If $q = O(1)$ the first term is $O(\varepsilon^2)$ relative to the other two terms. As long as α is small, the second term does not enter the lowest order, and the solution $\psi(r, z)$ can be obtained by a regular expansion in powers of ε , therefore not exhibiting any remarkable behaviour (for instance, the flux surfaces have approximately concentrically circular cross-sections if the boundary is circular). However, once α has become big enough for the second term to be comparable to the third term, ψ becomes, to lowest order in ε , a function of τ only. Hence the boundary condition cannot be satisfied, and the expansion fails. As a consequence, a boundary layer must be introduced, and the poloidal cross-sections assume the shapes indicated in Fig. 1. Thus, the various profiles remain essentially unchanged, but the flux surfaces undergo a dramatic change in that the magnetic axis moves towards the wall as the pressure increase and the temperature along with α. Needless to say, this effect, being due to the toroidal curvature, is not present in cylindrical equilibria [5] , but is well visible in computer plots of tokamak calculations [6] .

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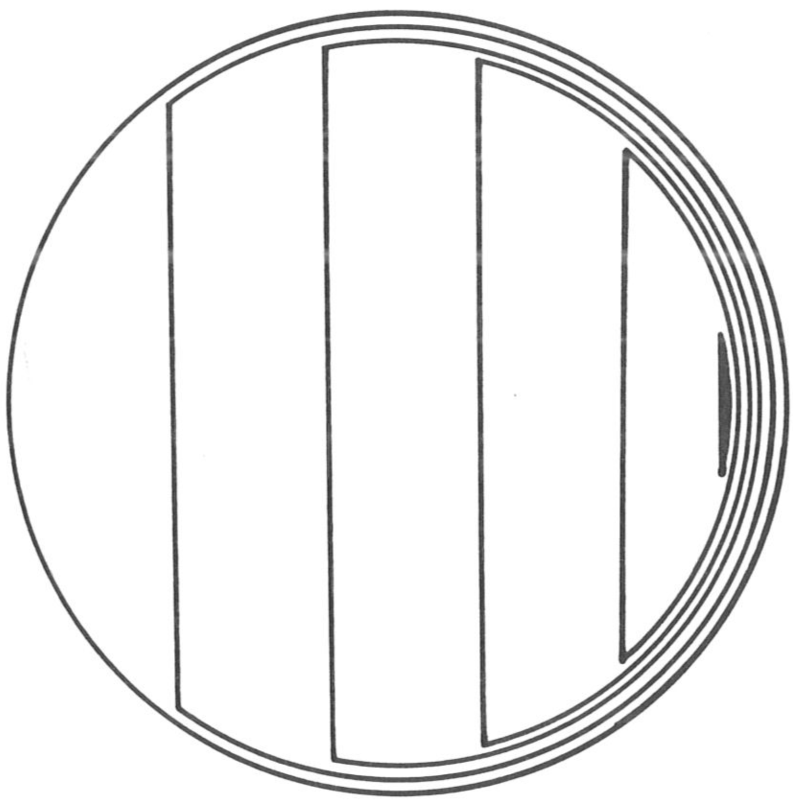

Figure 1: Poloidal cross-sections at large aspect ratio and high beta